2. An isosceles triangle ABC with $|AB| = |BC|$ in the xy-plane satisfies the following conditions:

❶ The side $[BC]$ lies along the x-axis

❷ The side $[AB]$ passes through the point $P(0, 6)$

❸ The side $[AC]$ passes through the point $Q(5, 6)$

Determine the largest and smallest possible values of the area of the triangle ABC.

Let y be the y-coordinate of A and let S be the area of ABC.
The area of APQ is $\frac{1}{2}\cdot 5\cdot(y-6)$. Since ABC is similar to APQ with a ratio of $\frac{y}{y-6}$, the area of ABC is $\left(\frac{y}{y-6}\right)^2\cdot\frac{5}{2}(y-6) = \frac{5}{2}\cdot\frac{y^2}{y-6}$.

(Although there are two different triangles for any given y with $6<y<11$, they both have the same area.)

Since $|AP| = |PQ| = 5$ and $|AP| \geqslant y-6$, y can be at most $5+6=11$.

Hence we want to:

$$\text{Maximize/Minimize } S = \frac{5}{2}\cdot\frac{y^2}{y-6} \text{ for } 6 < y \leq 11$$

Critical points: $\frac{dS}{dy} = \frac{5}{2}\cdot\left(\frac{2y}{y-6} - \frac{y^2}{(y-6)^2}\right) = \frac{5}{2}\cdot\frac{y(y-12)}{(y-6)^2} = 0 \Rightarrow \underbrace{y=0 \text{ or } y=12}_{\text{Not in the interval}}$

Endpoints: $y=6$: $\lim_{y\to 6^+} S = \infty$

$y = 11 \Rightarrow S = \frac{121}{2}$

The smallest possible area is $\frac{121}{2}$. There is no largest possible area as the area can be made arbitrarily large by choosing y sufficiently close to 6.