Introduction to the English Edition of Hilbert's *Zahlbericht*

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1 The Report

David Hilbert's (1862–1943) so-called *Zahlbericht* of 1897,[1] which appears here for the first time in English, was the principal textbook on algebraic number theory for a period of at least thirty years after its appearance. Emil Artin, Helmut Hasse, Erich Hecke, Hermann Weyl and many others learned their number theory from this book. Even beyond this immediate impact Hilbert's *Zahlbericht* has served as a model for many standard textbooks on algebraic number theory through the present day—cf. for instance Samuel's little book,[2] or the first chapter of Neukirch's textbook.[3] As a matter of fact, except for minor details (see Section 4 below), at least the first two parts of Hilbert's text can still today pass for an excellent introduction to classical algebraic number theory.

But even though Hilbert's presentation definitely left its mark on this material, the text does remain a *Bericht*, i.e., a report, commissioned as it was by the *Deutsche Mathematiker-Vereinigung* (D.M.V., the German Mathematical Society), on the state of the theory around 1895. During the first years of its existence, the commissioning of comprehensive reports on all parts of mathematics was an important part of the activities of the D.M.V. The first ten volumes of the *Jahresbericht* contain thirteen such reports. David Hilbert and Hermann Minkowski were asked to write a joint report covering all of number theory. Eventually Hilbert and Minkowski decided to split the report, and it was agreed that Minkowski's part should cover the elementary aspects of number theory like continued fractions, quadratic forms and the geometry of numbers. But in the end,

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[1] D. Hilbert, *Die Theorie der algebraischen Zahlkörper*, Jahresbericht der Deutschen Mathematiker-Vereinigung **4** (1897), pp. 175–546, re-edited in D. Hilbert, *Gesammelte Abhandlungen*, vol. 1, Berlin, Heidelberg, etc.: Springer, 1932 (second edition 1970), pp. 63–363. — A French edition appeared in 1910 (never mind its printed date): D. Hilbert, *Théorie des corps de nombres algébriques*, traduit par M.A. Lewy (professeur au lycée Voltaire), Annales de la Faculté des Sciences de l'Université de Toulouse, 1909, 3[e] fasc.

[2] P. Samuel, *Théorie algébrique des nombres*, Paris: Hermann, 1967.

[3] J. Neukirch, *Algebraische Zahlentheorie*, Heidelberg etc.: Springer, 1992.

only Hilbert's part was actually written. It is interesting to compare Hilbert's *Zahlbericht* for instance to the monumental report by A. Brill and M. Noether "on the development of the theory of algebraic functions in former and recent times,"[4] which is much more obviously historically oriented than Hilbert's *Zahlbericht*, and much less of a systematic introduction to the field. On the applied side, there was for example a report on the development and the main tasks of the theory of simple frameworks.[5]

Because it is above all a report, it is not in the *Zahlbericht* that one finds Hilbert's most important, original contributions to number theory—although it does include Hilbert's proof of the Kronecker-Weber theorem[6] as well as the theory of higher ramification groups, i.e., Hilbert's development from the early 1890s of Dedekind's arithmetic theory of Galois extensions of number fields. Indeed, Hilbert's most impressive original contribution to number theory came after the *Zahlbericht*; it was his conjectural anticipation of most of the theorems of Class Field Theory, based on a remarkably deep analysis of the arithmetic of quadratic extensions of number fields. This work appeared in two articles, in 1899 and 1902. The reader interested in this aspect of Hilbert's *œuvre* will still have to read German—see the first volume of Hilbert's *Gesammelte Abhandlungen* and Hasse's appreciation of Hilbert's number theoretical achievements therein.[7]

Coming back to the *Zahlbericht*, for Hilbert reporting on the state of algebraic number theory did not mean writing an inventory of theorems amassed in the course of the nineteenth century,[8] but rather the production of a conceptually coherent theory, which would then also point the way to further research. It is because of this goal: to indicate, and thereby influence future directions of research, that Hilbert's undertaking is most ambitious, and most open to criticism. Such criticism is of course easy to voice today, because we can base our judgement on another one hundred years of number theoretic research. Still, it is neither futile, nor is it *a priori* unjust, to put Hilbert's text into this perspective, because the *Zahlbericht* was precisely written with the idea of steering the

[4] A. Brill, M. Noether, *Bericht über die Entwicklung der Theorie der algebraischen Funktionen in älterer und neuerer Zeit*, Jahresbericht der Deutschen Mathematiker-Vereinigung **3** (1892–93), pp. 107–565.

[5] L. Henneberg, *Bericht über die Entwicklung und die Hauptaufgaben der Theorie der einfachen Fachwerke*, Jahresbericht der Deutschen Mathematiker-Vereinigung **3** (1892–93), pp. 567–599.

[6] For a review of early (purported) proofs of this theorem, see O. Neumann, *Two proofs of the Kronecker-Weber theorem "according to Kronecker, and Weber"*, J. Reine Angew. Math. **323** (1981), 105–126, in particular p. 125, and N. Schappacher, *On the History of Hilbert's Twelfth Problem, I: Paris 1900 – Zürich 1932: The Comedy of Errors, in: Matériaux pour l'histoire des mathématiques au XX siècle*, Actes du colloque à la mémoire de Jean Dieudonné (Nice, 1996), "Séminaires et Congrès" **3** (1998), 243–273.

[7] D. Hilbert, *Gesammelte Abhandlungen*, vol. 1, Berlin, Heidelberg, etc.: Springer, 1970. Hasse's note: "Zu Hilberts algebraisch-zahlentheoretischen Arbeiten," is on pp. 528–535.

[8] This may be what the D.M.V. had in mind, as they certainly were aware of H.J.S. Smith's report on the theory of numbers [Collected Mathematical Papers 1, 38–364] presented to the London Mathematical Society, which is an invaluable document for anyone interested in the number theory of the 19th century, but lacks the coherence of Hilbert's *Zahlbericht*. — One should also mention Paul Bachmann's work in five volumes, the fifth volume of which was already written under the influence of Hilbert's *Zahlbericht*: P. Bachmann, *Zahlentheorie, Versuch einer Gesammtdarstellung*, Leipzig: Teubner; vol. I, Die Elemente der Zahlentheorie (1892); vol. II, Die Analytische Zahlentheorie (1894); vol. III, Die Lehre von der Kreistheilung (1872); vol. IV, Die Arithmetik der quadratischen Formen (part 1, 1898; part 2, 1923); vol. V, Allgemeine Arithmetik der Zahlkörper (1905).

further development of the theory.

2 Later Criticism

Probably the most outspoken criticism of tendencies expressed in the *Zahlbericht* that has been published by an eminent number theorist is due to André Weil and concerns specifically Hilbert's treatment, in the latter parts of the work, of Kummer's arithmetic theory of the cyclotomic field of ℓ-th roots of unity, for a regular prime number ℓ. In fact, Weil's introduction to Kummer's Collected Papers begins:[9]

> The great number-theorists of the last century are a small and select group of men. ... Most of them were no sooner dead than the publication of their collected papers was undertaken and in due course brought to completion. To this there were two notable exceptions: Kummer and Eisenstein. Did one die too young and the other live too long? Were there other reasons for this neglect, more personal and idiosyncratic perhaps than scientific? Hilbert dominated German mathematics for many years after Kummer's death [in 1893]. More than half of his famous *Zahlbericht* (*viz.*, parts IV and V) is little more than an account of Kummer's number-theoretical work, with inessential improvements; but his lack of sympathy for his predecessor's mathematical style, and more specifically for his brilliant use of p-adic analysis, shows clearly through many of the somewhat grudging references to Kummer in that volume.

It seems difficult to prove—or to disprove, for that matter—Weil's suspicion that it was Hilbert's influence which prevented the timely publication of Kummer's collected papers. Hilbert and Minkowski in their correspondence about the *Zahlbericht*—at a time when Minkowski was still trying to contribute his share to a more extensive report on number theory—deplore the fact (and blame the Berlin Academy for it) that Eisenstein's collected papers had not been published.[10] An analogous comment by Hilbert or Minkowski about Kummer's collected papers is not known; instead, there is direct evidence that Hilbert complained to Minkowski about how unpleasant he found it to have to work through Kummer's papers.[11] This is also evident in the *Zahlbericht*; for instance in the preface:

[9]E.E. Kummer, *Collected Papers*, edited by André Weil, vol. 1, Heidelberg etc.: Springer, 1975, p. 1.

[10]See H. Minkowski, *Briefe an David Hilbert, mit Beiträgen und herausgegeben von L. Rüdenberg und H. Zassenhaus*, Berlin, Heidelberg, New York: Springer, 1973, p. 72, Minkowski to Hilbert 12-4-1895: "Eisensteins Werke gesammelt herauszugeben, wäre wirklich eine Pflicht und ausgeführt ein Verdienst; und für die Berliner Akademie wäre es eine ewige und wohlverdiente Schande, wenn jemand sonst dieses Unternehmen angriffe."

[11]See *loc. cit.*, p. 75f, Minkowski to Hilbert 22-1-1896: "Ich bin doch etwas zu spät an das Referat gegangen. Jetzt finde ich natürlich viele hübsche Probleme, von denen es ganz schön gewesen wäre, wenn ich sie erledigt hätte. Die Jordanschen Aufsätze sind eigentlich recht interessant. Aber wenn Dir schon die Kummerschen Rechnungen unangenehm waren, so würden die Jordanschen Operationen, sein 'pour fixer les idées' Dir einen wahren Ekel einjagen."

"I have tried to avoid Kummer's elaborate computational machinery, so that here too Riemann's principle may be realized and the proofs completed not by calculations but by pure thought."[12]

Readers of the first generation, echoing Hilbert's announcement, have praised the *Zahlbericht* in particular for having simplified Kummer.[13] Thus Hasse in 1932 mentions Kummer's "complicated and less than transparent proofs", which Hilbert replaced by new ones.[14] In 1951, however, Hasse added an afterthought describing the two rival traditions in the history of number theory: on the one hand, there is the Gauss-Kummer tradition which aims at explicit, constructive control of the objects studied, and which was carried on in particular by Kronecker and Hensel. The Dedekind-Hilbert tradition, on the other hand, aims above all at conceptual understanding. Although less critical of the *Zahlbericht* than Weil, Hasse saw the need to plead for an "organic equilibrium" of both approaches to number theory, as opposed to the "domination" of Hilbert's approach.[15]

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[12] D. Hilbert, *Gesammelte Abhandlungen*, vol. 1, Berlin, Heidelberg, etc.: Springer, 1970, p. 67 (cf. p. VIII of the translation below): "Ich habe versucht, den großen rechnerischen Apparat von Kummer zu vermeiden, damit auch hier der Grundsatz von Riemann verwirklicht würde, demzufolge man die Beweise nicht durch Rechnungen, sondern lediglich durch Gedanken zwingen soll." The picture of Riemann that Hilbert alludes to here was central in Felix Klein's campaign to elevate Riemann beyond comparison. For instance, Klein confronted Riemann with Eisenstein, whom he called a *Formelmensch*, a man of formulas. A nice tit for tat is to be found in Carl Ludwig Siegel's article *Über Riemanns Nachlaß zur analytischen Zahlentheorie* from 1932—see C.L. Siegel, *Werke*, edited by Chandrasekharan and Maaß, New York, Berlin, etc.: Springer, vol. I, p. 276, where Siegel, after having gone through Riemann's formidable handwritten notes on the zeta function, remarks: "The legend according to which Riemann found the results of his mathematical work via 'big general' ideas, without using the formal tools of analysis, does not seem to be as widespread any more as in Klein's time."

[13] Hilbert's proofreader and friend Minkowski, however, was less enthusiastic when he read Hilbert's treatment of Kummer's theory for the first time: He found the calculations still rather "involved," and Hilbert's proof of the reciprocity law "not easy to read." — see H. Minkowski, *Briefe an D. Hilbert, mit Beiträgen und herausgegeben von L. Rüdenberg und H. Zassenhaus*, Berlin, Heidelberg, New York: Springer, 1973, p. 86: Minkowski to Hilbert 17-11-1896: "... 20 Seiten habe ich bereits durchgesehen, bis zu der Stelle [around §126?], wo die langen Rechnungen anfangen. Sie sind doch noch ziemlich verwickelt." and p. 92, Minkowski to Hilbert 31-1-1897: "Dein Beweis des Reziprozitätsgesetzes ist nicht leicht zu lesen." Minkowski's subsequent sentence indicates that this refers to the proof given in §§155–161 of the *Zahlbericht*. At this point, Minkowski may not have known yet that Hilbert was going to give an alternative proof in §§166–171 of the *Zahlbericht*.

[14] D. Hilbert, *Gesammelte Abhandlungen*, vol. 1, Berlin, Heidelberg, etc.: Springer, 1970, p. 529: "Hilbert gibt hier neue, von den umständlichen und wenig durchsichtigen Rechnungen Kummers freie Beweise."

[15] See H. Hasse, *Über die Klassenzahl abelscher Zahlkörper*, Berlin: Akademie-Verlag, 1952; Reprint Berlin etc.: Springer, 1985, p. VII – VIII: "Der Sinn dafür, daß die explizite Beherrschung des Gegenstandes bis in alle Einzelheiten mit der allgemeinen Fortentwicklung der Theorie Schritt halten sollte, war bei Gauss und später vor allem noch bei Kummer in ganz ausgeprägter Weise vorhanden. ... Unter dem beherrschenden Einfluß, den Hilbert auf die weitere Entwicklung der algebraischen Zahlentheorie ausgeübt hat, ist jedoch dieser Sinn mehr und mehr verlorengegangen. Es ist typisch für Hilberts ganz auf das Allgemeine und Begriffliche, auf Existenz und Struktur gerichteten Einstellung, daß er in seinem Zahlbericht alle mit der expliziten Beherrschung des Gegenstandes sich befassenden Untersuchungen und Ergebnisse von Kummer und anderen durch kurze Hinweise oder Andeutungen abtut, ohne sich mit ihnen im einzelnen zu beschäftigen, wie er ja auch die mehr rechnerischen, konstruktiven und daher der expliziten Durchführung leicht zugänglichen Beweismethoden Kummers systematisch und folgerichtig durch mehr begriffliche, numerisch schwerer zugängliche und kaum kontrollierbare Schlußweisen ersetzt

Emil Artin, in an address for a general audience given on the occasion of Hilbert's 100th birthday in 1962, acknowledged the "great simplification" that Hilbert achieved in presenting Kummer's results.[16] According to Olga Taussky's recollections, however, "it was only ... at Bryn Mawr, that Emmy [Noether] burst out against the *Zahlbericht*, quoting also Artin as having said that it delayed the development of algebraic number theory by decades."[17] We have no way of knowing what Emmy Noether might have had in mind when she made the remark remembered by Taussky, and we do not want to speculate on this. Let us look instead a bit more closely at Hilbert's treatment of Kummer's theory.

3 Kummer's Theory

The results of Kummer's theory which both Hilbert and Weil, in rare unison, refer to as the highest peak of Kummer's number theoretic work[18] were the general reciprocity law for ℓ-th power residues, in the case of a regular odd prime number ℓ, together with the theory of genera in a Kummer extension of $\mathbb{Q}(\zeta_\ell)$ which had to be developed along the way. Hilbert refers to both as the arithmetic theory of Kummer fields. The general reciprocity law is first dealt with in the *Zahlbericht* in §§154–165, based on Eisenstein's reciprocity law which relates a rational to an arbitrary cyclotomic integer. The reciprocity law and its *Ergänzungssätze* (complementary theorems) are stated as Satz 161. Hilbert's presentation

hat. Auch Dedekind mit seiner stark begrifflichen, schon tief ins Axiomatische vorstoßenden Methodik hat an dieser Entwicklung entscheidenden Anteil, während auf der andern Seite die mehr konstruktiven Methoden von Kronecker und Hensel die Kummersche Tradition weiterführen, sich aber gegenüber dem beherrschenden Einfluß Hilberts nur schwer durchsetzen, bis dann allerdings in letzter Zeit entscheidende Erfolge gerade dieser Methodik den Boden für die Rückkehr zu einem organischen Gleichgewicht beider Richtungen bereitet haben."

Zassenhaus shares Hasse's point of view in his note *Zur Vorgeschichte des Zahlberichts*, in: H. Minkowski, *Briefe an D. Hilbert, mit Beiträgen und herausgegeben von L. Rüdenberg und H. Zassenhaus*, Berlin, Heidelberg, New York: Springer, 1973, p. 17–21): "Andererseits hat D. Hilbert mit charakteristischer Energie, aber auch, historisch gesehen, mit einer gewissen Einseitigkeit praktisch die Disziplin der algebraischen Zahlkörpertheorie aus dem historisch gegebenen Material herausmodelliert."

[16] E. Artin, *Collected Papers*, New York, Berlin, etc.: Springer, 1965, p. 549: Hilbert "machte durch große Vereinfachung die Ergebnisse Kummers einem größeren Leserkreis zugänglich."

[17] See *Emmy Noether, a tribute to her life and work*, edited by Brewer and Smith, M. Dekker 1981, p. 82, cf. p. 90. — Olga Taussky was one of the young mathematicians who helped with the editing of the first volume of Hilbert's *Gesammelte Abhandlungen*. She claims *loc. cit.* that Hilbert's original publications were full of errors of all kinds. But the only changes made by the editors of the collected works that we are aware of is the unfortunate replacement, in Hilbert's article *Über den Dirichlet'schen biquadratischen Zahlkörper* [Math. Ann. **45** (1894), 309–340], of the somewhat unusual abbreviation "bzglf." for "bezüglichenfalls" in Hilbert's original text, meaning "respectively," by the word "bezüglich," which means "with respect to".

[18] D. Hilbert, *Gesammelte Abhandlungen*, vol. 1, Berlin, Heidelberg, etc.: Springer, 1970, p. 66 (p. VIII of the following translation): "Es ist die Theorie dieses Kummerschen Körpers offenbar auf der Höhe des heutigen arithmetischen Wissens die äußerste erreichte Spitze ..." — E.E. Kummer, Collected Papers, edited by André Weil, vol. 1, Heidelberg etc.: Springer, 1975, p. 9: "This is the peak he [Kummer] sighted for the first time in January 1848, and whose summit he reached ten years later."

makes sytematic use of the power residue symbol (see in particular the momentous and original Satz 150 of the *Zahlbericht*), as well as the local Hilbert symbols. At the bad places the latter are given explicitly by Kummer's logarithmic derivatives—see §131. Today we would call this an "explicit reciprocity law," in the sense of the research development given new impetus in the late 1920s by Artin and Hasse and which continues to the present day.[19] The product formula for the local Hilbert symbols, i.e., "Hilbert's reciprocity law," appears in this treatment almost negligently as an auxiliary result, Satz 163 in §162.

However, once Hilbert has established all of Kummer's theory, he sets out afresh, in §§166–171, to reprove all of it, but avoiding this time Kummer's explicit formulæ for the symbols at the bad places, and controlling them instead via the product theorem, by information gathered at the primes not dividing ℓ. It may have been especially this second presentation of Kummer's theory that struck readers as a simplification, or at least as a way to get rid of Kummer's computational approach. Weil's wholesale claim quoted above, to the effect that Hilbert had only "inessential improvements" to offer on Kummer's theory, does not seem to do justice to Hilbert's double attempt to come to grips with this theory, even though, of course, whether or not his presentation is seen as an improvement over Kummer may be in the mind of the beholder.

Working up towards his general reciprocity law, Kummer's central achievement was to build up genus theory for Kummer extensions of $\mathbb{Q}(\zeta_\ell)$. Accordingly, genus theory plays a very important role in the *Zahlbericht*. This is quite different from what we are used to nowadays: for instance, the books by Samuel or Neukirch cited above never even mention genus theory. One reason for its disappearance in our time was Chevalley's introduction of idèles, which made it possible to develop class field theory without going through the notorious index calculus of the ideal-theoretic approach via genus theory.[20] Let us take a closer look at how genus theory is treated in Hilbert's *Zahlbericht*.

[19] P. Furtwängler, *Die Reziprozitätsgesetze für Potenzreste mit Primzahlexponenten in algebraischen Zahlkörpern I, II, III*, Math. Ann. **67** (1909), 1–31; **72** (1912), 346–386; **74** (1913), 413–429; T. Takagi, *Über das Reziprozitätsgesetz in einem beliebigen algebraischen Zahlkörper*, J. of the College of Science Tokyo **4** (1922); Coll. Papers, 179–216; E. Artin & H. Hasse, *Die beiden Ergänzungssätze zum Reziprozitätsgesetz der l^n-ten Potenzreste im Körper der l^n-ten Einheitswurzeln*, Abh. Math. Sem. Hamburg **6** (1928), 146–142; Coll. Papers Artin (1965), 142–158; H. Hasse, *Über das Reziprozitätsgesetz der m-ten Potenzreste*, J. Reine Angew. Math. **158** (1927), 228–259; Math. Abh. I, 294–325; K. Iwasawa, *On explicit formulas for the norm residue symbol*, J. Math. Soc. Japan **20** (1968), 151–165; I. Shafarevich, *A general reciprocity law*, Am. Math. Soc. Transl. **4** (1956), 73–106; J. Coates, A. Wiles, *Explicit reciprocity laws*, Journées arithm. Caen 1976, Astérisque **41/42** (1977), 7–17; A. Wiles, *Higher reciprocity laws*, Ann. Math. **107** (1978), 235–254; A. Brückner, *Explizites Reziprozitätsgesetz und Anwendungen*, Vorlesungen aus dem Fachbereich Mathematik der Universität Essen, 1979; G. Henniart, *Sur les lois de réciprocité explicites*, J. Reine Angew. Math. **329** (1981), 177–203; B. Perrin-Riou, *Théorie d'Iwasawa des représentations p-adiques sur un corps local*, Inventiones Math. **115** (1994), 81–149; K. Kato, *General explicit reciprocity laws*; to appear in the forthcoming first volume of the Korean journal: Advanced Studies in Mathematics; as well as recent unpublished work by P. Colmez as well as by F. Cherbonnier & P. Colmez [to appear in Journal AMS]. — With these last papers, history has come around full circle in that the "explicit" laws have themselves been absorbed into a new general abstract theory.

[20] See for instance Hasse's Zahlbericht, I §6, p. 23–24; Ia, §10, §§12–16. — Helmut Hasse's follow-up to Hilbert's *Zahlbericht*, also called "Hasse's Zahlbericht," appeared in three parts [Part I, Jahresbericht D.M.V. **35** (1926), 1–55; part Ia, ibid. **36** (1927), 233–311; part II, Ergänzungsband **6** (1930), 1–201]. It contains a presentation of Takagi's class field theory including Artin's reciprocity law and Furtwängler's principal ideal theorem, which appeared just in time for inclusion.

Genus theory has changed dramatically since it was created by Gauss (who himself followed in the footsteps of Euler and Lagrange).[21] Given an extension K/k of number fields, we can define the genus class group of K as the factor group of the ideal class group of K corresponding by class field theory to the extension FK/K, where F is the maximal abelian extension of k such that FK/K is unramified. Genus theory in Hilbert's *Zahlbericht* is the special case where K/k is a Kummer extension of prime degree ℓ, with $k = \mathbb{Q}(\zeta_\ell)$ a regular cyclotomic field.

Whereas Kummer, in the tradition of Gauss and Dirichlet, used the power residue symbol to define genera,[22] Hilbert built his theory on the norm residue symbol. Using the concept of factor groups (which Hilbert is so successful in avoiding as a theoretical notion—see below), his definition[23] boils down to the following: Let ℓ be an odd prime, k a number field containing the group μ_ℓ of ℓ-th roots of unity, assume that the class number h of k is not divisible by ℓ, and let $K = k(\sqrt[\ell]{\mu}\,)$ be a Kummer extension of k; assume moreover that the class number h of k is not divisible by ℓ and let h^* be an integer such that $hh^* \equiv 1 \bmod \ell$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ denote the primes of k that ramify in K/k, and define a homomorphism $X : k^\times \longrightarrow \mu_\ell^t$ by the rule $\alpha \longmapsto \{(\frac{\alpha,\mu}{\mathfrak{p}_1}), \cdots, (\frac{\alpha,\mu}{\mathfrak{p}_t})\}$. Let $X(E)$ denote the image of the unit group E of $k^\times$ under X, and define the homomorphism χ from the group of invertible ideals of K to $\mu_\ell^t/X(E)$ by putting $\chi(A) = X(\alpha) \cdot X(E)$, where α is a generator of the principal ideal $N_{K/k}A^{hh^*}$. This is clearly well defined, and ideals of K having the same image under χ are said to belong to the same genus. The kernel of χ is called the principal genus. The fact that norms are norm residues implies that each genus is a collection of ideal classes.

This definition is also valid for $\ell = 2$ and equivalence in the usual sense if one is willing to use infinite primes; these were, however, introduced by Hilbert only after his *Zahlbericht* was written.[24]

The main problem of genus theory was to find a formula for the number of genera. Traditionally, this is accomplished by proving a lower and an upper bound which are called the first and the second inequality, respectively (these are special cases of the corresponding inequalities of class field theory).

To see the connection of genus theory with the general reciprocity law, recall Gauss's second proof of the quadratic reciprocity law: Let t denote the number of distinct primes dividing the discriminant of k. Then the first inequality of genus theory says that there are at most 2^{t-1} genera, i.e., that $\#\,\mathrm{Cl}^+(k)/\mathrm{Cl}^+(k)^2 \le 2^{t-1}$. Using this information, a special case of the quadratic reciprocity law can be proved as follows. Let $p \equiv 1 \bmod 4$ and q be odd primes and assume that $(p/q) = +1$. Then q splits in $k = \mathbb{Q}(\sqrt{p}\,)$, and by

[21]See e.g. G. Frei, *On the development of the genus group in number fields*, Ann. Sci. Math. Quebec **3** (1979), 5–62. For modern presentations of genus theory see the books of D. Zagier [*Zetafunktionen und quadratische Körper. Eine Einführung in die höhere Zahlentheorie*, Berlin, Heidelberg, etc.: Springer, 1981] for the quadratic and A. Fröhlich [*Central extensions, Galois groups, and ideal class groups of number fields*, AMS 1983.] for the general case.

[22]E.E. Kummer, *Über die allgemeinen Reciprocitätsgesetze der Potenzreste*, Collected Papers I, 673–687, in particular p. 678

[23]See *Zahlbericht*, §66 and §84 for quadratic fields and equivalence in the usual and strict sense, respectively, and §150 for Kummer fields.

[24]D. Hilbert, *Über die Theorie der relativ-Abelschen Zahlkörper*, Gesammelte Abhandlungen I, 483–509, in particular §6.

the first inequality k has odd class number h. Thus $\pm q^h$ is the norm of an integer from k, and considering this equation as a congruence modulo p yields $(\pm q^h/p) = +1$; but since $(-1/p) = +1$ and h is odd, this implies $(q/p) = +1$.

Kummer's line of attack was similar (the basic idea in both proofs is the observation that norms are norm residues modulo ramified primes), but he soon found[25] that the first inequality would not suffice for a proof. In fact, Kummer's as well as Hilbert's first proof (see §160 of the *Zahlbericht*) both use that the number of genera in certain extensions is exactly ℓ. The three main differences of the proofs are:

1. Kummer's proof uses the first supplementary law, which he had derived earlier using cyclotomy,[26] whereas Hilbert can do without it.

2. Kummer works in a certain subring of the ring of integers $\mathcal{O}_K$ of the Kummer extensions K; Hilbert transfers the whole theory to $\mathcal{O}_K$.

3. Although Kummer occasionally derived necessary conditions for an integer in k to be a norm from K, the theory of the norm residue symbol is a genuine achievement of Hilbert, and it is here that he goes beyond Kummer: see e.g. Satz 167, which is a special case of Hasse's norm theorem.

4 A few Noteworthy Details

Let us now look more closely at a few places where Hilbert's treatment in the *Zahlbericht* differs from what one is used to today. The first such instance occurs when Hilbert proves the unique factorization of ideals into prime ideals in the ring of integers of an algebraic number field via Leopold Kronecker's theory of forms and Dedekind's so-called "Prague Theorem"—see §§5–6.[27] Hilbert does, however, also point out alternative proofs of this key result, at the end of §6.

Since we are today looking back on the *Zahlbericht* from a distance of 100 years, a few proofs there strike us as somewhat roundabout. For instance, in the proof that precisely the prime divisors of the discriminant are ramified as well as in his discussion of the splitting of primes, in particular of those dividing the discriminant of a generating polynomial (see §§10–13 of the *Zahlbericht*), Hilbert switches again from Dedekind's ideal-theoretic language to Kronecker's theory of forms. Modern proofs built upon Hensel's p-adic theory may be more appealing.

[25] "Wegen dieses Umstandes reicht auch der gefundene Satz, daß die Anzahl der wirklichen *Genera* nicht größer ist, als der λte Teil der möglichen, nicht aus, sondern es ist nöthig, daß die Anzahl der wirklichen *Genera* genau gefunden werde, wozu andere Prinzipien erforderlich sind, als welche Gauß zur Ergründung dieser wichtigen Frage für die quadratischen Formen angewendet hat." [*Über die allgemeinen Reciprocitätsgesetze der Potenzreste*, Collected Papers I, 673–687, in particular p. 682].

[26] "und zwar unter einem erheblichen Aufwande von Rechnung", as Hilbert remarks in §161.

[27] For the background of this argument, see H.M. Edwards, *The Genesis of Ideal Theory*, Archive Hist. Exact Sciences **23** (1980), 321–378; here in particular: section 13, pp. 364–368.

A special feature of Hilbert's *Zahlbericht* which the unsuspecting reader may not be prepared for is the very uneven usage of notions from abstract algebra. Thus, while the notion of fields and their arithmetic is at the very heart of Hilbert's concept of algebraic number theory, and even though Hilbert uses the word "(Zahl)ring" for orders in algebraic number fields, this must not be taken as evidence that Hilbert employs here parts of our current algebraic terminology the way we would do it; rather than referring to a general algebraic structure,[28] the word "ring" is used for sets of algebraic integers which form a ring in our modern sense of the word. Similarly, but much more obviously for the modern reader, Hilbert does not have at his disposal general abstract notions from group theory that could unify the discussions of situations that we immediately recognize as analogous.[29] Most striking is the absence from the *Zahlbericht* of the notion of quotient groups.[30] Thus, when we would say that "G/H is cyclic of order h," Hilbert has to write elaborate prose: "the members of G are each obtained precisely once when we multiply the members of H by $1, g, \dots, g^{h-1}$ where g is a suitably chosen member of G"(see e.g. *Zahlbericht*, Sätze 69, 71 and 75).

Contrary to these points, which, from our vantage point, may be considered as shortcomings, but which the *Zahlbericht* shares with most other texts on the subject before Hecke's book from 1923, some other unusual features of the text have to be counted among its pearls. In fact, theorems 89–94 may be regarded as the first highlight of the *Zahlbericht*. Here Hilbert *did* "point the way to further research."

Hilbert's Satz 89 says that the ideal class group is generated by the classes of prime ideals of degree 1. In the special case of cyclotomic fields, this result is due to Kummer.[31] It is a very special case of the density theorems of class field theory, such as Chebotarev's, but it is accessible without analytic methods. Already Max Deuring[32] remarked that

[28] According to I. Kleiner, *From numbers to rings: the early history of ring theory*, Elem. Math. **53** (1998), 18–35, the modern axiomatic definition of a field is due to Steinitz (1910), that of a ring to Fraenkel (1914) and Sono (1917).

[29] This omission seems to be deliberate: in his letter from 21-7-1896 (loc. cit.), Minkowski suggested that the inclusion of a few lemmas on Abelian groups in §100 would allow the reader to enjoy Hilbert's proof of the theorem of Kronecker-Weber without distractions ["Damit der Leser zu einem völlig ungestörten Genusse desselben komme, möchte ich empfehlen, die gebrauchten Hilfssätze über Abelsche Gruppen mit Andeutungen ihrer Beweise vorweg in §100 zu absolviren."]

[30] Factor groups were first defined by O. Hölder, *Zurückführung einer algebraischen Gleichung auf eine Kette von Gleichungen*, Math. Ann. **34** (1889), 26–56; they are discussed in Weber's *Algebra 2*, Braunschweig 1896, but authors ranging all the way from Heinrich Weber himself (*Ueber Zahlengruppen in algebraischen Körpern*, Math. Ann. **48** (1897), 433–473) to Erich Hecke (*Vorlesungen über die Theorie der algebraischen Zahlen*, Leipzig 1923, where Chapters II and III are devoted to Abelian groups) found it necessary to explain to their readers the concept of groups in general, and of factor groups in particular. — See L. Corry, *Modern algebra and the rise of mathematical structures*; Science Networks, vol 17, Basel, Boston (Birkhäuser) 1996, in particular Chapter III, for an extensive discussion of Hilbert's position in the development of modern abstract algebra.

[31] E.E. Kummer, *Über die Zerlegung der aus Wurzeln der Einheit gebildeten complexen Zahlen in ihre Primfaktoren*, Collected Papers I, 211–251, in particular pp. 241–243.

[32] In M. Deuring, *Neuer Beweis des Bauerschen Satzes*, J. Reine Angew. Math. 173 (1935), 1–4, one reads in reference to Satz 89: "...ist eine naturgemäße Verallgemeinerung eines wenig beachteten Satzes in Hilbert's *Zahlbericht*."

this theorem has been somewhat neglected. This is also demonstrated by the fact that Hilbert's proof of this theorem has not been simplified in 100 years. Even worse: it was only in 1987 that Lawrence Washington[33] noticed a gap in it — which is, however, easy to fix.

Hilbert's Satz 90. Again it was Kummer[34] who discovered (in the special case of Kummer extensions of $\mathbb{Q}(\zeta_p)$) what is probably the most famous theorem from the *Zahlbericht*. It is hardly necessary today to recall that E. Noether generalized Satz 90 considerably; in cohomological formulation, her result reads $H^1(G, K^\times) = 1$, where K/k is a normal extension of number fields with Galois group G. Satz 90 is the statement $\widehat{H}^{-1}(G, K^\times) = 1$ for cyclic G which follows from $\widehat{H}^1(G, K^\times) = 1$ using the periodicity of Tate's cohomology groups for cyclic G.

Hilbert's Satz 91 on the existence of relative units was to be the first in a series of generalizations of Dirichlet's unit theorem.[35] A similar result had been proved by Kummer[36] in a slightly different form. The theorem is contained implicitly in the index computations in Hasse's Zahlbericht (Ia, §12; Fußnote 25).

Hilbert's Satz 92. Once more, this is a result which was inspired by work of Kummer.[37] In cohomological form, the statement reads $H^{-1}(G, E_K) \neq 1$ (here K/k is a cyclic extension of prime order p and Galois group G). Today this is part of the standard theorems in class field theory known as the "Herbrand index of the unit group", which says that in fact $\#\, \widehat{H}^{-1}(G, E_K) = p \cdot (E_k : NE_K)$ for cyclic extensions of number fields that are unramified everywhere.

Hilbert's Satz 94. This is a forerunner of the principal ideal theorem of class field theory. It does not, however, contain Satz 94 as a special case. In contrast to Theorems 89–92, Satz 94 cannot be traced back to Kummer. In fact, it is very likely that Kummer did not even think about the possibility of ideal classes becoming principal in extensions: his invalid proof [38] that the class number of cyclotomic fields is divisible by the class number

[33] L.C. Washington, *Stickelberger's theorem for cyclotomic fields, in the spirit of Kummer and Thaine*, Théorie des nombres, Quebec, Canada 1987, 990–993 (1989)

[34] E.E. Kummer, *Über eine besondere Art, aus complexen Einheiten gebildeter Ausdrücke*, Collected Papers I, 552–572; in particular p. 553, and *Über die allgemeinen Reciprocitätsgesetze unter den Resten und Nichtresten der Potenzen, deren Grad eine Primzahl ist*, Collected Papers I, 699–839; in particular p. 754

[35] See for instance J. Herbrand, *Sur les unités d'un corps algébrique*, C. R. Acad. Sci. Paris **192** (1931), 24–27; and E. Artin, *Über Einheiten relativ galoisscher Zahlkörper*, J. Reine Angew. Math. **167** (1932), 153–156.

[36] E.E. Kummer, *Über die allgemeinen Reciprocitätsgesetze unter den Resten und Nichtresten der Potenzen, deren Grad eine Primzahl ist*, Collected Papers I, 699–839; in particular pp. 785–792

[37] E.E. Kummer, *Zwei neue Beweise der allgemeinen Reciprocitätsgesetze unter den Resten und Nichtresten der Potenzen, deren Grad eine Primzahl ist*, Collected Papers I, 842–882.

[38] E.E. Kummer, *Bestimmung der Anzahl nichtäquivalenter Klassen für die aus λten Wurzeln der Einheit gebildeten complexen Zahlen und die idealen Factoren derselben*, Collected Papers I, 299–322; in particular, p. 320–322. Weil remarks in this connection (ibid, p. 955): "In other contexts, of course, he

of any subfield is based on the implicit assumption that ideals do not capitulate.

It seems that the phenomenon of capitulation was discovered by Kronecker in connection with his investigations of his Jugendtraum: Kronecker[39] noticed that $k = \mathbb{Q}(\sqrt{-31}\,)$ admitted an unramified cyclic cubic extension K in which every ideal is principal and remarked, referring to the splitting behaviour of prime ideals of k in this extension, that "it is exactly this fact which contains a clear hint at the future development in the theory of power residues, in particular for the cases excluded in Kummer's investigations."[40]

Hilbert's remarks after the proof of Satz 94 are quite mysterious: we do not know what to make of his claim that the results of Satz 94 can easily be extended to relatively abelian extensions with relative different 1 whose degree is not necessarily a prime. It is easy to make his proof work for cyclic extensions of prime power degree, and an elementary argument shows that Satz 94 then holds for cyclic extensions of arbitrary degree. But there are problems even with extensions K/k of type (p,p): of course the statement that there is a nonprincipal ideal in k that becomes principal in K continues to hold, but the claim that the class number of k is divisible by p^2 does not follow easily.[41] In fact, the correct generalization of Satz 94, namely the claim that in unramified abelian extensions K/k there are at least $(K:k)$ ideal classes that become principal, was only proved in 1991 by reduction to a group theoretic problem via Artin's reciprocity law.[42] Incidentally, Kronecker in his earlier writings used the word 'Abelian' for what is called 'cyclic' today.

Stickelberger's Theorem (Satz 138) gives the prime ideal decomposition of Gauss sums; its treatment in Hilbert's *Zahlbericht* is remarkable in two ways: first, Hilbert does not state (let alone prove) Stickelberger's theorem at all, but is content with the special case[43] already known to Jacobi and Kummer. In fact, he does not even mention Stickelberger's article in this connection although he lists it in his references. The reasons for this omission are not clear; one may speculate that Hilbert had read Stickelberger's article only superficially.

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had many examples of non-principal ideals in a given field which become principal in some bigger field." What Weil has in mind here is probably Kummer's observation on p. 209 of the article *Zur Theorie der komplexen Zahlen*, Collected Papers I, 203–210: "Ich bemerke nur noch, daß sie [die idealen Zahlen] ... immer die Form $\sqrt[h]{(\Phi(\alpha))}$ annehmen, wo $\Phi(\alpha)$ eine wirkliche complexe Zahl ist, und h eine ganze Zahl." Of course this can be read as "every ideal $\mathfrak{a}$ of k becomes principal in the extension $k(\sqrt[h]{\mu}\,)$, where $\mathfrak{a}^h = (\mu)$," but one may want to be careful about assuming that this was Kummer's way of seeing things.

[39] L. Kronecker, *Ueber die Potenzreste gewisser complexer Zahlen*, Monatsber. Berlin (1880), 404–407; Werke II, 95–101.

[40] *loc. cit.*, p. 101: "Es ist genau dieser Umstand, welcher einen deutlichen Hinweis auf die Weiterentwicklung der Theorie der Potenzreste namentlich auch für die in den Kummerschen Untersuchungen ausgeschlossenen Fälle enthält."

[41] P. Furtwängler could prove Satz 94 for unramified extensions of type (p,p) in *Über das Verhalten der Ideale des Grundkörpers im Klassenkörper*, Monatsh. Math. Phys. **27** (1916), 1–15, and a cohomological proof for the more general case (p^n,p) can be found e.g. in R. Bond, *Unramified abelian extensions of number fields*, J. Number Theory **30** (1988), 1–10.

[42] See H. Suzuki, *A generalization of Hilbert's theorem 94*, Nagoya Math. J. 121 (1991), 161–169; for more on capitulation, see K. Miyake, *Algebraic investigations of Hilbert's Theorem 94, the principal ideal theorem, and the capitulation problem*, Expos. Math. **7** (1989), 289–346.

[43] This is actually the only place in the *Zahlbericht* that we are aware of where Hilbert does not go to the borders of what was known.

In 1933, while working on their paper on the congruence zeta function, in particular of Fermat curves, Hasse and Davenport proved a general version of the result due to Jacobi and Kummer. In February 1934, Davenport then discovered this generalization in Stickelberger's paper, and Hasse[44] promptly blamed Hilbert for not including it in the *Zahlbericht*. This episode indicates how much the generation of number theorists after Hilbert relied on the *Zahlbericht* as a comprehensive reference for the subject.

The second observation concerning Hilbert's approach to Stickelberger's theorem is that he ties it in with his study of normal integral bases (simply called 'normal bases' by Hilbert). A normal basis of a normal extension $k/\mathbb{Q}$ is an integral basis of the form $\{t\nu : t \in \mathrm{Gal}(\mathrm{K}/\mathbb{Q})\}$ for some integral $\nu \in k$ (see §105). For fields k of prime degree ℓ Hilbert associates to each such normal basis a "root number" $\Omega = \nu + \zeta \cdot t\nu + \ldots \zeta^{l-1} \cdot t^{l-1}\nu$, where ζ is a primitive ℓth root of unity (see §106). Satz 133 and Satz 134 characterize root numbers by four properties,[45] and Satz 135 gives the prime ideal factorization of root numbers up to conjugacy. In the special case where the normal basis is generated by Gaussian periods, the root number is a Gauss sum ("Lagrangian root number" in Hilbert's terminology).[46]

None of these results related to Stickelberger's theorem made it into the list of theorems that Hilbert considered to be "departure points for further advances" (see the Preface to the *Zahlbericht*), and at first history seemed to agree with Hilbert. In fact, after the contributions of Andreas Speiser and Emmy Noether,[47] the topic of normal integral bases lay dormant for 40 years. Then, however, things changed dramatically; for a description of the activities during the 1970s, the reader may consult Section I §1 in Albrecht Fröhlich's book.[48] Research in this area is continuing vigorously at the moment.

We hope that the special features of Hilbert's *Zahlbericht* justify this first english edition of it, and will enable it to hold its own as a classical text that readers will be happy to consult along with the many excellent textbooks on algebraic number theory which are available today, 100 years after the *Zahlbericht* was written.

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[44] Hasse to Davenport, 22-2-1934: "Moreover the old proof and the whole matter seems to have slipped from the minds of our generation, presumably owing to Hilbert's inconceivable not giving it in his Zahlbericht." We thank Peter Roquette for this information. The letters from Hasse to Davenport are kept in Trinity College, Cambridge, UK.

[45] The fourth property, namely that ω is not an ℓth power in k, should be added to Satz 133; it follows at once from $\zeta \cdot t\Omega = \Omega$.

[46] Cf. A. Weil's ironical remarks in *La cyclotomie jadis et naguère*, Sém. Bourbaki 1973/74, no. 452 (juin 1974), Springer Lecture Notes Math. **431**, or *Œuvres scientifiques* **3** (1980), 311–328.

[47] A. Speiser, *Gruppendeterminante und Körperdiskriminante*, Math. Annalen **77** (1916), 546–562; E. Noether, *Normalbasis bei Körpern ohne höhere Verzweigung*, J. Reine Angew. Math. **167** (1932), 147–152.

[48] A. Fröhlich, *Galois module structure of algebraic integers*, Berlin, Heidelberg, etc.: Springer, 1983.