3. Determine whether each of the following series converges or diverges.

In this question you may use the fact that $\lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}$.

a. $\sum_{n=1}^{\infty} (\cos(1/n))^{n^2}$

$$a_n = (\cos(1/n))^{n^2} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (\cos(1/n))^{n^2} = \lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2} \neq 0$$

$$\Rightarrow \sum_{n=1}^{\infty} (\cos(1/n))^{n^2} \text{ diverges by nTT.}$$

b. $\sum_{n=1}^{\infty} (\cos(1/n))^{n^3}$

$$a_n = (\cos(1/n))^{n^3} \Rightarrow L = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\cos(1/n)^{n^3}\right|^{1/n} = \lim_{n\to\infty} (\cos(1/n))^{n^2}$$

$$= \lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}$$

$$L = e^{-1/2} < 1 \Rightarrow \sum_{n=1}^{\infty} (\cos(1/n))^{n^3} \text{ converges by nRT.}$$

c. $\sum_{n=1}^{\infty} \ln(\cos(1/n)) = -\sum_{n=1}^{\infty} (-\ln(\cos(\frac{1}{n}))$

$$c = \lim_{n\to\infty} \frac{-\ln(\cos(1/n))}{\frac{1}{n^2}} = \lim_{n\to\infty} (-n^2 \ln(\cos(1/n))) = \lim_{n\to\infty} \left(-\ln\left((\cos(1/n))^{n^2}\right)\right)$$

$$= \lim_{x\to 0} \left(-\ln\left((\cos(x))^{1/x^2}\right)\right) = -\ln(e^{-1/2}) = \frac{1}{2}$$

Since $c = \frac{1}{2} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$),

$\sum_{n=1}^{\infty} (-\ln(\cos(\frac{1}{n})))$ converges by LCT.

Hence $\sum_{n=1}^{\infty} \ln(\cos(\frac{1}{n}))$ converges.