DISCRETE MATHEMATICS

HOMEWORK 4

(1) Check which of the following relations are equivalence relations: (For showing that a relation is not an equivalence relation it is sufficient to show that one of the three conditions fails to hold.)

(a) On the set L of all lines in the plane $\mathbb{R}^2$, call two lines ℓ_1 and ℓ_2 related ($\ell_1 \mathcal{R} \ell_2$) if ℓ_1 is perpendicular to ℓ_2.

(i) reflexivity: no line is perpendicular to itself, so this relation is not reflexive.

(ii) symmetry: If $\ell_1 \perp \ell_2$, then $\ell_2 \perp \ell_1$, so this relation is symmetric.

(iii) transitivity: If $\ell_1 \perp \ell_2$ and $\ell_2 \perp \ell_3$, then $\ell_1 \parallel \ell_3$, so this relation is not transitive.

(b) Define a relation $\mathcal{R}$ on $\mathbb{Z}$ by saying $x\mathcal{R}y$ for integers x, y if $x + y$ is even.

(i) reflexivity: $x \sim x$ since $x + x = 2x$ is even.

(ii) symmetry: If $x \sim y$, then $x + y$ is even, hence so is $y + x$, and this means $y \sim x$.

(iii) transitivity: Assume that $x \sim y$ and $y \sim z$. Then $x + y$ and $y + z$ are even. Adding shows that $x + 2y + z$ is even, hence so is $x + z$. But then $x \sim z$, and the relation is transitive.

(c) Define a relation $\mathcal{R}$ on $\mathbb{Z}$ by saying $x\mathcal{R}y$ for integers x, y if $x + y$ is odd.

(i) reflexivity: does not hold, since $x + x$ is even, hence $x \sim x$ is false.

(ii) symmetry: holds, because if $x + y$ is odd, then so is $y + x$.

(iii) transitivity: does not hold: if $x + y$ and $y + z$ are odd, then $x + 2y + z$ and $x + z$ are even.

(d) Let T be the set of triangles in $\mathbb{R}^2$, and call two triangles related if they have an angle of the same measure (that is, the same size).

(i) reflexivity: every triangle is equivalent to itself.

(ii) symmetry: If A and B have an equal angle in common, then so do B and A.

(iii) transitivity: Assume that A and B have an equal angle in common, and that the same is true for B and C. Then A and C need not have an equal angle in common: for example, let A be equilateral (three angles of 60^o), let B have angles 30^o, 60^o, and 90^o, and let C have angles 30^o, 70^o and 80^o. Then $A \sim B$ and $B \sim C$, but not $A \sim C$.

(2) Draw the digraph with vertices $\{a, b, c, d, e, f\}$ and edges $\{(a, b), (a, d), (b, c), (b, e), (d, b), (d, e), (e, c), (e, f), (f, d)\}$. Also determine the adjacency matrix of this digraph.

The adjacency matrix is
$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

(3) Let A be a set with 5 elements. How many relations from A to A are there? How many of them are symmetric?

There are as many relations as there are subsets of $A \times A$, and there are exactly 2^{25} of them (for choosing a subset, you have two choices for each element: include it or exclude it; now observe that $A \times A$ has 25 elements).

How many of these 2^{25} relations are symmetric? For selecting a subset of $A \times A$ that represents a symmetric relation we proceed as follows: assume without loss of generality that $A = \{1, 2, 3, 4, 5\}$. Consider the set S of all pairs $(a, b) \in A \times A$ with $a \leq b$. For choosing a symmetric relation, select elements from S, and when selecting (a, b) with $a < b$, automatically include (b, a). Since you have 2 choices for each element in S, and since $\#S = 2^{15}$, there are exactly 2^{15} symmetric relations from A to A.

(4) For $A = \mathbb{R}^2$, define a relation $\mathcal{R}$ on A by $(x_1, y_1)\mathcal{R}(x_2, y_2)$ if $x_1 = x_2$. Check that $\mathcal{R}$ is an equivalence relation, and describe the equivalence classes geometrically.

(a) reflexivity: $(x_1, y_1)\mathcal{R}(x_1, y_1)$ since $x_1 = x_1$.
(b) symmetry: If $x_1 = y_1$, then $y_1 = x_1$ etc.
(c) transitivity: If $(x_1, y_1)\mathcal{R}(x_2, y_2)$ and $(x_2, y_2)\mathcal{R}(x_3, y_3)$, then $x_1 = x_2$ and $x_2 = x_3$, hence $x_1 = x_3$ and therefore $(x_1, y_1)\mathcal{R}(x_3, y_3)$.

An equivalence class consists of all points (x, y) with the same x-coordinate, hence forms a vertical line in the Euclidean plane.