**5.** The towns $A$ and $B$ lie inland and the town $C$ lies on the coast as shown in *Figures 1* and *2*. A port $P$ will be built on the coast and connected to the towns $A$, $B$, $C$ with straight roads. The cost of constructing the roads connecting $P$ to $A$ and $P$ to $B$ is $10^5$ ₺/km, and the cost of constructing the road connecting $P$ to $C$ is $k \times 10^5$ ₺/km. The port will be built at the location for which the total cost of constructing these roads is the smallest.

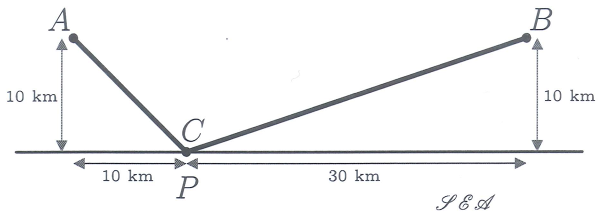


Figure 1

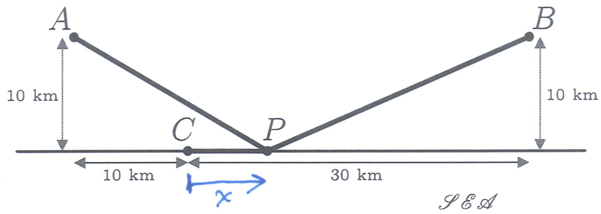


Figure 2

For some values of $k$ the smallest total cost will be achieved when $P$ is built at $C$, as in *Figure 1*; whereas for others it will be achieved, if at all, when $P$ is built somewhere else, as in *Figure 2*. Determine the smallest value of $k$ for which the lowest total cost is achieved by building $P$ at $C$.

The total cost is given by

$$K(x) = (1^2+(x+1)^2)^{1/2} + (1^2+(3-x)^2)^{1/2} + k\cdot|x| \quad \text{for } -\infty < x < \infty$$

in units of $10^5$ ₺, where the distances are measured in units of 10 km and $x$ is the directed distance from $C$ to the right. Then:

$$K'(x) = \begin{cases} (x+1)\cdot(1+(x+1)^2)^{-1/2} + (x-3)\cdot(1+(x-3)^2)^{-1/2} + k & \text{if } x>0 \\ (x+1)\cdot(1+(x+1)^2)^{-1/2} + (x-3)\cdot(1+(x-3)^2)^{-1/2} - k & \text{if } x<0 \end{cases}$$

1 [ If $k < \frac{3}{\sqrt{10}} - \frac{1}{\sqrt{2}}$, then $\lim_{x\to 0^+} K'(x) = -\left(\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right) + k < 0$ and therefore $K'$ is negative on some interval $(0,a)$. Then $K$ is decreasing on $[0,a]$ as $K$ is continuous. Therefore, the absolute minimum of $K$ does not occur at $x=0$.

$$K''(x) = (1+(x+1)^2)^{-1/2} - (x+1)^2\cdot(1+(x+1)^2)^{-3/2} + (1+(x-3)^2)^{-1/2} - (x-3)^2\cdot(1+(x-3)^2)^{-3/2}$$
$$= (1+(x+1)^2)^{-3/2} + (1+(x-3)^2)^{-3/2} > 0 \quad \text{for all } x \neq 0$$

$\Rightarrow K'$ is increasing on $(-\infty,0)$ and $(0,\infty)$. In particular:

2 [ If $k = \frac{3}{\sqrt{10}} - \frac{1}{\sqrt{2}}$, then $\lim_{x\to 0^+} K'(x) = 0 \Rightarrow K'(x) > 0$ for $x>0$, and $\lim_{x\to 0^-} K'(x) = -2\cdot\left(\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right) < 0 \Rightarrow K'(x) < 0$ for $x<0$. Therefore, $K$ has its absolute minimum at $x=0$.

1 and 2 $\Longrightarrow \frac{3}{\sqrt{10}} - \frac{1}{\sqrt{2}}$ is the smallest value of $k$ for which the absolute minimum of $K$ occurs at $C$.