4a. A sequence $\{a_n\}_{n=0}^{\infty}$ satisfies $a_0 = 2$ and $a_n = 3 - \dfrac{2}{(a_{n-1})^2}$ for $n \geq 1$.

i. Fill in the boxes: $a_1 = \boxed{\frac{5}{2}}$ and $a_2 = \boxed{\frac{67}{25}}$

ii. Show that the sequence $\{a_n\}_{n=0}^{\infty}$ converges and find its limit.

$\dfrac{2}{a_{n-1}^2} > 0 \Rightarrow a_n = 3 - \dfrac{2}{a_{n-1}^2} < 3$ for all $n \geq 1 \Rightarrow \{a_n\}_{n=0}^{\infty}$ is bounded above.

$\circledast$ $1 \leq a_0 < a_1$

$\circledast\circledast$ If $1 \leq a_k < a_{k+1}$ for some $k \geq 0$, then $1 \leq a_k^2 < a_{k+1}^2 \Rightarrow 1 \geq \dfrac{1}{a_k^2} > \dfrac{1}{a_{k+1}^2}$

$\Rightarrow -2 \leq -\dfrac{2}{a_k^2} < -\dfrac{2}{a_{k+1}^2} \Rightarrow 1 \leq 3 - \dfrac{2}{a_k^2} < 3 - \dfrac{2}{a_{k+1}^2} \Rightarrow 1 \leq a_{k+1} < a_{k+2}$

$\Rightarrow \{a_n\}_{n=0}^{\infty}$ is increasing.

$\{a_n\}_{n=0}^{\infty}$ is bounded above and increasing $\Rightarrow \{a_n\}_{n=0}^{\infty}$ converges by MST.

Let $L = \lim\limits_{n\to\infty} a_n$. $a_n = 3 - \dfrac{2}{a_{n-1}^2}$ for $n \geq 1 \Rightarrow L = 3 - \dfrac{2}{L^2} \Rightarrow L^3 - 3L^2 + 2 = 0$

$\Rightarrow (L-1)(L^2 - 2L - 2) = 0 \Rightarrow L = 1,\ L = 1 - \sqrt{3}$ or $L = 1 + \sqrt{3}$.

$a_n \geq 2$ for $n \geq 0 \Rightarrow L \geq 2 \Rightarrow L = 1 + \sqrt{3}$

4b. A sequence $\{c_n\}_{n=0}^{\infty}$ satisfies $c_0 = 1$, $c_1 = 2$, and $c_{n+1} = 3c_n - \dfrac{2(c_{n-1})^2}{c_n}$ for $n \geq 1$.

i. Fill in the boxes: $c_2 = \boxed{5}$ and $c_3 = \boxed{\frac{67}{5}}$

ii. Find the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} c_n x^n$.

Let $a_n = \dfrac{c_{n+1}}{c_n}$ for $n \geq 0$. Then $\{a_n\}_{n=0}^{\infty}$ is the sequence in <u>Part a</u>.

$\dfrac{1}{R} = \lim\limits_{n\to\infty} \dfrac{|c_{n+1}|}{|c_n|} = \lim\limits_{n\to\infty} |a_n| = 1 + \sqrt{3}$ by <u>Part a</u>.

Hence $R = \dfrac{1}{1+\sqrt{3}}$.