ELEMENTARY NUMBER THEORY

HOMEWORK 3

(1) Fermat repeatedly challenged English mathematicians by sending them problems he claimed to have solved and asking for proofs. Two of them were the following that he sent to Wallis:
   - Prove that the only solution of $x^2 + 2 = y^3$ in positive integers is given by $x = 5$ and $y = 3$;
   - Prove that the only solution of $x^2 + 4 = y^3$ in positive integers is given by $x = 11$ and $y = 5$.

   In a letter to his English colleague Digby, Wallis called these problems trivial and useless, and mentioned a couple of problems that he claimed were of a similar nature:
   - $x^2 + 12 = y^4$ has unique solution $x = 2$, $y = 2$ in integers;
   - $x^4 + 9 = y^2$ has unique solution $x = 2$, $y = 5$ in integers;
   - $x^3 - y^3 = 20$ has no solution in integers;
   - $x^3 - y^3 = 19$ has unique solution $x = 3$, $y = 2$ in integers.

   When Fermat learned about Wallis's comments, he called Wallis's problems mentioned above "amusements for a three-day arithmetician" in a letter to Digby. In fact, while Fermat's problems were hard (and maybe not even solvable using the mathematics known in his times), Wallis's claims are easy to prove. Do this.

(2) Show that there are infinitely many primes of the form $p \equiv \pm 1 \bmod 8$ by modifying Euclid's proof.

(3) Use Gauss's Lemma to show that -2 is a quadratic residue of an odd prime p if $p \equiv 1, 3 \bmod 8$. Imitate the proof of Fermat's 2-squares-theorem to show that primes $p \equiv 1, 3 \bmod 8$ can be written in the form $p = c^2 + 2d^2$ for integers c, d.

The homework will be collected in class next Monday.