5. Suppose that a function f satisfies the following conditions:

❶ f is positive and continuous on $[0, \infty)$.

❷ $f'(x) < 0$ for all $x > 0$.

❸ The improper integral $\displaystyle\int_0^\infty x f(x)\, dx$ converges.

Let R be the region between the graph of $y = f(x)$ and the x-axis for $x \geq 0$, and let $V(a)$ be the volume of the solid generated by revolving the region R about the line $x = a$ where a is a nonnegative real number.

a. Show that the improper integral $\displaystyle\int_0^\infty f(x)\, dx$ converges.

$$\int_0^\infty x f(x)\, dx \text{ converges} \Rightarrow \left.\begin{array}{l}\displaystyle\int_1^\infty x f(x)\, dx \text{ converges} \\ 0 \leq f(x) \leq x f(x) \text{ for } x \geq 1\end{array}\right\} \Rightarrow \int_1^\infty f(x)\, dx \text{ converges by Comparison Test}$$

$$\Rightarrow \int_0^\infty f(x)\, dx \text{ converges as } f \text{ is continuous on } [0,1].$$

b. Express $V(a)$ using the cylindrical shells method.

$$V(a) = 2\pi \int_0^a (a-x) f(x)\, dx + 2\pi \int_{2a}^\infty (x-a) f(x)\, dx$$

c. Show that $V''(a) > 0$ for all $a > 0$.

$$V(a) = 2\pi \left(a \int_0^a f(x)\, dx - \int_0^a x f(x)\, dx + \int_{2a}^\infty x f(x)\, dx - a \int_{2a}^\infty f(x)\, dx \right)$$

FTC1

$$V'(a) \overset{\downarrow}{=} 2\pi \left(\int_0^a f(x)\, dx + a f(a) - a f(a) - 2a f(2a) \cdot 2 - \int_{2a}^\infty f(x)\, dx + a f(2a) \cdot 2 \right)$$

$$= 2\pi \left(\int_0^a f(x)\, dx - \int_{2a}^\infty f(x)\, dx - 2a f(2a) \right)$$

FTC1

$$V''(a) \overset{\downarrow}{=} 2\pi \left(f(a) + f(2a) \cdot 2 - 2 f(2a) - 2a f'(2a) \cdot 2 \right)$$

$$= 2\pi \left(f(a) - 4a f'(2a) \right) > 0 \quad \text{for } a > 0$$

as $f(a) > 0$ and $f'(2a) < 0$ for $a > 0$.