1a. Let $f(x) = \underbrace{x + \frac{x^3}{2} + \frac{x^5}{4} + \frac{x^7}{8} + \cdots + \frac{x^{2n+1}}{2^n} + \cdots}$. Find all x satisfying the equation $f(x) = -1$.

This is a geometric series with $a = x$ and $r = \frac{x^2}{2}$.

$$\Rightarrow \begin{cases} f(x) = \dfrac{x}{1 - \frac{x^2}{2}} & \text{if } |r| = \left|\frac{x^2}{2}\right| < 1 \quad [\Leftrightarrow |x| < \sqrt{2}] \\ f(x) \text{ is undefined} & \text{if } |r| = \left|\frac{x^2}{2}\right| \geq 1 \quad [\Leftrightarrow |x| \geq \sqrt{2}] \end{cases}$$

$$f(x) = -1 \Longleftrightarrow \frac{x}{1 - \frac{x^2}{2}} = -1 \Leftrightarrow x^2 - 2x - 2 = 0 \Leftrightarrow x = 1 - \sqrt{3} \text{ or } x = 1 + \sqrt{3}.$$

$$1 + \sqrt{3} > 1 + 1 = 2 > \sqrt{2} \Rightarrow f(1+\sqrt{3}) \text{ is undefined.}$$

$$0 < \sqrt{3} - 1 < 1 < \sqrt{2} \Rightarrow x = 1 - \sqrt{3} \text{ is the only solution.}$$

1b. Let $A = \frac{1}{2} + \underbrace{\frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} + \cdots + \frac{1}{(2n+1) \cdot 2^{2n+1}} + \cdots}_{B}$. Show that $A < \frac{5}{9}$.

$$B < \underbrace{\frac{1}{3 \cdot 2^3} + \frac{1}{3 \cdot 2^5} + \cdots + \frac{1}{3 \cdot 2^{2n+1}} + \cdots} = \frac{\frac{1}{3 \cdot 2^3}}{1 - \frac{1}{2^2}} = \frac{1}{18}$$

This geometric series with $a = \frac{1}{3 \cdot 2^3}$ and $r = \frac{1}{2^2}$ converges as $|r| = \left|\frac{1}{2^2}\right| = \frac{1}{4} < 1$, and

Therefore,

$$A = \frac{1}{2} + B < \frac{1}{2} + \frac{1}{18} = \frac{10}{18} = \frac{5}{9}$$