# GAUSS BOUNDS OF QUADRATIC EXTENSIONS

FRANZ LEMMERMEYER

Abstract. We give a simple proof of results of Lubelski and Lakein on Gauss bounds for quadratic extensions of imaginary quadratic Euclidean number fields.

## 1. Preliminaries

Let $k$ be a number field with class number 1; in the following, $N$ will denote the absolute value of the norm, i.e. $N\alpha = |N_{k/\mathbb{Q}}\alpha|$. We define the *Euclidean minimum* $M(k)$ by $M(k) = \inf\{\delta > 0 : \forall \xi \in k\ \exists \eta \in \mathbb{Z}_k$ such that $N(\xi - \eta) < 1.\}$ An ideal $I$ in the maximal order $\mathbb{Z}_K$ of a quadratic extension $K/k$ is called *primitive* if it is not divisible by any non-unit $a \in \mathbb{Z}_k$. Since $h(k) = 1$, there exists a relative integral basis $\{1, \omega\}$ of $\mathbb{Z}_K$.

The following lemma and its proof are well known for $k = \mathbb{Q}$ ([2], 14.12):

**Lemma 1.** *Let $k$ be a number field with class number 1, and suppose that $K/k$ is a quadratic extension. Then every primitive ideal $I$ has the form $I = (a+\omega)\mathbb{Z}_k + c\mathbb{Z}_k$ for algebraic integers $a, c \in \mathbb{Z}_k$, where $c$ is a generator of the ideal $c\mathbb{Z}_k = N_{K/k}I$.*

*Proof.* Choose $\alpha = a + b\omega$ such that $I = (\alpha, c)$ (cf. [2], 6.19). Writing $c\omega \in I$ as a linear combination of $a + b\omega$ and $c$ shows easily that $b \mid a$ and $b \mid c$. Since $I$ is primitive, $b$ must be a unit, and we may assume without loss of generality that $b = 1$. □

## 2. Quadratic Number Fields

The following theorem is well known (see e.g. Holzer [3]); we will give a very simple proof which we will generalize in the next section.

**Theorem 2.** *Let $K = \mathbb{Q}(\sqrt{m}\,)$ be a quadratic number field with ring of integers $\mathbb{Z}_K = \mathbb{Z}[\omega]$ and discriminant $\Delta$, where*

$$\omega = \begin{cases} \sqrt{m}, & \text{if } m \equiv 2,3 \bmod 4, \\ \frac{1+\sqrt{m}}{2}, & \text{if } m \equiv 1 \bmod 4. \end{cases} \quad \text{and } \Delta = \begin{cases} 4m, & \text{if } m \equiv 2,3 \bmod 4, \\ m, & \text{if } m \equiv 1 \bmod 4. \end{cases}$$

*Let $\mu_K$ be defined by* $$\mu_K = \begin{cases} 1, & \text{if } \Delta = 5 \\ \sqrt{\Delta/8}, & \text{if } \Delta \ge 8 \\ \sqrt{-\Delta/3}, & \text{if } \Delta < 0. \end{cases}$$

*Then each ideal class of $K$ contains an integral ideal of norm $\le \mu_K$.*

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*Proof.* Let $[I]$ be an ideal class generated by an integral ideal $I$ which we may assume to be primitive. Then $I = (\gamma, c)$ with $(c) = N_{K/\mathbb{Q}} I$ and $\gamma = a + \omega = s + \frac{1}{2}\sqrt{\Delta}$, where $2s \in \mathbb{Z}$. Applying the Euclidean algorithm to the pair $(s, c)$ we see that there exists a $\gamma = r + \frac{1}{2}\sqrt{\Delta} \in I$ such that

$$\begin{array}{ccccll} & & |r| & \le & \frac{c}{2} & \text{if } \Delta < 0, \\ \frac{c}{2} & \le & |r| & \le & c & \text{if } c^2 > \frac{\Delta}{5}, \\ c & \le & |r| & \le & \frac{3}{2}c & \text{if } \frac{\Delta}{8} < c^2 < \frac{\Delta}{5} \end{array}$$

We claim that $|N\gamma| \le \frac{1}{4}(c^2 - \Delta) < c^2$ provided that $c^2 > \mu_K$; this shows that $I_1 = \gamma' c^{-1} I \sim I$ (where $\gamma'$ denotes the algebraic conjugate of $\gamma$) is an integral ideal such that $[I_1] = [I]$ and $NI_1 < NI$. Repeating this procedure if necessary we eventually arrive at an integral ideal $I_n \sim I$ with norm $\le \mu_K$.

The claimed inequality is proved by going through all the cases:

(1) $\Delta < 0$: here $|N\gamma| = \Big| r^2 - \dfrac{\Delta}{4} \Big| \le \dfrac{c^2 + |\Delta|}{4} < 1$ since $c^2 > \mu_K = \frac{|\Delta|}{3}$.

(2) $c^2 > \frac{\Delta}{5}$: here $-c^2 = \dfrac{c^2 - 5c^2}{4} < r^2 - \dfrac{\Delta}{4} < c^2$.

(3) $\frac{\Delta}{8} < c^2 < \frac{\Delta}{5}$: then $-c^2 = c^2 - \dfrac{8c^2}{4} < r^2 - \dfrac{\Delta}{4} < \dfrac{9c^2 - 5c^2}{4} = c^2$.

The only possibility not covered by the proof is $c^2 = \Delta/5$; since the odd part of $\Delta$ is squarefree, this will happen if and only if $\Delta = 5$ and $c = \pm 1$. This completes the proof of the theorem. □

## 3. 2. Quadratic Extensions of imaginary Quadratic Fields

Let $k = \mathbb{Q}(\sqrt{-n})$, where $n \in \{-1, -2, -3, -7, -11\}$. These are the Euclidean among the imaginary quadratic fields, and it is known (cf. [5]) that for all $\xi \in k$ there exist integers $\eta \in \mathbb{Z}_k$ such that $N(\xi - \eta) \le M$, where the Euclidean minimum $M = M(k)$ is given by

$$M = \begin{cases} \frac{|n|+1}{4}, & \text{if } \Delta \equiv 0 \bmod 4, \\ \frac{(|n|+1)^2}{16|n|}, & \text{if } \Delta \equiv 1 \bmod 4. \end{cases}$$

Fix an embedding of $k$ into $\mathbb{C}$; then $N\xi = |\xi|^2$ for all $\xi \in k$, and the above result translates into

**Lemma 3.** *Let $k = \mathbb{Q}(\sqrt{-n})$ be Euclidean; then for all $\xi \in k$ there exist $\eta \in \mathbb{Z}_k$ such that $|\xi - \eta|^2 \le M$.*

Now we redo our computations in the proof of Theorem 1, assuming $a, c, m$, etc. to be integers (resp. half-integers) in $k$; the discriminant $\Delta$ is now replaced by the relative discriminant $d = \operatorname{disc}_{K/k}(1, \omega)$, and we have $\Delta = \operatorname{disc}(K/\mathbb{Q}) = d_0^2 N d$, where $d_0 = \operatorname{disc}(k/\mathbb{Q})$. Now

$$\frac{|r^2 - d/4|}{|c|^2} \le \frac{4|r^2| + |d|}{4|c|^2} \le \frac{4M|c|^2 + |d|}{4|c|^2},$$

and this expression is $< 1$ if and only if

$$|c|^2 > \frac{|d|}{4(1-M)} = \frac{\sqrt{\Delta}}{4|d_0|(1-M)}. \qquad (1)$$

For $k = \mathbb{Q}(\sqrt{-1})$ we have $M(k) = \frac{1}{2}$ and $d_0 = -4$, hence $\mu_K = \sqrt{\Delta}/8$. Evaluating (1) for the other fields gives

**Theorem 4.** *Let $k = \mathbb{Q}(\sqrt{-n})$ be Euclidean, and let $K/k$ be a quadratic extension with absolute discriminant $\Delta$. Then every ideal class of $K$ contains an integral ideal of norm $\leq \mu_K$, where*

$$\mu_K = \frac{\sqrt{\Delta}}{4|d_0|(1-M)} = \begin{cases} \sqrt{\Delta}/8, & \text{if } n \in \{-1,-2,-3,-11\}; \\ \sqrt{\Delta}/12, & \text{if } n = -7. \end{cases}$$

These are exactly the bounds given by Lakein [4]; another proof is due to Mordell [7]. The result in the special case $k = \mathbb{Q}(\sqrt{-1})$ was already known to S. Kuroda and J. A. Nyman (cf. [4]). After the completion of this article I discovered that S. Lubelsky (in his posthumously published paper [6]) had already found the formula connecting the bounds given in Theorem 2 with the Euclidean minima of imaginary quadratic number fields; his results remained unnoticed, probably because he used the language of quadratic forms.

In [1], Robin Chapman has generalized Theorem 2 to quadratic extensions of imaginary quadratic fields with class number 1.

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Bilkent University, Department of Mathematics, 06800 Bilkent, Ankara, Turkey
*E-mail address*: franz@fen.bilkent.edu.tr