Discrete Mathematics

Midterm 1, March 11, 2007

March 13, 2007

1. At a meeting of 17 delegates from 5 countries from America, 3 from Africa, 4 from Asia and 5 from Europe, a committee of 7 persons has to be elected. In how many ways can this be done

   (a) if there are no further conditions?

   (b) if the USA insists on being represented?

   (c) if at least one country from each continent has to be represented?

   (a) Choosing a committee of 7 from 17 can be done in $\binom{17}{7}$ ways.

   (b) We select the USA and then choose 6 out of 16, so there are $\binom{16}{6}$ ways of doing this.

   Alternatively, there are $\binom{16}{7}$ ways of picking a committee without a delegate from the USA, hence the number we are looking for is $\binom{17}{7} - \binom{16}{7} = \binom{16}{6}$.

   (c) The following solution that I had in mind is wrong: Picking 5 delegates, one from each continent, can be done in $5 \cdot 3 \cdot 4 \cdot 5 = 300$ ways. Then we have to choose 3 more delegates from the remaining 13, so the overall number of possibilities is $3 \cdot 4 \cdot 5^2 \cdot \binom{13}{3}$.

   Here's why: assume for simplicity that we have 4 delegates from America, and one from each of the other continents; picking a committee of 5 with at least one from each continent can obviously done in 6 ways, since we have to select two from America as well as the other three. The above solution, however, would give us $4 \cdot \binom{3}{1} = 12$ possibilities. The additional factor 2 comes from the fact that we have counted each committee twice because we have distinguished between the American country that gets picked first, and the one that gets picked second.

   A correct solution in the simple case proceeds like this: there are $\binom{7}{5} = 21$ ways of picking a committee of 5; we have to subtract the number of committees with no one from a single continent, namely

$3\binom{6}{5} = 18$ (there are 6 ways to choose a committee with no one from Africa, and similarly for Asia and Europe). But now we have subtracted the committees with no one from two continents twice, so we have to add $3\binom{5}{5} = 3$ again. This gives us $21 - 18 + 3 = 6$ as the number of possible committees.

In our situation, the correct solution would proceed like this: there are $\binom{17}{7}$ possible committees. We subtract $\binom{12}{7} + \binom{14}{7} + \binom{13}{7} + \binom{12}{7}$, which is the number of committees with no delegate from at least one continent. Since we have subtracted the number of committees with no delegate from at least two continents twice, we have to add $\binom{9}{7} + \binom{8}{7} + \binom{7}{7} + \binom{10}{7} + \binom{9}{7} + \binom{8}{7}$. Since there are no committees with no delegates from at least three continents, we are done now, and the number of possibilities is $\binom{17}{7} - \binom{12}{7} - \binom{14}{7} - \binom{13}{7} - \binom{12}{7} + \binom{9}{7} + \binom{8}{7} + \binom{7}{7} + \binom{10}{7} + \binom{9}{7} + \binom{8}{7} = 63193$.

This technique is called inclusion-exclusion; we will deal with it soon.

2. (a) How many integral solutions $x_1 > 0$, $x_2, x_3, x_4, x_5 \geq 0$, does the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 9$$

have?

The equation is equivalent to $y_1 + x_2 + x_3 + x_4 + x_5 = 8$ with $y_1 = x_1 - 1 \geq 0$, $x_j \geq 0$. If we think of this as distributing eight Units to 5 variables, this number is the same as the number of 12-letter words made from 8 U's and 4 bars, so there are $\binom{12}{4}$ solutions.

Alternatively, count the number of solutions $x_i \geq 0$ and subtract the number of solutions with $x_1 = 0$; this gives the answer $\binom{13}{4} - \binom{12}{3} = \binom{12}{4}$.

(b) How many 5-digit numbers (no leading zeros) are there whose digits add up to 9?

If the number is given in the decimal system as $x_1x_2x_3x_4x_5$, then the digits satisfy the above equations; thus there are $\binom{12}{4}$ such numbers.

If you count the numbers with leading digits 1, 2, ..., your answer will be

$$\binom{11}{3} + \binom{10}{3} + \binom{9}{3} + \ldots + \binom{3}{3} = \binom{12}{4}.$$

3. How many different 12-digits numbers N (no leading zeros) are there with the following two properties?

(a) N contains each of the digits 1, 2, ..., 9 exactly once;

(b) N does not have any consecutive zeros.

There are two correct solutions.

- Count the numbers with properties (a) and (b), which is the reading I had intended.
  The counting is done in two steps:
  1. Write down the digits from 1 to 9 in some order; there are 9! ways of doing this.
  2. Think of the spaces between these digits as boxes; each box to the right of the first digit either gets one of the three 0's, or no zero. Thus there are as many possibilities as there are 9-letter words with three 0 and 6 N, that is, $\binom{9}{3}$.
  Thus there are $9! \cdot \binom{9}{3}$ of these numbers.
  Alternatively, there are 9 ways of picking the first digit; the remaining number is an 11-letter word made out of the remaining 8 nonzero digits as well as three 0s. There are $\frac{11!}{3!}$ of them. Among these, 10! have at least a double 0 (count 10-letter words with the letters 0, 00, and the eight nonzero digits), and from these 10! numbers, 9! have a triple 0; since these were counted twice by the 10! (once as 0 00, and once as 00 0), there are exactly $9(\frac{11!}{3!} - 10! + 9!) = 9! \cdot 84 = 9!\binom{9}{3}$ such numbers.
- Count the numbers with property (a), and those with property (b).
  There are $\frac{12!}{3!}$ words with 12-letters consisting of 0, 0, 0, 1, 2, ..., 9. From this number we have to subtract the number of words with leading 0, and there are $\frac{11!}{2!}$ of them (write down a 0 and then an arbitrary 11-letter word with 0, 0, 1, ..., 9). Thus the correct answer to (a) is $\frac{12!}{3!} - \frac{11!}{2!} = \frac{12!-3\cdot 11!}{3!} = 9 \cdot \frac{11!}{3!}$.
  Alternatively, there are 9 ways to pick the first digit, and there are $\frac{11!}{3!}$ words made out of the remaining 8 digits and the three 0s. Thus we find $9 \cdot \frac{11!}{3!}$ such numbers.

  Now we count the 12-digit numbers with 0, 1, 2, ... zeros, none of them consecutive.
  (a) Clearly there are 9^{12} such numbers without any 0.
  (b) The numbers with exactly 1 zero consist of 11 nonzero digits, which can be chosen in 9^{11} ways; if we regard the spaces between the digits as boxes, there are 11 places where we can put the 0; thus there are $11 \cdot 9^{11}$ such numbers.
  (c) Numbers with exactly two zeros: Here we have 9^{10} ways of picking the nonzero digits, and $\binom{10}{2}$ ways of placing the 0s.
  (d) Similarly, there are $\binom{9}{3} \cdot 9^9$ numbers with exactly three zeros etc.
  Thus there are exactly
  $$9^{12} + \binom{11}{1}9^{11} + \binom{10}{2}9^{10} + \binom{9}{3}9^9 + \binom{8}{4}9^8 + \binom{7}{5}9^7 + \binom{6}{6}9^6$$
  12-digit numbers with no consecutive zeros.

4. Consider the recurrence sequences a_n and F_n defined as follows:

   - $a_1 = a_2 = 1$, $a_3 = 2$; $a_{n+1} = a_n + a_{n-2}$.
   - $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$.

   Use induction to prove that $a_n \le F_n$ for all $n \ge 1$.

   We first show that the claim is true for $n = 1, 2, 3$ (it is **not** sufficient to check it for $n = 1$ only); in fact we have $a_1 = 1 = F_1$, $a_2 = 1 = F_2$, $a_3 = 2 = F_3$.

   Now assume the claim holds for n, $n-1$, and $n-2$; then

   $$a_{n+1} = a_n + a_{n-2} \le F_n + F_{n-2} \le F_n + F_{n-1} = F_{n+1}$$

   since $F_{n-2} \le F_{n-3} + F_{n-2} = F_{n-1}$ for all $n \ge 3$.

5. Compute $\gcd(8n+10, 6n+8)$, and represent the gcd as a linear combination of $8n+10$ and $6n+8$. We find

   $$\begin{aligned} 8n + 10 &= 1(6n+8) + 2n + 2 \\ 6n + 8 &= 3(2n+2) + 2 \\ 2n + 2 &= (n+1)2 + 0, \end{aligned}$$

   hence $\gcd(8n+10, 6n+8) = 2$.

   Working backwards we find

   $$\begin{aligned} 2 &= 6n + 8 - 3(2n+2) \\ &= 6n + 8 - 3((8n+10) - (6n+8)) \\ &= 4(6n+8) - 3(8n+10). \end{aligned}$$

6. Show that $5 \mid 7^n - 2^n$ for all integers $n \geq 0$

   (a) *by induction.*

   The claim holds for $n = 0$ and $n = 1$. Assume it holds for n; then $7^{n+1} - 2^{n+1} = 7 \cdot 7^n - 2 \cdot 2^n = 2(7^n - 5^n) + 5 \cdot 7^n$. The first term is divisible by 5 by induction assumption, hence $5 \mid 7^{n+1} - 2^{n+1}$ as claimed.

   (b) *using congruences.*

   This is trivial: we have $7 \equiv 2 \bmod 5$, hence $7^n - 2^n \equiv 2^n - 2^n \equiv 0 \bmod 5$.

   You could also use the binomial theorem: we have

$$\begin{aligned} 7^n - 2^n &= (5+2)^n - 2^n \\ &= 5^n + \binom{n}{1} 5^{n-1} 2 + \ldots + \binom{n}{n-1} 5^1 2^{n-1} + 2^n - 2^n, \end{aligned}$$

   and this is clearly a multiple of 5.

   A fourth proof uses

$$7^n - 2^n = (7-2)(7^{n-1} + 7^{n-2}2 + \ldots + 7 \cdot 2^{n-2} + 2^{n-1}).$$

7. Starting in 2007, ISBNs will consist of 12 digits and one check digit, which is computed as follows: multiply the digits alternately by 1 and 3 (first digit by 1, second by 3, third by 1, ..., check digit by 1); the sum must then be 0 mod 10.

   Compute the check digit $*$ of the book "Compact Lie Groups" published in early 2007, whose ISBN is $978 - 0 - 387 - 30263 - *$.

   $1 \cdot 9 + 3 \cdot 7 + 1 \cdot 8 + 3 \cdot 0 + \ldots + 3 \cdot 3 + 1 \cdot * \equiv 2 + * \equiv 0 \bmod 10$ implies that the check digit must be 8.