ELEMENTARY NUMBER THEORY

HOMEWORK 7

(1) Let

$$a = 1 + 3 \cdot 5 + 4 \cdot 5^2 + \dots,$$
$$b = 3 + 2 \cdot 5 + 2 \cdot 5^2 + \dots$$

Compute approximations modulo 5^2 for $a + b$, $a - b$, ab and a/b.

(2) Show that $\sqrt{2} \in \mathbb{Z}_{17}$.
   (a) First solve $x_1^2 \equiv 2 \bmod 17$.
   (b) Write $x_2 = x_1 + 17y$ and determine $y \bmod 17$ in such a way that $x_2^2 \equiv 2 \bmod 17^2$.
   (c) Prove by induction that you can solve $x_k^2 \equiv 2 \bmod 17^k$ for every $k \geq 1$.
   (d) Prove that the sequence x_k is a Cauchy sequence with respect to $|\cdot|_{17}$.
   (e) Let x be the 17-adic number defined by the Cauchy sequence x_k. Show that $x^2 = 2$.

(3) Show that the equation $x^3 = 2$ has no solution in $\mathbb{Z}_7$.