**5.** Find all values of the constant $k$ for which the function $f(x) = \dfrac{k}{(e^x + e^{-x})^2}$ satisfies the equation

$$f'(x) = f(x) \int_0^x f(t)\,dt$$

for all $x$.

If $k=0$, then $f(x)=0$ for all $x$ and the equality holds for all $x$.

If $k \neq 0$, then:

$$f'(x) = -\frac{2k}{(e^x+e^{-x})^3} \cdot (e^x - e^{-x}) \Rightarrow \frac{f'(x)}{f(x)} = -2\,\frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\Rightarrow \left(\frac{f'(x)}{f(x)}\right)' = -2 \cdot \frac{(e^x+e^{-x})^2 - (e^x - e^{-x})^2}{(e^x+e^{-x})^2} = \frac{-8}{(e^x+e^{-x})^2}$$

Let $g(x) = \dfrac{f'(x)}{f(x)} - \int_0^x f(t)\,dt$.

$g(x)=0$ for all $x \iff g'(x)=0$ for all $x$ and $g(0)=0$

$$\underset{\text{FTC1}}{\iff} \left(\frac{f'(x)}{f(x)}\right)' - f(x) = 0 \text{ for all } x \text{ and } \frac{f'(0)}{f(0)} - \int_0^0 f(t)\,dt = 0$$

$$\iff \frac{-8}{(e^x+e^{-x})^2} - \frac{k}{(e^x+e^{-x})^2} = 0 \text{ for all } x \text{ and } \frac{0}{k/4} - 0 = 0$$

$$\iff k = -8$$

Hence the equality holds for all $x$ exactly when $k=0$ or $k=-8$.