HIGHER DESCENT ON PELL CONICS.
II. TWO CENTURIES OF MISSED OPPORTUNITIES

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Introduction

It was already observed by Euler [Eul1773] that the method of continued fractions occasionally requires a lot of tedious calculations, and even Fermat knew – as can be seen from the examples he chose to challenge the English mathematicians – a few examples with large solutions. To save work, Euler suggested a completely different method, which allows to compute even very large solutions of certain Pell equations rather easily; its drawback was that the method worked only for a specific class of equations. Although Euler's tricks were rediscovered on an almost regular basis, nobody really took this approach seriously or generalized it to arbitrary Pell equations.

The main goal of [Lem2003a] and this article is to discuss certain results that have been obtained over the last few centuries and which will be put into a bigger perspective in [Lem2003b]. This is opposite to what Dickson aimed at when he wrote his history; in [Dic1920, vol II, preface] he says

> What is generally wanted is a full and correct statement of the facts, not an historians personal explanation of those facts.

Dickson's books have occasionally been criticised for putting trivial results next to important ideas, that is, for not separating the wheat from the chaff; observe, however, that a history concentrating only on important ideas would hardly have mentioned most of the references on Pell's equation that are important for us. This is not because Dickson failed to see their importance, but because without the framework of a general theory they were hardly more than "mildly amusing", as van der Poorten [vdP2003] puts it in his review of [Art2002], the most recent paper containing examples of what we will call second 2-descents on certain Pell conics.

In the follow-up [Lem2003b] to this article, I will explain the theory of the first 2-descent, study parts of the Selmer and Tate-Shafarevich groups attached to Pell conics, and interpret the results discussed here from this modern point of view.

1. Euler

1.1. **The Content of Euler's Article.** In [Eul1773], Euler remarked that the method for solving Pell's equation $x^2-dy^2=1$ he has given in [Eul1765] (essentially equivalent to the CFM) is very powerful, but that there are certain values of d for which even the CFM produces the solution only after tedious calculations; he mentions the example $d=61$, where the smallest positive solution is given by $x=1766319049$ and $y=226153980$.

Euler then goes on to describe how solutions of certain auxiliary equations lead to solutions of the Pell equation. His first equation is

$$q^2 - ap^2 = -1. \tag{1}$$

He writes

> Problema 1. Si fuerit $app - 1 = qq$, invenire numeros x et y, ut fiat $axx + 1 = yy$.[1]

Euler multiplies $ap^2 - 1 = q^2$ through by $4q^2$ and adds 1 to get $4ap^2q^2 + 1 = 4q^4 + 4q^2 + 1 = (2q^2+1)^2$. He has proved:

Lemma 1.1. *If $-1 = q^2 - ap^2$, then $y^2 - ax^2 = 1$ for $x = 2pq$ and $y = 2q^2 + 1$.*

Now Euler investigates the new equation $ap^2 - 1 = q^2$ more closely:

> Problema 2. Investigare numeros a, pro quibues fieri potest $app - 1 = qq$, hincque ipsos numeros x et y assignare, ut fiat $axx+1 = yy$.[2]

Euler writes $ap^2 = q^2 + 1$ in the form $a = \frac{q^2+1}{p^2}$ and observes that he has to find p and q in such a way that this fraction becomes an integer. Since a and p (and therefore also p^2) are sums of two squares, there exist integers b, c, f, g such that $p^2 = b^2+c^2$ and $q^2+1 = (b^2+c^2)(f^2+g^2)$, so in particular $a = f^2+g^2$. Comparing both sides he deduces that, for an appropriate choice of these numbers, we must have $q = bf + cg$ and $\pm 1 = bg - cf$.

Now Euler assigns values to b, c, f, g in such a way that $\pm 1 = bg - cf$ (which can be done in infinitely many ways), and then computes a, p, $q = bf + cg$ and finally $x = 2pq$ and $y = 2q^2 + 1$. Actually, he starts by fixing $p = 5$, which leads to $b = 3$, $c = 4$ in view of $p^2 = 25 = 3^2 + 4^2$. Then he computes the following table:

f	1	2	4	5	7	8
g	1	3	5	7	9	11
a	2	13	41	74	130	185
q	7	18	32	43	57	68
x	70	180	320	430	570	680
y	99	649	2049	3699	6499	9249

TABLE 1.

He also remarks that the cases in this table are not very difficult, and gives the additional examples $p = 13$, $p = 17$, and $p = 25$.

For ease of reference, let us collect Euler's result in the following

Proposition 1.2. *The equation $-1 = q^2 - ap^2$ is solvable if and only if there exist $f, g \in \mathbb{N}$ with $a = f^2+g^2$ such that there are $b, c \in \mathbb{N}$ with $p^2 = b^2+c^2$, $bg-cf = \pm 1$, and $q = bf + cg$.*

Now Euler turns to the next equation, namely

$$q^2 - ap^2 = -2. \tag{2}$$

He first observes ([Eul1773, Problema 4])

[1] Problem 1. Given $ap^2 - 1 = q^2$, to find numbers x and y such that $axx + 1 = yy$.

[2] Problem 2. To investigate the numbers a for which we can solve $ap^2 - 1 = q^2$, and then to assign the numbers x and y satisfying $ax^2 + 1 = y^2$.

Lemma 1.3. *If* $-2 = q^2 - ap^2$*, then* $y^2 - ax^2 = 1$ *for* $x = pq$ *and* $y = q^2 + 1$.

As above he then deduces from $ap^2 = q^2 + 2$ that $a = f^2 + 2g^2$ for some $f, g \in \mathbb{N}$, and that $p^2 = b^2 + 2c^2$, and then concludes that $cf - bg = \pm 1$.

Proposition 1.4. *The equation* $-2 = q^2 - ap^2$ *is solvable if and only if there exist* $f, g \in \mathbb{N}$ *with* $a = f^2 + 2g^2$ *such that there are* $b, c \in \mathbb{N}$ *with* $p^2 = b^2 + 2c^2$, $bg - cf = \pm 1$*, and* $q = bf + 2cg$.

Euler's first example is $p = 3$, which in view of $3^2 = 1^2 + 2 \cdot 2^2$ implies $b = 1$ and $c = 2$. Euler solves $2f - g = \pm 1$, giving $a = f^2 + 2g^2$, $q = f + 4g$, hence the solutions $x = 3q$ and $y = q^2 + 1$ of the Pell equation $1 = y^2 - ax^2$:

f	1	1	2	2	3	3	4
g	1	3	3	5	5	7	7
a	3	19	22	54	59	107	114
q	5	13	14	22	23	31	32
x	15	39	42	66	69	93	96
y	26	170	197	485	530	962	1025

TABLE 2.

In addition, Euler discusses the examples $p = 9, 11, 17, 19$ and then (Problema 5) goes on to investigate the equation

$$q^2 - ap^2 = +2. \tag{3}$$

Lemma 1.5. *If* $2 = q^2 - ap^2$*, then* $y^2 - ax^2 = 1$ *for* $x = pq$ *and* $y = q^2 - 1$.

His main result in this case is

Proposition 1.6. *The equation* $2 = q^2 - ap^2$ *is solvable if and only if there exist* $f, g \in \mathbb{N}$ *with* $a = f^2 - 2g^2$ *such that there are* $b, c \in \mathbb{N}$ *with* $p^2 = b^2 - 2c^2$, $bg - cf = \pm 1$*, and* $q = bf - 2cg$.

As examples, Euler treats the cases $p = 7, 17, 23$.

In Problema 7, Euler studies the equation $app + 4 = qq$:

Lemma 1.7. *If* $4 = q^2 - ap^2$*, and if* p *and* q *are odd, then* $y^2 - ax^2 = 1$ *for the integers* $x = p\,\frac{q^2-1}{2}$ *and* $y = q\,\frac{q^2-3}{2}$.

This case will not be of interest to us, so let us go right to Euler's final case, the equation

$$s^2 - ar^2 = -4. \tag{4}$$

Lemma 1.8. *If* $-4 = s^2 - ar^2$*, then* $y^2 - ax^2 = 1$ *for* $x = p\,\frac{q^2-1}{2}$ *and* $y = q\,\frac{q^2-3}{2}$*, where* $p = rs$ *and* $q = s^2$.

The results of Euler's problema 10 are collected in the following

Proposition 1.9. *The equation* $-4 = q^2 - ap^2$ *is solvable if and only if there exist* $f, g \in \mathbb{N}$ *with* $a = f^2 - 2g^2$ *such that there are* $b, c \in \mathbb{N}$ *with* $p^2 = b^2 - 2c^2$, $bg - cf = \pm 1$*, and* $q = bf - 2cg$.

Euler's fourth example concerns $a = 109 = 10^2 + 3^2$,

> qui methodo vulgari molestissimos calculos requirit,[3]

Here he takes $25^2 = 625 = 24^2 + 7^2$, and finds $f = 10$, $g = 3$, $b = 24$ and $c = 7$, hence $s = 261$, giving $p = 6525$ and $q = 261^2 + 2 = 68123$ and finally

$$\begin{aligned} x &= 6525\big(\tfrac{68123^2-1}{2}\big) = 15140424455100,\\ y &= 68123\big(\tfrac{68123^2-3}{2}\big) = 158070671986249. \end{aligned}$$

1.2. **Interpretation with Continued Fractions.** In the article discussed above, Euler presents solutions of the Pell equation for a variety of discriminants. The solutions given in Table 1 correspond to two families, namely $d = (3k-1)^2+(4k-1)^2$ and $d = (3k+1)^2+(4k+1)^2$. Developing $\sqrt{d}$ into continued fractions (which Euler did not do) we find:

d	$\sqrt{d}$
$(3k-1)^2 + (4k-1)^2 = 5^2k^2 - 14k + 2$	$[5k-2, \overline{1, 1, 1, 1, 10k-4}]$
$(3k+1)^2 + (4k+1)^2 = 25k^2 + 14k + 2$	$[5k+1, \overline{2, 2, 10k+2}]$
$(5k-2)^2 + (12k-5)^2 = 13^2k^2 - 140k + 29$	$[13k-6, \overline{1, 1, 1, 1, 1, 1, 26k-12}]$

The last line comes from Euler's second table, which we did not reproduce here.

In general, assume that (r, s, t) is a primitive Pythagorean triple with s even. Then there exist m, n such that $r = m^2 - n^2$, $s = 2mn$ and $t = m^2 + n^2$. Euler has to solve the linear equation $(m^2 - n^2)f - 2mng = \pm 1$.

Let us look at families that admit $n = 1$ as a solution. Then we have to consider $(m^2 - 1)f - 2mg = \pm 1$; if we put $m = 2k$, we find $(4k^2 - 1)f - 4kg = \pm 1$. The solutions of this equation are $f = 1 + 4ku$, $g = k + (4k^2 - 1)u$ and $f = -1 + 4ku$, $g = -k + (4k^2 - 1)u$ for $u = 0, 1, 2, \ldots$. As above we find

d	$\sqrt{d}$	
$[(4k^2+1)u+k]^2 + 4ku + 1$	$[e, \overline{2k, 2k, 2e}]$	$e = (4k^2+1)u + k$
$[(4k^2+1)u-k]^2 - 4ku + 1$	$[e, \overline{1, 2k+1, 2k+1, 1, 2e}\,]$	$e = (4k^2+1)u - k - 1$

2. Hart

2.1. **Hart's Article.** In a short note [Har1878a], Hart studied the solvability of the diophantine equation $X^2 - dY^2 = -1$; let us quote the first half of [Har1878a]:

> To find general values of x and y to solve the problem
> $$x^2 - Ay^2 = -1.$$
> Let A be a non-quadrate number = the sum of two squares = r^2+s^2, then we shall have
> $$x^2 - (r^2 + s^2)y^2 = -1,$$
> and by transposition
> $$x^2 - r^2y^2 = s^2y^2 - 1 = (sy - 1)(sy + 1);$$
> whence
> $$x^2 = r^2y^2 + (sy - 1)(sy + 1) = \Box; \quad \therefore\ x = [r^2y^2 + (sy - 1)(sy + 1)]^{1/2}.$$

[3]for which the usual method requires the most tedious calculations.

Let $r^2y^2 + (sy-1)(sy+1) = [ry - (m \div n)(sy-1)]^2$. Reducing this we get

$$y = \frac{m^2+n^2}{sm^2 - 2nrm - sn^2}.$$

Here, in order to have y integral, put $sm^2 - 2nrm - sn^2 = \pm 1$; whence, transposing and dividing by s, we have

$$m^2 - \frac{2nr}{s}m = \frac{sn^2 \pm 1}{s},$$

and by quadratics

$$m = \frac{rn \pm \sqrt{(r^2+s^2)n^2 \pm s}}{s}, \quad \therefore \ y = \pm(m^2+n^2).$$

In the general value of m, r and s may represent any numbers one of which is even, and the other any odd number except 1, and n can be found by trial, or, if large, by the solution of the formula $P^2 - (r^2+s^2)n^2 = \pm s$.

Let us now give a modern interpretation of Hart's idea. Assume we want to solve

$$x^2 - Ay^2 = -1 \tag{5}$$

for some squarefree integer A. It was already known to Brahmagupta (see Whitford [Whi1912]) that this implies that A is the sum of two squares. Thus $A = r^2 + s^2$ for integers r, s, and we have to solve $x^2 = r^2y^2 + (sy-1)(sy+1)$. Now parametrize this conic using the rational point $P = (x, y) = (\frac{r}{s}, \frac{1}{s})$: the equation $x - ry = t(sy-1)$ describes lines through P with slope $\frac{1}{r+st}$; if we pick $t = \frac{m}{n}$ rational, such a line will intersect the conic in another rational point, and a simple calculation shows that its y-coordinate is $y = \frac{m^2+n^2}{sm^2-2nrm-sn^2}$. We want values m and n for which y is integral; thus we are led to consider

$$sm^2 - 2nrm - sn^2 = \pm 1. \tag{6}$$

Any integral solution of (6) will give an integral solution of (5). Euler and Lagrange have shown how to reduce (6) to a Pell equation: interpreting (6) as a quadratic polynomial in m, a necessary condition for the existence of an integral solution is that the discriminant $4r^2n^2 + 4s(sn^2 \mp 1)$ be a square, i.e., that one of the equations

$$P^2 - An^2 = \pm s \tag{7}$$

have an integral solution. This shows

Proposition 2.1. *Let $d = r^2 + s^2$ be a sum of two squares; if the equation $sm^2 - 2rmn - sn^2 = \pm 1$ has an integral solution (m, n), then so does $x^2 - dy^2 = -1$; in fact, we can put $y = m^2 + n^2$.*

Thus Hart has proved that the solvability of (6) for some choice of r, s implies the solvability of (5); it does not follow (at least not directly) from Hart's proof that the condition is also necessary. The necessity actually was proved by Euler: Hart's result is nothing but a special case of Proposition1.2. In fact, if (5) is solvable, then by Euler there exist r, s with $A = r^2 + s^2$ such that $y^2 = b^2 + c^2$ and $bs - cr = \pm 1$. Since (b, c, y) is a Pythagorean triple, we can write $b = m^2 - n^2$, $c = 2mn$ and $y = m^2 + n^2$; plugging the values of b and c into $bs - cr = \pm 1$ then yields (6). Thus we have shown that combining the results by Euler and Hart gives

Proposition 2.2. *The equation $x^2 - Ay^2 = -1$ is solvable if and only if there exist $r, s \in \mathbb{N}$ with $A = r^2 + s^2$ such that the diophantine equation (6) has an integral solution.*

2.2. **The Negative Pell Equation.** Hart's equations were mentioned by Dickson [Dic1920] and rediscovered by Sansone [San1925a, San1925b] and Epstein [Eps1934]; these two authors used the fact that solvability of the negative Pell equation implies the solvability of $x^2 - dy^2 = a$ (Hart's equation (7)), where $d = a^2 + b^2$ and a is odd. Reducing this equation modulo primes p dividing d shows that $(a/p) = +1$:

Proposition 2.3. *Let d be a squarefree integer. If $x^2 - dy^2 = -1$ is solvable, then $d = a^2 + b^2$ with a odd, and $(a/p) = +1$ for all primes $p \mid d$.*

Escott [Esc1905] had already proved that if $x^2 - Dy^2 = -1$ is solvable in integers and $D = a^2 + b^2$, then $(a/D) = 1$ or $(b/D) = 1$. He also showed that the condition is not sufficient by using the example $2306 = 41^2 + 25^2$. Recently, Proposition 2.3 was rediscovered by Khessami Pilerud [KP1999].

Grytczuk, Luca, & Wojtowicz [GLW2000] proved the following result, which is just a slightly reformulated version of Euler's Proposition 1.2:

Proposition 2.4. *The equation $s^2 - dr^2 = -1$ has an integral solution if and only if there is some odd integer $B \in \mathbb{Z}$ with $d = A^2 + B^2$ and a Pythagorean triple (a, b, c) such that $aA - bB = \pm 1$.*

3. Sylvester

Sylvester [Syl1881] proved the following

Proposition 3.1. *Let $A = 2f^2 + g^2$ be a prime with f odd; then the equation $fy^2 + 2gxy - 2fx^2 = \pm 1$ is solvable in integers.*

For the proof, let (u, v) be the minimal solution of the Pell equation $u^2 - Av^2 = 1$. Playing the usual game, Sylvester arrives at $p^2 - Aq^2 = 1$ or $p^2 - Aq^2 = -2$, the other signs being excluded because of $A \equiv 3 \bmod 8$. The minimality of the solution implies that we have $p^2 - Aq^2 = -2$, and using unique factorization in $\mathbb{Z}[\sqrt{-2}\,]$ we get $p + \sqrt{-2} = (g + f\sqrt{-2}\,)(y + x\sqrt{-2}\,)^2$. Comparing the imaginary parts shows that $fy^2 + 2gxy - 2fx^2 = \pm 1$.

Sylvester's result can be derived from Euler's Proposition 1.4 in the same way we deduced Hart's result from Proposition 1.2; all we have to do is replace Pythagorean triples by solutions of the equation $x^2 + 2y^2 = z^2$.

4. Günther

In [Gue1882], S. Günther discussed the comments of Theon Smyrnaeus on Plato's work and concluded that he must have been familiar with the Pell equation

$$2x^2 - 1 \;=\; y^2. \tag{8}$$

He then tries to reconstruct a possible approach to the solution of this equation.

Günther suggests writing (8) as $x^2 - 1 = y^2 - x^2$, substitutes $x + 1 = \frac{p}{q}(y + x)$ and $x - 1 = \frac{q}{p}(y - x)$, then sets

$$p^2 + 2pq - q^2 = z, \tag{9}$$

solves this equation for p, and then deduces

$$x = \frac{4q^2 + z \mp 2q\sqrt{2q^2+z}}{z}, \quad y = \frac{-4q^2 - z \pm 4q\sqrt{2q^2+z}}{z}.$$

Thus any solution of (9) with $z = \pm 1$ will lead to an integral solution of the negative Pell equation (8).

The same method, he remarks, works for the more general equation

$$(a^2 + b^2)x^2 - 1 \ = \ y^2,$$

which can be written in the form $(ax-1)(ax+1) = (y-bx)(y+bx)$. Substituting $ax+1 = \frac{p}{q}(y+bx)$ and $ax-1 = \frac{q}{p}(y-bx)$ leads to equations equivalent to Hart's (6) and (7), but the formulas derived by Günther are incorrect, and his final conclusion is unclear.

5. Gérardin

While extending existing tables of solutions of the Pell equation by Legendre, Bickmore, and Whitford, A. Gérardin [Ger1917] complained about the tedious work necessary when using the theory of continued fractions:

> La recherche pratique de la solution minima était faite jusqu'à présent sur les fractions continues, ce qui demande en général beaucoup de soins et de temps.[4]

He then presents, by giving a few examples, a 'new method' for solving Pell equations; he uses the auxiliary equations

$$x^2 - Ay^2 = \pm 4, \pm 2, -1 \tag{10}$$

and remarks that from their solutions one can pass easily to the solution of the corresponding Pell equation.

For $A = 941 = 29^2 + 10^2$ he solves the pair of equations

$$31^2 - 941 \cdot 1^2 = +2 \cdot 10,$$
$$184^2 - 941 \cdot 6^2 = -2 \cdot 10,$$

and then constructs a solution of $x^2 - 941y^2 = -4$ by setting $y = 1^2 + 6^2 = 37$, giving $x = 1135$.

The general cases are given by the following formulas:

a) $z^2 - At^2 = -1$, $A = m^2 + n^2$, $t = \alpha^2 + \beta^2$

$$(m\alpha - n\beta)^2 - A\beta^2 = \pm m, \tag{11}$$
$$(m\beta + n\alpha)^2 - A\alpha^2 = \mp m. \tag{12}$$

b) $z^2 - At^2 = +2$, $A = m^2 - 2n^2$, $t = \alpha^2 - 2\beta^2$

$$(n\alpha - m\beta)^2 - A\beta^2 = \pm n, \tag{13}$$
$$(n\beta - m\alpha)^2 - A\alpha^2 = \pm 2n. \tag{14}$$

[4]The practical search for the minimal solution was made up until now via continued fractions, which demands in general a lot of care and time.

c) $z^2 - At^2 = -4$, $A = m^2 + n^2$, $t = \alpha^2 + \beta^2$

$$(n\alpha - m\beta)^2 - A\beta^2 = \pm 2n, \tag{15}$$

$$(n\beta + m\alpha)^2 - A\alpha^2 = \mp 2n. \tag{16}$$

Gérardin also remarks that the case $z^2 - At^2 = -2$ can be treated similarly.

Thus in case a), Gérardin writes $A = m^2 + n^2$ and then tries to solve the two equations $r^2 - As^2 = m$ and $t^2 - Au^2 = -m$; putting $t = \alpha^2 + \beta^2$ then gives a solution of $z^2 - At^2 = -1$, from which a solution to the Pell equation $X^2 - AY^2 = 1$ is easily derived. Observe the similarity with the result of Hart [Har1878a] discussed above.

Gérardin does not give any proofs, but his claims are easily verified. Let us first consider equation (11). Plugging in $A = m^2 + n^2$ and simplifying we get

$$m\alpha^2 - 2n\alpha\beta - m\beta^2 = \pm 1. \tag{17}$$

Observe that taking (12) would lead to the very same equation; in particular, (11) and (12) are equivalent. Also note that m must be odd for (17) to be solvable. A simple calculation now shows that

$$\begin{aligned}(n\alpha^2 - 2m\alpha\beta - n\beta^2)^2 - A(\alpha^2 + \beta^2)^2 &= \\ -(m\alpha^2 - 2n\alpha\beta - m\beta^2)^2 &= -1. \end{aligned}\tag{18}$$

Similarly, in cases b) and c) we get the equations

$$n\alpha^2 - 2m\alpha\beta + 2n\beta^2 = \pm 1 \tag{19}$$

$$n\alpha^2 - 2m\alpha\beta - n\beta^2 = \pm 2 \tag{20}$$

(it is easy to see that, in (20), m must be odd) as well as

$$\begin{aligned}(m\alpha^2 - 4n\alpha\beta + 2m\beta^2)^2 - A(\alpha^2 - 2\beta^2)^2 &= \\ 2(n\alpha^2 - 2m\alpha\beta + 2n\beta^2)^2 &= +2 \end{aligned}\tag{21}$$

$$\begin{aligned}(m\alpha^2 - 2n\alpha\beta + m\beta^2)^2 - A(\alpha^2 + \beta^2)^2 &= \\ -(n\alpha^2 - 2m\alpha\beta - n\beta^2)^2 &= -4. \end{aligned}\tag{22}$$

Comparing (10) with Euler's equations (1) – (4) from Section 1 it will come as no surprise that Gérardin's formulas are an easy consequence of Euler's results.

6. Hardy & Williams, Bapoungué, Arteha

6.1. **Hardy & Williams.** In [HW1986], K. Hardy & K. Williams investigate the solvability of the diophantine equation

$$fx^2 - 2gxy - fy^2 = 1. \tag{23}$$

Their main result is

Proposition 6.1. *The equation $-1 = q^2 - ap^2$ is solvable if and only if there exist $f, g \in \mathbb{N}$ with $a = f^2 + g^2$ such that there are $x, y \in \mathbb{Z}$ with (23). In this case, the pair (f, g) with f odd is unique.*

Apart from the uniqueness assertion, this result is an almost trivial consequence of Euler's Proposition 1.2: we start with the observation that (b, c, p) is a Pythagorean triple. Now $-1 = q^2 - ap^2$ implies that a and p are odd. Assuming that c is odd, we find from $bg - cf = \pm 1$ that f is also odd. Changing the signs of f, g if necessary we may assume that $bg - cf = -1$. The parametrization of Pythagorean triples shows that $b = 2xy$ and $c = x^2 - y^2$; the equation $bg - cf = -1$ then becomes $f(x^2 - y^2) - 2gxy = 1$. This proves Proposition 6.1 except for the uniqueness part.

The claim that there is essentially only one such pair (f, g) is an important contribution: as we will see later, it should be seen as (part of) an analogue of Dirichlet's Theorem [Lem2003a, Thm. 3.3].

6.2. **Bapoungué.** Inspired by the work of Hardy & K. Williams [HW1986], Bapoungué [Bap1989, Bap1998, Bap2000a, Bap2000b, Bap2002] started investigating the solvability of the diophantine equation

$$ax^2 + 2bxy - kay^2 = \pm 1 \tag{24}$$

for values of k for which $\mathbb{Q}(\sqrt{-k}\,)$ has class number 1. The identity

$$(-bx^2 + 2akxy + kby^2)^2 - (b^2 + ka^2)(x^2 + ky^2)^2 = -k(ax^2 + 2bxy - kay^2)^2$$

yields, upon substituting a solution of (24) for (x, y), the equation

$$(-bx^2 + 2akxy + kby^2)^2 - \delta(x^2 + ky^2)^2 = -k. \tag{25}$$

This shows ([Bap1998, Thm. 2])

Theorem 6.2. *If (24) has an integral solution, then so does (25).*

Multiplying (24) through by a and completing the square we get

$$(ax + by)^2 - \delta y^2 = \pm a, \quad \text{where } \delta = ka^2 + b^2. \tag{26}$$

Thus the solvability of (24) implies the solvability of the Pell equation $X^2 - \delta Y^2 = a$, where solvability denotes solvability in integers.

Similarly, multiplying (24) through by $-ka$ and completing the square we get

$$(kay - bx)^2 - \delta x^2 = \mp ka. \tag{27}$$

The special cases $k = 1, 2$ of these equations go back to Euler, Hart, and Gérardin.

The main result of Bapoungué's thesis [Bap1989] is

Theorem 6.3. *Let* $k \in \{2, 3, 7, 11, 19, 43, 67, 163\}$ *(this implies that* $\mathbb{Q}(\sqrt{-k}\,)$ *has class number* 1*). If* a, b *are positive integer with* a *odd such that* $p = ka^2 + b^2$ *is prime, then (24) is solvable if and only if (25) is solvable.*

A similar result holds for $k = 1$. The case $k = 2$ is Sylvester's Proposition 3.1. In [Bap2000b], the following result is proved:

Theorem 6.4. *Let* k *be as above. Among all pairs* (a, b) *of coprime natural numbers with* a *odd and* $d = ka^2 + b^2$*, there is exactly one pair for which (24) is solvable.*

This generalizes Theorem 6.1.

6.3. **Arteha.** The last rediscovery of the method of Euler-Hart-Gérardin so far is due to Arteha [Art2002]; his results are

Proposition 6.5. *Consider the Pell equation*

$$x^2 - dy^2 = 1 \tag{28}$$

for primes d.

(1) *If* $d \equiv 1 \bmod 4$, *and write* $d = a^2 + b^2$ *with* a *odd. Then the minimal positive solution of Pell's equation (28) is given by*

$$y = 2|2amn + b(m^2 - n^2)|(m^2 + n^2),$$

where m *and* n *satisfy*

$$a(m^2 - n^2) - 2bmn = \pm 1.$$

(2) *If* $d \equiv 3 \bmod 8$, *write* $d = a^2 + 2b^2$. *Then the minimal positive solution of Pell's equation (28) is given by*

$$y = \big|4bmn + a|m^2 - 2n^2|\big|(m^2 + 2n^2),$$

where m *and* n *satisfy*

$$b|m^2 - 2n^2| - 2amn = \pm 1.$$

(3) *If* $d \equiv 7 \bmod 8$, *write* $d = a^2 - 2b^2$. *Then the minimal positive solution of Pell's equation (28) is given by*

$$y = |(a(m^2 + 2n^2) - 4bmn)(m^2 - 2n^2)|,$$

where m *and* n *satisfy*

$$2amn - b(m^2 + 2n^2) = \pm 1.$$

7. Summary

Starting with Euler in [Eul1773], many authors have come up with essentially the same idea: solving the Pell equation

$$X^2 - dY^2 = 1 \tag{29}$$

becomes easier by looking at certain auxiliary equations.

The first step is writing down Legendre's equations

$$rx^2 - sy^2 = 1, 2 \quad \text{for } d = rs. \tag{30}$$

There is one nontrivial equation among these with a solution, and the smallest solution will have about half as many digits as the smallest solution of (29).

The second step is to look at equations whose solutions give rise to a solution of one of the equations (30); but this second step has never been completed in full generality. What we have are specific equations applicable only in special situations; these were first discovered by Euler, rediscovered by Hart, Sylvester, Günther, Gérardin, Hardy & Williams, and Arteha, and slightly generalized by Bapoungué.

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