DISCRETE MATHEMATICS

HOMEWORK 4

(1) Take 101 points in the unit square. Show that three of them are vertices of a triangle with area at most 0.01.

Let me first discuss approaches that do not give the desired result. The most obvious idea is to divide the square into 100 squares of side lenght 0.1. The pigeonhole principle then guarantees that there is a little square containing at least two points. Their maximal distance is $0.1\sqrt{2}$. Now pick any of the other 99 points; the height of the triangle fromed from these is at most $\sqrt{2}$, hence the area of this triangle is $\leq \frac{1}{2} \cdot 0.1\sqrt{2} \cdot \sqrt{2} = 0.1$. This is not good enough.

What if we divide the square into 25 subsquares of length 0.2? Then there is a square containing at least 3 points; the area of this triangle is at most half the area of the little square, that is, $\leq \frac{1}{2}0.2^2 = 0.02$. This is a lot better than the last result, but still not good enough.

In order to get the most out of the pigeonhole principle we have to divide the square into 50 equal parts: then one of them will contain at least three points. This can be achieved by dividing the square into 50 rectangles of side lenghts 0.1 and 0.2; the area of a triangle inside such a rectangle is $\leq \frac{1}{2}0.1 \cdot 0.2 = 0.01$.

(2) The set of all points (x, y) in the plane with integers x, y is called a lattice, and the points (x, y) are called lattice points. Show that among five lattice points, there are always two points such that the segment connecting them contains another lattice point.

Consider the pairs residue classes $(x \bmod 2, y \bmod 2)$ (equivalently, use pigeonholes of the form (even,even), (even,odd) etc.). Since the only possibilities are $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, there must be a pair of points (x, y), (x', y') whose coordinates have the same parity (i.e., $x \equiv x'$, $y \equiv y' \bmod 2$). But then the midpoint $(\frac{x+x'}{2}, \frac{y+y'}{2})$ is a lattice point.

(3) For each of the following functions $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, determine whether the function is injective or surjective; determine its range (the image of f), and find out which of these functions have an inverse.
   (a) $f(x) = x + 7$ is injective and surjective. Injectivity: $f(x) = f(x') \Longrightarrow x + 7 = x' + 7 \Longrightarrow x = x'$. Surjectivity: Let $y \in \mathbb{Z}$; then $y = x + 7$ for $x = y - 7$. This also gives the inverse function $g(x) = x - 7$.
   (b) $f(x) = -x + 5$ is also bijective, with inverse function $g(x) = 5 - x$.
   (c) $f(x) = x^2 + x$ is not injective since $f(0) = f(-1) = 0$. f is not surjective since $x^2 + x = 1$ does not have a solution in $\mathbb{Z}$.
   (d) $f(x) = 2x - 3$ is injective, but not surjective since $2x - 3 \neq 0$ for $x \in \mathbb{Z}$.
   (e) $f(x) = x^2$ is neither injective nor surjective ($f(-1) = f(1)$; $f(x) \neq 2$).

(f) $f(x) = x^3$ is injective, but not surjective.

(4) Find sets A, B, subsets $A_1, A_2 \subseteq A$, and a function $f : A \longrightarrow B$ with the property that $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$.

Let $A_1 = \{1\}$, $A_2 = \{2\}$, and $A = \{1, 2\}$, and consider the function f defined by $f(1) = f(2) = 1$. Then $A_1 \cap A_2 = \varnothing$ and $f(\varnothing) = \varnothing$, but $f(A_1) = f(A_2) = \{1\}$.