Discrete Mathematics

Midterm 2, April 29, 2007

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1. *Determine (with explanations!) whether the following functions are injective, surjective, or bijective.*

   (a) $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, $f(x) = 2x + 1$;

   (b) $f : \mathbb{Q} \longrightarrow \mathbb{Q}$, $f(x) = 2x + 1$.

   (a) $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, $f(x) = 2x + 1$. If $f(x) = f(y)$, then $2x + 1 = 2y + 1$, hence $x = y$. This shows that f is injective. Since the image of f only contains odd numbers, it is not surjective, and therefore not bijective.

   (b) $f : \mathbb{Q} \longrightarrow \mathbb{Q}$, $f(x) = 2x + 1$. As above, f is injective. Given $y \in \mathbb{Q}$, we find $y = 2x + 1$ for $x = \frac{y-1}{2}$. Since x is rational if y is rational, f is surjective, and therefore bijective.

2. *If* 11 *integers are selected from* $\{1, 2, 3, \dots, 100\}$*, show that there are at least two, say* x *and* y*, such that*

$$0 \leq \sqrt{x} - \sqrt{y} < 1.$$

Assume we want to find a counterexample. We take $x = 1$; for $\sqrt{y} - \sqrt{x}$ to be ≥ 1 we need to take $y \geq 4$, so let us assume $y = 4$. For $\sqrt{z} - \sqrt{y}$ to be ≥ 1 we should take $z \geq 9$ etc. Thus the squares 1, 4, 9, ..., 100 play an important role. After these preparations, it is easy to write down a conclusive proof:

Since there are 10 squares between 1 and 100 and we have 11 numbers, two of them must be in the same interval bounded by successive squares, say $n^2 \leq x \leq y < (n+1)^2$. But then $n \leq \sqrt{x} \leq \sqrt{y} < n + 1$, hence $0 \leq \sqrt{x} - \sqrt{y} < 1$ as desired.

3. *Determine the number of integer solutions of*

$$x_1 + x_2 + x_3 + x_4 = 19$$

with $0 \leq x_j < 8$ *for* $j = 1, 2, 3, 4$.

We first solve the problem using inclusion-exclusion.

- The number of solutions with $x_j \geq 0$ is $\binom{22}{3}$ (distribute 19 apples to 4 kids, i.e., count words with 19 A's and 3 bars).
- The number of solutions where at least one x_j is ≥ 8 is $\binom{14}{3}$ (replace e.g. x_1 by $y_1 = x_1 - 8$).
- The number of solutions where at least two x_j are ≥ 8 is $\binom{6}{3}$.
- There are no solutions in which three x_j are ≥ 8.

By the inclusion-exclusion principle, the number of solutions is

$$\binom{22}{3} - \binom{4}{1}\binom{14}{3} + \binom{4}{2}\binom{6}{3} = 204.$$

The second solution starts with the observation that the number of solutions we are looking for is the coefficient of x^{19} in the expression

$$(1 + x + x^2 + \ldots + x^7)^4 = \Big(\frac{x^8 - 1}{x - 1}\Big)^4.$$

Since the exponents of x in $(x^8 - 1)^4$ are divisible by 8, we only have the following possibilities:

- $x^{19} = x^0 x^{19}$, where the first factor x^0 is taken from $(x^8 - 1)^4$, the second one from $(x - 1)^{-4}$. The coefficient of x^0 in $(x^8 - 1)^4$ is 1, that of x^{19} in $(x - 1)^{-4}$ is $\binom{-4}{19}(-1)^{-23} = \binom{22}{3}$.
- $x^{19} = x^8 x^{11}$; the coefficient of x^8 in $(x^8 - 1)^4$ is $-\binom{4}{1}$, that of x^{11} in $(x - 1)^{-4}$ is $-\binom{-4}{11} = \binom{14}{3}$.
- $x^{19} = x^{16} x^3$; here the coefficients are $\binom{4}{2}$ and $-\binom{-4}{3} = \binom{6}{3}$, respectively.

Thus the coefficient of x^{19} we are looking for is

$$\binom{22}{3} - \binom{4}{1}\binom{14}{3} + \binom{4}{2}\binom{6}{3} = 204.$$

4. *Find the coefficient of the constant term in*

   (a) $(2x^3 + \frac{1}{3x})^{12}$.
   (b) $(2x^3 + \frac{1}{3x})^{13}$.
   (c) $(2x^3 + \frac{1}{3x})^n$ for a general integer $n \geq 0$.

   The binomial theorem tells us that

$$(2x^3 + \frac{1}{3x})^n = \sum_{j=0}^{n} \binom{n}{j} (2x^3)^j \Big(\frac{1}{3x}\Big)^{n-j}.$$

   The term $(2x^3)^j(3x)^{j-n}$ has degree $3j + j - n = 4j - n$. Thus we only get constant terms if n is a multiple of 4, and this means that the answer to question b) is 0.

   If $n = 4m$, however, then we have to look at the term with $j = m$, and we find that it equals

$$\binom{4m}{m} (2x^3)^m \Big(\frac{1}{3x}\Big)^{3m} = 2^m 3^{-3m} \binom{4m}{m}.$$

   For $m = 3$, this equals $\binom{12}{3}\frac{8}{27}$.

5. *Use generating functions to solve the recurrence relation* $a_0 = 2$*,* $a_1 = 5$*, and* $a_{n+1} = 5a_n - 6a_{n-1}$ *for* $n \geq 1$*.*

   We have

$$a_{n+1}x^{n+1} - 5a_n x^{n+1} + 6a_{n-1}x^{n+1} = 0.$$

   Adding these up for $n = 1, 2, \ldots$ and setting $F(x) = \sum_{n=0}^{\infty} a_n x^n$ we get

$$F(x) - a_0 - a_1 x - 5x(F(x) - a_0) + 6x^2 F(x) = 0,$$

   hence

$$F(x) = \frac{2 - 5x}{1 - 5x + 6x^2}.$$

   Since $1 - 5x + 6x^2 = (1 - 2x)(1 - 3x)$, a partial fraction decomposition gives

$$\frac{2 - 5x}{1 - 5x + 6x^2} = \frac{1}{1 - 2x} + \frac{1}{1 - 3x}.$$

   Using the geometric series yields

$$F(x) = \sum_{n=0}^{\infty} (2x)^n + \sum_{n=0}^{\infty} (3x)^n,$$

   and comparing the coefficient of x^n we get

$$a_n = 2^n + 3^n$$

   for all $n \geq 0$.

6. *Use generating functions to solve the recurrence relation* $a_0 = 2$, *and* $a_{n+1} = 2a_n - 1$ *for* $n \geq 0$.

   As above, we find $a_{n+1}x^{n+1} - 2xa_nx^n = x^{n+1}$, hence

$$F(x) - a_0 - 2xF(x) = x\sum_{n=0}^{\infty} x^n = \frac{x}{1-x}.$$

   This shows that

$$F(x)(1-2x) = 2 + \frac{x}{1-x},$$

   hence

$$\begin{aligned} F(x) &= \frac{2}{1-2x} + \frac{x}{(1-x)(1-2x)} \\ &= \frac{2}{1-2x} + \frac{1}{1-x} - \frac{1}{1-2x} = \frac{1}{1-2x} + \frac{1}{1-x}. \end{aligned}$$

   Comparing the coefficients of x^n we find

$$a_n = 2^n + 1.$$

Remark. The formulas for a_n in problems 5 and 6 were so simple that you could actually have guessed them. But as I said repeatedly, what you should demonstrate is not how to solve recurrence relations but that you know how to use generating functions.