DISCRETE MATHEMATICS

HOMEWORK 7

(1) Solve the recurrence relation $a_0 = a$, $4a_{n+1} = 5a_n$.

We find $a_1 = \frac{5}{4}a_0 = \frac{5}{4}a$, $a_2 = \frac{5}{4}a_1 = (\frac{5}{4})^2 a$, ..., hence $a_n = (\frac{5}{4})^n a$ by induction.

(2) Solve the recurrence relation $a_0 = 1$, $a_1 = 3$, $a_n = 5a_{n-1} + 6a_{n-2}$.

Write down the generating function

$$F(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a - nx^n + \ldots.$$

Then

$$F(x)(1 - 5x - 6x^2) = a_0 + (a_1 - 5a_0)x = 1 - 2x,$$

hence

$$F(x) = \frac{1 - 2x}{1 - 5x - 6x^2}.$$

Now $1 - 5x - 6x^2 = (1 + x)(1 - 6x)$, and a partial fraction decomposition gives

$$\frac{1 - 2x}{1 - 5x - 6x^2} = \frac{A}{1 + x} + \frac{B}{1 - 6x}$$

with $A = \frac{3}{7}$ and $B = \frac{4}{7}$. Using the geometric series then yields

$$F(x) = \frac{3}{7}(1 - x + x^2 - x^3 + \ldots) + \frac{4}{7}(1 + 6x + 6^2x^2 + 6^3x^3 + \ldots),$$

and comparing the coefficients of x^n finally provides us with

$$a_n = \frac{3}{7}(-1)^n + \frac{4}{7} \cdot 6^n.$$

Important Remark. What I tried to teach you the last few weeks was not how to solve recurring sequences; this is not something that I regard as being important. What I taught you was the concept of generating functions, and this is what I want to see in the solutions. Correct solutions built on memorized recipes will not earn you full credit, neither in the homework nor in the midterm.

(3) Solve the recurrence relation $a_0 = 2$, $a_1 = -8$, $2a_{n+2} - 11a_{n+1} + 5a_n = 0$.

As before we find

$$F(x)(2 - 11x + 5x^2) = 2a_0 + (2a_1 - 11a_0)x = 4 - 38x.$$

Since $2 - 11x + 5x^2 = (1 - 5x)(2 - x)$, a partial fraction decomposition gives us

$$F(x) = \frac{-2}{1 - 5x} + \frac{8}{2 - x} = \frac{-2}{1 - 5x} + \frac{4}{1 - \frac{x}{2}}.$$

Comparing the coefficients of x^n yields

$$a_n = 4 \cdot 2^{-n} - 2 \cdot 5^n.$$

(4) Solve the recurrence relation $a_0 = 1$, $a_1 = 6$, $a_{n+2} = 2a_{n+1} + a_n$.

This time we find

$$F(x) = \frac{1+4x}{1-2x-x^2}.$$

With $1 - 2x - x^2 = (1 - \alpha x)(1 - \beta x)$ we get $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Setting

$$F(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

we find $A = \frac{\sqrt{2}+5}{2\sqrt{2}} = \frac{2+5\sqrt{2}}{4}$ and $B = \frac{2-5\sqrt{2}}{4}$, as well as

$$a_n = A\alpha^n + B\beta^n.$$