DISCRETE MATHEMATICS

HOMEWORK 6

(1) How many 8-digit numbers are there whose digits are from $\{1,2,3,4,5,6\}$ such that each digit occurs at least once?

We count all numbers, subtract those made from only 5 digits, add those made from 4 digts, etc.

Thus there are $6^8 - \binom{6}{1}\cdot 5^8 + \binom{6}{2}\cdot 4^8 - \binom{6}{3}\cdot 3^8 + \binom{6}{4}\cdot 2^8 - \binom{6}{5}$ such numbers.

(2) How many numbers $1 \le n \le 1000$ are not a perfect square? How many are neither perfect squares nor cubes?

The squares are 1^2, 2^2, ..., $31^2 = 961$; thus there are $1000 - 31 = 969$ nonsquares.

The cubes are 1^3, 2^3, ..., $10^3 = 1000$; among these cubes, there are three sixth powers that have been counted among the squares, namely $1 = 1^6$, $4^3 = 2^6$, and $9^3 = 3^6$. Thus there are $1000 - 31 - 10 + 3 = 962$ numbers that are neither squares nor cubes.

(3) Find the number of integer solutions for the following equations:
(a) $x_1 + x_2 + x_3 + x_4 = 20$, $0 \le x_i \le 7$.
(b) $x_1 + x_2 + x_3 + x_4 = 20$, $x_i \ge 0$, x_2 and x_3 even.
(c) $x_1 + x_2 + x_3 + x_4 + x_5 = 30$, $2 \le x_1 \le 4$, $3 \le x_i \le 8$ for $i = 2,3,4,5$.
(d) $x_1 + x_2 + x_3 + x_4 + x_5 = 30$, $x_i \ge 0$, x_2 even, x_3 odd.

Actually finding the number of solutions is difficult in some of the cases. I'm content with identifying the number of solutions with a coefficient in some power series (and I should have said so, but ...). Let N denote the number of solutions; then N is the coefficient of
(a) x^{20} in $(1 + x + x^2 + \ldots + x^7)^4$.
(b) x^{20} in $(1 + x + x^2 + \ldots)^2(1 + x^2 + x^4 + \ldots)^2$.
(c) x^{30} in $(x^2 + x^3 + x^4)(x^3 + x^4 + \ldots + x^8)^4$.
(d) x^{30} in $(1 + x + x^2 + \ldots)^3(1 + x^2 + x^4 + \ldots)(x + x^3 + x^5 + \ldots)$.

(4) Find the coefficient of x^{50} in $(x^7 + x^8 + x^9 + \ldots)^6$.

This is the same as the coefficient of x^8 in $(1+x^2+x^3+\ldots)^6 = (1-x)^{-6}$, and this coefficient is $\binom{-6}{8} = \binom{13}{8} = \binom{13}{5}$.

(5) Find the coefficient of x^{15} in $\frac{x^3-5x}{(1-x)^3}$.

This is the same as the coefficient of x^{14} in $\frac{x^2-5}{(1-x)^3}$. Now $(1-x)^{-3} = \sum_{n\geq 0}(-1)^n\binom{-3}{n}x^n$, so

$$\begin{aligned}(x^2-5)(1-x)^{-3} &= \sum_{n\geq 0}(-1)^n\binom{-3}{n}x^{n+2} - 5\sum_{n\geq 0}(-1)^n\binom{-3}{n}x^n \\ &= \sum_{n\geq 2}(-1)^n\binom{-3}{n-2}x^n - 5\sum_{n\geq 0}(-1)^n\binom{-3}{n}x^n,\end{aligned}$$

from which we see that the coefficient of x^{14} is $\binom{-3}{12} - 5\binom{-3}{14} = \binom{14}{2} - 5\binom{16}{2} = -509$.