GALOIS ACTION ON CLASS GROUPS

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Abstract. It is well known that the Galois group of an extension L/F puts constraints on the structure of the relative ideal class group $\mathrm{Cl}(L/F)$. Explicit results, however, hardly ever go beyond the semisimple abelian case, where L/F is abelian (in general cyclic) and where $(L:F)$ and $\#\mathrm{Cl}(L/F)$ are coprime. Using only basic parts of the theory of group representations, we give a unified approach to these as well as more general results.

It was noticed early on that the action of the Galois group puts constraints on the structure of the ideal class groups of normal extensions; most authors exploited only the action of cyclic subgroups of the Galois groups or restricted their attention to abelian extensions. See e.g. Inaba [11], Yokoyama [27], Iwasawa [13], Smith [23], Cornell & Rosen [6], and, more recently, Komatsu & Nakano [16]. Our aim is to describe a simple and general method that is applicable to arbitrary finite Galois groups.

1. Background from Algebraic Number Theory

Let L be a number field, $\mathcal{O}_L$ its ring of integers, I_L its group of fractional ideals $\neq (0)$, and $\mathrm{Cl}(L)$ the ideal class group of L in the usual (wide) sense. In this section we will collect some well known results and techniques that will be generalized subsequently.

For relative extensions L/F of number fields, the relative norm $N_{L/F} : I_L \longrightarrow I_F$ induces a homomorphism $\mathrm{Cl}(L) \longrightarrow \mathrm{Cl}(F)$, which we will also denote by $N_{L/F}$. The kernel $\mathrm{Cl}(L/F)$ of this map is called the *relative class group*. The p-Sylow subgroup $\mathrm{Cl}_p(L/F)$ of $\mathrm{Cl}(L/F)$ is the kernel of the restriction of $N_{L/F}$ to the p-Sylow subgroup $\mathrm{Cl}_p(L)$ of $\mathrm{Cl}(L)$.

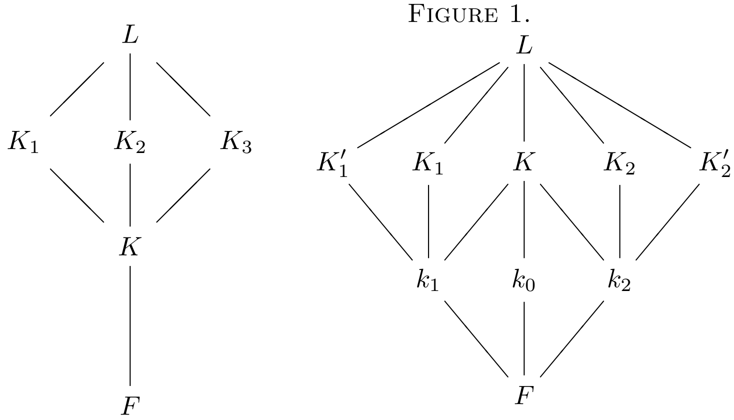
$$\begin{tikzcd} L \arrow[d, dash] & & \mathrm{Cl}(L) \arrow[d, shift right, "N_{L/F}"'] \\ F & & \mathrm{Cl}(F) \arrow[u, shift right, "j_{F\to L}"'] \end{tikzcd}$$

Let $j_{F\to L} : \mathrm{Cl}(F) \longrightarrow \mathrm{Cl}(L)$ denote the transfer of ideal classes induced by mapping ideals $\mathfrak{a}\mathcal{O}_F$ to $\mathfrak{a}\mathcal{O}_L$. Then $N_{L/F} \circ j_{F\to L}(\mathfrak{a}) = \mathfrak{a}^{(L:F)}$. Therefore, the map $\mathrm{Cl}_p(F) \longrightarrow \mathrm{Cl}_p(F) : c \longmapsto c^{(L:F)}$ is an isomorphism for primes $p \nmid (L:F)$, hence $j_{F\to L} : \mathrm{Cl}_p(F) \longrightarrow \mathrm{Cl}_p(L)$ is injective and $N_{L/F} : \mathrm{Cl}_p(L) \longrightarrow \mathrm{Cl}_p(F)$ is surjective; in other words, $j_{F\to L}$ is a section (modulo an isomorphism of $\mathrm{Cl}_p(F)$) of the exact sequence

$$1 \longrightarrow \mathrm{Cl}_p(L/F) \longrightarrow \mathrm{Cl}_p(L) \xrightarrow{N_{L/F}} \mathrm{Cl}_p(F) \longrightarrow 1.$$

Thus this sequence splits, and we conclude $\mathrm{Cl}_p(L) \simeq \mathrm{Cl}_p(L/F) \times \mathrm{Cl}_p(F)$.

Figure 1.



Proposition 1. *Let L/F be an extension of number fields and p a prime not dividing $(L:F)$. Then the transfer of ideal classes $j_{F\to L}:\mathrm{Cl}_p(F)\longrightarrow\mathrm{Cl}_p(L)$ is injective, the relative norm $N_{L/F}:\mathrm{Cl}_p(L)\longrightarrow\mathrm{Cl}_p(F)$ is surjective, and $\mathrm{Cl}_p(L)\simeq\mathrm{Cl}_p(L/F)\times\mathrm{Cl}_p(F)$.*

This result can be used to derive constraints on the class groups of normal extensions; for this type of constraints one needs the presence of conjugate subfields.

Let $V_4=C_2\times C_2$ denote Klein's four group, and C_n a cyclic group of order n. Moreover, let A_4 denote the alternating group of order 12. Hasse diagrams of subfields of extensions with Galois groups A_4 and D_4 (the dihedral group of order 8) are given in Fig. 1.

Corollary 2. *Let K/F be a V_4-extension of number fields with quadratic subextensions k_i, $i=0,1,2$. Then, for any odd prime p,*

$$\mathrm{Cl}_p(K/F)\simeq\mathrm{Cl}_p(k_0/F)\times\mathrm{Cl}_p(k_1/F)\times\mathrm{Cl}_p(k_2/F) \tag{1}$$

and

$$\mathrm{Cl}_p(K/k_0)\simeq\mathrm{Cl}_p(k_1/F)\times\mathrm{Cl}_p(k_2/F). \tag{2}$$

Proof. First we observe that squaring is an automorphism on every finite abelian group of odd order. Next, let σ_i denote the nontrivial automorphism of k_i/F. The identity $2+(1+\sigma_0+\sigma_1+\sigma_2)=(1+\sigma_0)+(1+\sigma_1)+(1+\sigma_2)$ in the group ring $\mathbb{Z}[G]$, $G=V_4$, gives rise to a homomorphism $\mathrm{Cl}_p(L/F)\longrightarrow\mathrm{Cl}_p(k_1/F)\times\mathrm{Cl}_p(k_2/F)\times\mathrm{Cl}_p(k_3/F)$: in fact, write $c\in\mathrm{Cl}_p(L/F)$ in the form $c=d^2$, observe that $d^{(1+\sigma_0+\sigma_1+\sigma_2)}=1$, and map c to $(d^{(1+\sigma_3)},d^{(1+\sigma_1)},d^{(1+\sigma_2)})$. Checking that this is an isomorphism is easy, hence we have (1). Putting together $\mathrm{Cl}_p(K)\simeq\mathrm{Cl}_p(K/k_0)\times\mathrm{Cl}_p(k_0)$ and $\mathrm{Cl}_p(k_0)\simeq\mathrm{Cl}_p(k_0/F)\times\mathrm{Cl}_p(F)$ gives $\mathrm{Cl}_p(K)\simeq\mathrm{Cl}_p(K/k_0)\times\mathrm{Cl}_p(k_0/F)\times\mathrm{Cl}_p(F)$. Comparing this with $\mathrm{Cl}_p(K)\simeq\mathrm{Cl}_p(K/F)\times\mathrm{Cl}_p(F)$ we deduce $\mathrm{Cl}_p(K/F)\simeq\mathrm{Cl}_p(K/k_0)\times\mathrm{Cl}_p(k_0/F)$. Together with (1) this gives (2) as claimed. □

Corollary 3. *Let L/F be a dihedral extension of degree 8, with subextensions denoted as in Fig. 1. Then $\mathrm{Cl}_p(L/K)\simeq\mathrm{Cl}_p(K_1/k_1)\times\mathrm{Cl}_p(K_1/k_1)$ for every odd prime p.*

Proof. Since K_1 and K_1' are conjugate fields (over F), they have isomorphic ideal class groups, and by Corollary 2 this implies that $\mathrm{Cl}_p(L/K)\simeq\mathrm{Cl}_p(K_1/k_1)\times$

$\mathrm{Cl}_p(K_1/k_1)$ for every odd prime p. This is the algebraic explanation for the fact (following easily from the analytic class number formula) that the order $h_p(L/K)$ of the relative p-class group is a square. □

Corollary 4. *Let L/F be an A_4-extension, with subextensions denoted as in Fig. 1. Then $\mathrm{Cl}_p(L/K) \simeq \mathrm{Cl}_p(K_1/K) \times \mathrm{Cl}_p(K_1/K) \times \mathrm{Cl}_p(K_1/K)$ for odd primes p.*

Proof. This follows at once from Corollary 2 by observing that the fields K_i are conjugate over F. □

2. Background from Representation Theory

Here we will review the basic notions of the small part of representation theory that we need. We use Isaacs [12] as our main reference (see also James & Liebeck [14] and Huppert [10]).

Let F be a field and G a finite group; any $F[G]$-module A gives rise to a representation $\Phi : G \longrightarrow \mathrm{Aut}(A)$. Conversely, every homomorphism $\Phi : G \longrightarrow \mathrm{Aut}(A)$ of G into the automorphism group of an F-vector space A makes A into an $F[G]$-module, and the submodules of A are exactly the subspaces of A invariant under the action of G. Such an $F[G]$-module A is called *irreducible* if 0 and A are the only submodules of A; it is called *completely reducible*, if, for every submodule $B \subseteq A$, there is a submodule $C \subseteq A$ such that $A = B \oplus C$. One of the most basic results of nonmodular representation theory is Maschke's theorem (cf. [12, Thm. 1.9]):

Theorem 5. *If G is a finite group, and if F is a field whose characteristic does not divide $\#G$, then all $F[G]$-modules are completely reducible.*

An F-representation Φ of G can be viewed as a homomorphism $G \longrightarrow \mathrm{GL}_n(F)$. In particular, $\Phi(g)$ is a matrix for every $g \in G$, and its trace $\chi(g)$ does not depend on the choice of the basis. We call χ the *character* associated to Φ. Characters are functions $G \longrightarrow F$ that are constant on conjugacy classes; the number h of conjugacy classes of G is called the *class number* of G.

An F-representation Φ is called *faithful* if $\ker \Phi = 0$. For fields $F \subseteq \mathbb{C}$, the kernel of Φ can be read off the character table since it can be shown that $\ker \Phi = \{g \in G : \chi(g) = \chi(1)\}$. For arbitrary fields, this characterization of the kernel does not hold in general, and recognizing faithful characters from the character table becomes a problem. If χ is linear, then $\chi(g) = \chi(1)$ clearly implies $g \in \ker \Phi$. For our purposes, the following lemma suffices:

Lemma 6. *Let χ be the character of a 2-dimensional F-representation of a finite group G, and assume that* $\mathrm{char}\, F$ *is odd. If $\chi(g) = \chi(1) = 2$ for some element $g \in G$ of order 2, then $g \in \ker \Phi$.*

Proof. Write $\Phi(g) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $\chi(g) = a + d = 2 \neq 0$, and from $\Phi(g)^2 = I$ we deduce that $b = c = 0$, hence $a^2 = d^2 = 1$ and finally $a = d = 1$. □

It would be nice to have more powerful criteria.

Let Φ be an F-representation of G, and suppose that $F \subseteq E$; then we can view Φ also as an E-representation, and we denote this extension by Φ^E. An F-representation Φ of G is called *absolutely irreducible*, if Φ^E is irreducible for every extension E of F. A field E is called a *splitting field* for G, if every irreducible E-representation of G is absolutely irreducible. Clearly, every algebraically closed field is a splitting field for G. For fields with prime characteristic it is not hard to show that finite splitting fields exist for every finite group ([12, Cor. 9.15]):

Proposition 7. *Let G be a group with exponent n, and let E be an extension of $\mathbb{F}_p$ in which $x^n - 1$ splits into linear factors. Then E is a splitting field for G.*

Let F be a field, and suppose that $E \supseteq F$ is a splitting field for the group G. For an irreducible E-character χ, the field $F(\chi)$ is defined to be the smallest extension of F containing all $\chi(g)$, $g \in G$. Two irreducible characters χ and ψ of some E-representations of G are called *Galois conjugate* over F if $F(\chi) = F(\psi)$, and if there is a $\sigma \in \mathrm{Gal}\,(F(\chi)/F)$ such that $\chi^\sigma = \psi$. This induces an equivalence relation on the set of irreducible E-characters. The cardinality of these equivalence classes is given by [12, Lemma 9.17.c)]:

Proposition 8. *Let Ω be the equivalence class of an E-character χ over F. Then Ω has cardinality $(F(\chi) : F)$.*

The notion of faithfulness is compatible with the extension of representations to splitting fields:

Lemma 9. *Let E be a splitting field for G, where char $E \nmid \#G$, let F be a subfield of E, and let Φ be an irreducible F-representation. If Φ is faithful, then so is each irreducible component of Φ^E.*

Proof. Let χ be the character afforded by Φ. Since χ^E is the sum of irreducible E-characters χ_j that are Galois conjugate, we have $\chi_j(g) = \chi_j(1)$ for one of these characters if and only if $\chi_j(g) = \chi_j(1)$ for all of them. Thus if one of the components of χ^E is not faithful, neither is χ. □

Proposition 7 shows that $F(\chi)/F$ is an abelian extension if F is finite. The set of irreducible E-characters of G will be denoted by $\mathrm{Irr}_E(G)$. It can be shown that this notion does not depend on the choice of the splitting field E (cf. [12, Lemma 9.13]).

The following theorem describes the behaviour of irreducible representations and their characters in splitting fields (cf. [12, Thm. 9.21]):

Theorem 10. *Let F be a field containing $\mathbb{F}_p$, G a finite group such that $p \nmid \#G$, and let $E \supseteq F$ be a splitting field for G. If Φ is an irreducible F-representation, then Φ^E is the direct sum of E-irreducible representations, all occurring with multiplicity 1. Moreover, the characters afforded by the E-irreducible components of Φ^E constitute a Galois conjugacy class over F.*

3. The Main Theorem

Before we give the main result of this paper, we take the opportunity to recall an almost forgotten result of Grün [8] on the structure of class groups of non-abelian normal extensions:

Proposition 11. *Let L/F be a normal extension of number fields, and let K denote the maximal subextension of L/F that is abelian over F.*

i) *If $\mathrm{Cl}(L/K)$ is cyclic, then $h(L/K) \mid (L : K)$;*
ii) *If $\mathrm{Cl}(L)$ is cyclic, then $h(L) \mid (L : K)e$, where e denotes the exponent of $N_{L/K}\mathrm{Cl}(L)$. Observe that $e \mid h(K)$, and that $e = 1$ if L contains the Hilbert class field K^1 of K.*

Similar results hold for the p-Sylow subgroups.

Proof. Let C be a cyclic group of order h on which $\Gamma = \mathrm{Gal}\,(L/F)$ acts. This is by definition equivalent to the existence of a homomorphism $\Phi : \Gamma \longrightarrow \mathrm{Aut}(C) \simeq \mathbb{Z}/(h-1)\mathbb{Z}$. Since $\operatorname{im}\Phi$ is abelian, $\Gamma' \subseteq \ker\Phi$, hence Γ' acts trivially on C. Now Γ' corresponds to the field K via Galois theory, and we find $N_{L/K}c = c^{(L:K)}$ for all $c \in C$. Putting $C = \mathrm{Cl}(L/K)$, we see at once that $(L:K)$ annihilates $\mathrm{Cl}(L/K)$. If we denote the exponent of $N_{L/K}\mathrm{Cl}(L)$ by e and put $C = \mathrm{Cl}(L)$, then we find in a similar way that $(L:K)e$ annihilates $\mathrm{Cl}(L)$. □

We now come to the main theorem:

Theorem 12. *Let L/F be a normal extension of number fields with Galois group $\Gamma = \mathrm{Gal}\,(L/F)$. Let p be a prime not dividing $\#\Gamma$, and assume that $\mathrm{Cl}_p(K/F) = 1$ for all normal subextensions K/F with $F \subseteq K \subsetneq L$. Let Φ denote the representation $\Phi : \Gamma \longrightarrow \mathrm{Aut}(C)$ induced by the action of Γ on $C = \mathrm{Cl}(L/F)/\mathrm{Cl}(L/F)^p$; if $C \neq 1$, then Φ is faithful.*

If the degrees of all irreducible faithful $\mathbb{F}_p$-characters χ of Γ are divisible by f, then $\operatorname{rank}\mathrm{Cl}_p(L) \equiv 0 \bmod f$. If, in addition, $(\mathbb{F}_p(\chi) : \mathbb{F}_p) = r$ for all these χ, then $\operatorname{rank}\mathrm{Cl}_p(L) \equiv 0 \bmod rf$.

Proof. Clearly $G = \ker\Phi$ is a normal subgroup of Γ which acts trivially on C. Let K be the subextension of L/F corresponding to G by Galois theory. Then, for every $c \in C$ we have $N_{L/K}c = c^{(L:K)}$ because the automorphisms of L/K leave c invariant. On the other hand $N_{L/K}c$ is an element of $\mathrm{Cl}(K)/\mathrm{Cl}(K)^p$, where K/F is a normal subextension of L. Since $\mathrm{Cl}_p(K/F) = 1$ by assumption, we conclude that $N_{L/K}c = 1$.

Now if $C \neq 1$ we can find a $c \in C$ of order p; then c is killed by $(L:K)$ and by p; since $p \nmid (L:F)$, we find that $(L:K) = 1$, i.e. $G = 1$, and Φ is faithful.

If all faithful $\mathbb{F}_p$-characters χ of Γ have degrees divisible by f, then clearly $\operatorname{rank}\mathrm{Cl}_p(L) \equiv 0 \bmod f$ since every character is the sum of irreducible characters; note that Lemma 9 guarantees that these characters are faithful. If, in addition, $(\mathbb{F}_p(\chi) : \mathbb{F}_p) = r$ for all these χ, and if Φ denotes an irreducible faithful representation, then Theorem 10 shows that Φ^E is the direct sum of r representations of degree f, and this in turn implies that Φ has degree rf. □

Remark. Instead of assuming that $\mathrm{Cl}_p(K/F) = 1$ for all normal subextensions $F \subseteq K \subsetneq L$ we might as well consider the group $C^* = \bigcap \mathrm{Cl}(L/F)$, where the intersection is over all normal subextensions $F \subseteq K \subsetneq L$. If we put $C = C^*/C^{*p}$, the proof goes through unchanged.

4. Applications

We first derive a couple of corollaries from Theorem 12:

Corollary 13. *Let L/F be a normal ℓ-extension with $\Gamma = \mathrm{Gal}\,(L/F)$, and assume that the center $Z(\Gamma)$ is not cyclic. If $p \neq \ell$ is a prime dividing $h(L)$, then p divides the class number of some normal subextension K/F of L/F.*

Proof. If not, then there exists a faithful irreducible representation of Γ. According to [10, V, §5, Ex. 15], an ℓ-group Γ possesses faithful irreducible representations if and only if $Z(\Gamma)$ is cyclic. This contradicts our assumption. □

Corollary 14. *Let L/F be a normal ℓ-extension with nonabelian Galois group $\Gamma = \mathrm{Gal}\,(L/F)$. If the prime $p \neq \ell$ does not divide the class number of any normal subextension K/F of L/F, then* $\operatorname{rank} \mathrm{Cl}_p(L) \equiv 0 \bmod \ell$.

Proof. It is sufficient to show that every irreducible faithful character of Γ has degree divisible by ℓ. To this end it is sufficient to show that linear characters cannot be faithful (recall that the degree of a character over a splitting field divides the group order). So assume that χ is linear and faithful; linear characters are lifts from characters of Γ/Γ', and this group contains a subgroup of type (ℓ, ℓ) since Γ is nonabelian. In particular, Γ/Γ' has noncyclic center and, a fortiori, no faithful characters. $\square$

The theorem below is well known in the case where Γ is a cyclic group; the other examples serve only to illustrate the method.

Theorem 15. *Let L/F be a normal extension with Galois group Γ, and let $p \nmid \#\Gamma$ be a prime. Suppose that $\mathrm{Cl}_p(K/F) = 1$ for all normal subextensions K/F of L/F with $K \neq L$. Then* $\operatorname{rank} \mathrm{Cl}_p(L/F)$ *is divisible by r, where r is given in the table below:*

Γ	$\#\Gamma$	r	$f > 0$ *minimal with*
C_n	n	f	$p^f \equiv 1 \bmod n$
D_n, $n \geq 3$	$2n$	$2f$	$p^f \equiv \pm 1 \bmod n$
H_{4n}, $n \geq 2$	$2n$	$2f$	$p^f \equiv \pm 1 \bmod 2n$
A_4	12	3	

Proof. This follows immediately from Theorem 12 and character tables for the corresponding groups.

For $\Gamma = C_n = \langle a \rangle$ (see [14, p. 82]), all irreducible characters χ_j $(0 \leq j \leq n-1)$ are linear and given by $\chi_j(a^k) = \zeta_n^{kj}$; the character χ_j is faithful if and only if $\gcd(j, n) = 1$, and then $\mathbb{F}_p(\chi_j) = \mathbb{F}_p(\zeta_n)$. By the decomposition law for cyclotomic extensions we know that $(\mathbb{F}_p(\zeta_n) : \mathbb{F}_p)$ is just the order of $p \bmod n$, and this implies the claim if $\Gamma = C_n$.

For $\Gamma = D_n$ ([14, p. 182, 183]), the faithful irreducible characters χ have degree 2 and satisfy $\mathbb{F}_p(\chi) = \mathbb{F}_p(\zeta_n + \zeta_n^{-1})$. Thus $(\mathbb{F}_p(\chi) : \mathbb{F}_p) = f$ where $f > 0$ is the minimal integer such that $p^f \equiv \pm 1 \bmod n$. Since each irreducible faithful Φ consists of f irreducible faithful characters χ with degree 2, the claim follows.

For generalized quaternion groups $\Gamma = H_{4n}$ ([14, p. 385]), the faithful irreducible characters χ have degree 2 and satisfy $\mathbb{F}_p(\chi) = \mathbb{F}_p(\zeta_{2n} + \zeta_{2n}^{-1})$. Thus $(\mathbb{F}_p(\chi) : \mathbb{F}_p) = f$ where $f > 0$ is the minimal integer such that $p^f \equiv \pm 1 \bmod 2n$.

Finally, consider the group $\Gamma = A_4$ with four conjugacy classes C_1, C_2, C_3, C_4 and the character table (see [14, p. 180, 181] or [5]; ρ denotes a primitive cube root of unity)

	C_1	C_2	C_3	C_4
χ_1	1	1	1	1
χ_2	1	1	ρ	ρ^2
χ_3	1	1	ρ^2	ρ
χ_4	3	-1	0	0

Clearly χ_4 is the only irreducible faithful character, which implies the claims. Note that, in this case, the presence of conjugate subfields is responsible for the divisibility of the p-rank of $\mathrm{Cl}(L/F)$ by 3 (compare Corollary 4). $\square$

Remark. By replacing A with $A = C^{p^m}/C^{p^{m+1}}$, we can give similar results on the p^m-rank of $\mathrm{Cl}(L/F)$.

We also can obtain results for groups with irreducible faithful characters of different degrees by adding assumptions about the triviality of $\mathrm{Cl}_p(K/F)$ for certain nonnormal subextensions.

4.1. **Cyclic Extensions.** The following proposition is well known (cf. [11], [7]):

Proposition 16. *Let p be an odd prime, and suppose that L/K is a cyclic extension of degree p. If $\mathrm{Cl}_p(L/K)$ is cyclic, then $\mathrm{Cl}_p(L/K)$ is trivial or $\simeq \mathbb{Z}/p\mathbb{Z}$.*

In fact there are even stronger results describing $\mathrm{Cl}_p(L/K)$ as a $\mathrm{Gal}\,(L/K)$-module. The following is a slight generalization for $p = 2$:

Proposition 17. *Let L/F be a cyclic quartic extension with Galois group $\Gamma = \mathrm{Gal}\,(L/F) = \langle\sigma\rangle$, and let K be its quadratic subextension, i.e. the fixed field of $\langle\sigma^2\rangle$. If C is a cyclic subgroup of $\mathrm{Cl}_2(L/K)$ and a Γ-module, then $\#C \mid 2$.*

Proof. Assume that C has an ideal class c of order 4. Then the order of c^σ is also 4, hence we must have $c^\sigma = c$ or $c^\sigma = c^3$. In both cases we get $c = c^{\sigma^2}$, hence $1 = N_{L/K}c = c^{1+\sigma^2} = c^2$, contradicting our assumption. Therefore, C is elementary abelian, and since it is cyclic by assumption, our claim follows. ☐

As an application, we note

Corollary 18. *Let $L/\mathbb{Q}$ be a cyclic quartic extension with quadratic subfield K. If $\mathrm{Cl}_2(L/K) \simeq (2^\alpha, 2^\beta, \ldots)$, where $\alpha \ge \beta \ge \ldots$, then $\alpha - \beta \in \{0, 1\}$.*

Proof. Let $c \in \mathrm{Cl}_2(L/K)$ be an ideal class of order 2^α. Then c^{2^β} generates a subgroup C of $\mathrm{Cl}_2(L/K)$. Since $c^\sigma = c^j \cdot d$ for some odd integer j and an ideal class d of order dividing 2^β, we conclude that C is a $\mathrm{Gal}\,(L/\mathbb{Q})$-module. Since C is cyclic, Proposition 17 implies that $\#C \le 2$, and this proves our claim. ☐

This strengthens results of [2, 3] on the structure of 2-class groups of certain quartic cyclic number fields.

4.2. **Quaternion Extensions.** Next assume that L/F is a normal extension with Galois group $\mathrm{Gal}\,(L/F) \simeq H_8$, the quaternion group of order 8. Let K be the unique quartic subextension of L/F. Louboutin [20] has computed some relative class numbers $h(L/K)$ in the special case where $F = \mathbb{Q}$ and L is a CM-field: it is known that, for primes $q \equiv 3 \bmod 8$, the real bicyclic biquadratic number field $K = \mathbb{Q}(\sqrt{2}, \sqrt{q}\,)$ admits a unique totally complex quadratic extension L_q/K with the following properties:

a) $L_q/\mathbb{Q}$ is normal, and $\mathrm{Gal}\,(L_q/\mathbb{Q}) \simeq H_8$;
b) L_q/K is unramified outside $\{\infty, 2, q\}$.

Here are his results:

q	3	11	19	43	59	67	83
$h(L_q)$	2	$2 \cdot 3^2$	$2 \cdot 7^2$	$2 \cdot 3^4$	$2 \cdot 3^2 \cdot 7^2$	$2 \cdot 3^6$	$2 \cdot 5^4$

Louboutin observed that the class numbers $h_p(L/K)$, p odd, were all squares. In contrast to the cases $G = A_4$ or $G = D_4$ (cf. Corollaries 3, 4), however, this cannot be explained by the presence of subfields.

For primes $p \equiv 3 \bmod 4$, we see at once that the p-part of $\mathrm{Cl}(L/K)$ has even rank, because L is a cyclic quartic extension over each of its three quadratic subextensions, and the action of any of the cyclic groups proves our claim. For primes $p \equiv 1 \bmod 4$, Proposition 11 shows that $\mathrm{Cl}_p(L/K)$ cannot be cyclic, but it does not allow us to deduce that its rank is even: this only follows from the results in the table of Theorem 15. In this special case of quaternion extensions of degree 8 this has already been noticed by Tate (cf. [24, p. 51, Lemme 2.2]).

4.3. **Dihedral Extensions.**

We conjecture that the result in the table of Theorem 15 pertaining to D_n also holds for $p = 2$, that is: if the relative class number of the quadratic subextension of L/F is odd, then the 2-rank of $\mathrm{Cl}(L/F)$ is divisible by $2f$, where $f > 0$ is minimal with $2^f \equiv \pm 1 \bmod n$. This is certainly true if $2^f \equiv -1 \bmod n$, because in this case the claim follows from the action of the cyclic subgroup of D_n alone. Ken Yamamura [26] gives ad hoc proofs for some special cases when $2^f \equiv 1 \bmod n$ and f is odd. The general case requires studying modular representations, and I hope to return to this question at another occasion.

Amazingly, our results on dihedral extensions can be used to show that there are constraints on the structure of class groups of non-normal extensions:

Corollary 19. *Let K/F be an extension of odd degree n such that its normal closure L/F has Galois group $G = D_n = \langle \sigma, \tau \mid \sigma^n = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$, and let r_p denote the rank of $\mathrm{Cl}_p(K/F)$. Then $r_p \equiv 0 \bmod f$ for every prime $p \nmid 2n$, where f is the smallest positive integer such that $p^f \equiv \pm 1 \bmod n$. This result is also valid for $p = 2$ if $2^f \equiv -1 \bmod n$.*

Proof. Put $A = \mathrm{Cl}_p(K/F)$ and let k be the quadratic subextension of L/F; the map $j_{K \to L} : \mathrm{Cl}_p(K/F) \longrightarrow \mathrm{Cl}_p(L/k)$ is known to be injective (for $p \nmid 2n$ this follows from Proposition 1; for $p = 2$ it is due to J.-F. Jaulent, cf. [4, Thm. 7.8]), so we can regard A as a subgroup in $\mathrm{Cl}_p(L/k)$. Let C be the G-module generated by $j(A)$. According to [9, Lemma 1], we have $C = AA^\sigma$, where σ generates $\mathrm{Gal}\,(L/k)$. Now we claim that $C = A \oplus A^\sigma$ as abelian groups (observe that A and A^σ in general are not G-modules). To this end we have to show that $A \cap A^\sigma = \{1\}$. Let $a \in A \cap A^\sigma \setminus \{1\}$; from $a = b^\sigma$ for some $b \in A$ we get $a = a^\tau = b^{\sigma\tau} = b^{\tau\sigma^{-1}} = b^{\sigma^{-1}} = (b^\sigma)^{\sigma^{-2}} = a^{\sigma^{-2}}$, hence $a = a^\sigma$. But then $1 = N_{K/F} a = a^n$ contradicting our assumption that $a \in \mathrm{Cl}_p(K/F) \setminus \{1\}$ and $p \nmid n$.

Thus $\mathrm{rank}\, C = \mathrm{rank}\, A + \mathrm{rank}\, A^\sigma = 2 \cdot \mathrm{rank}\, A$. On the other hand, we have $\mathrm{rank}\, C \equiv 0 \bmod 2f$ by the results of the table in Theorem 15. □

Examples: We have used PARI to compute class numbers of some dihedral extensions. Kondo [17] describes a family of quintic dihedral extensions whose normal closure is unramified over its quadratic subfield: it is given by $f(x) = x^5 + (a-3)x^4 + (b-a+3)x^3 + (a^2-a-1-2b)x^2 + bx + a$, where a and b are integers. The same family of dihedral quintics was also found by Brumer (see Lecacheux [18]) and Kihel [15]; see also Martinais & Schneps [21] and Roland, Yui, & Zagier [22].

Putting $a = 1$, we find a subfamily which can be viewed as a series of 'simplest quintic dihedral fields', since these have a parametrized system of independent units. Table 1 gives the parameter b, the discriminant of the quintic field K generated by a root of f, and the class group of K.

4.4. **The Class Field Tower of $\mathbb{Q}(\sqrt{-105}\,)$.**

TABLE 1. Simplest Dihedral Quintic Fields.

b	$\operatorname{disc} K$	$\mathrm{Cl}(K)$	b	$\operatorname{disc} K$	$\mathrm{Cl}(K)$
0	2209	[1]	10	50225569	[19]
1	10609	[1]	11	81414529	[4, 4]
2	57121	[1]	12	127215841	[11]
3	229441	[1]	13	192626641	[4, 4]
4	717409	[5]	14	283821409	[19]
5	1868689	[2, 2]	15	408322849	[10, 2]
6	4255969	[2, 2]	16	575184289	[61]
7	8755681	[3, 3]	17	795183601	[4, 4]
8	16638241	[2, 2]	18	450241	[1]
9	29669809	[11]	19	1447574209	[20, 4]

Corollary 20. *Let k be a quadratic number field, and let L/k be an unramified extension such that $\Gamma = \mathrm{Gal}\,(L/k) \simeq 32.40$, and assume that the two quaternion extensions of k inside L are normal over $\mathbb{Q}$. Let K be the fixed field of the commutator subgroup $\Gamma' \simeq (2,2)$. Then*

$$\operatorname{rank} \mathrm{Cl}_p(L/K) \equiv \begin{cases} 0 \bmod 2, & \text{if } p \equiv 1 \bmod 4, \\ 0 \bmod 4, & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

Proof. This follows from Theorem 12 and a simple look at the character tables in [5]. □

As an application we give a proof for the fact that the class field tower of $k = \mathbb{Q}(\sqrt{-105}\,)$ terminates with k^2 which differs from the one in [19]. We know that $\mathrm{Cl}(k) \simeq (2,2,2)$, $\mathrm{Cl}(k^1) \simeq (2,2)$, and $\Gamma = \mathrm{Gal}\,(k^2/k) \simeq 32.40$, and the other assumptions of Corollary 20 are satisfied as well. We also know that $h(k^2)$ is odd from Proposition 21. If $\mathrm{Cl}(k^2)$ were non-trivial, Corollary 20 would show that $h(k^2) \geq 25$. This contradicts the unconditional Odlyzko bounds, which show that $h(k^2) \leq 10$.

The proposition referred to is

Proposition 21. *If k is a number field such that $\mathrm{Cl}_2(k_2^1) \simeq (2,2)$, then $k_2^3 = k_2^2$.*

Proof. Suppose that $k_2^3 \neq k_2^2$ and let $G = \mathrm{Gal}\,(k_2^3/k)$. Then a result of Taussky [25] shows that $G' \simeq \mathrm{Gal}\,(k_2^3/k_2^1)$ is dihedral, semidihedral or quaternionic, and all these groups have cyclic centers. Burnside [1] has shown that p-groups G with cyclic $Z(G')$ have cyclic G'. But if k_2^3/k_2^1 is cyclic, we must have $k_2^3 = k_2^2$ in contradiction to our assumption. □

ACKNOWLEDGEMENT

I would like to thank A. Brandis and R. Schoof for several helpful discussions, T. Kondo for sending me [17], K. Yamamura for referring me to [18], G. Boeckle for his comments, and the referee for his careful reading of the manuscript.

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