DISCRETE MATHEMATICS

HOMEWORK 2

(1) Find the coefficient of xyz^2 in $(2x - y - z)^4$.

The coefficient is -24. In fact, we get xyz^2 by choosing from the 4 brackets x and y once, and z twice. This choice can be represented by a 4-letter word with x, y, z, z, and there are $\frac{4!}{2!} = 12$ of them. The coefficient of a single term xyz^2 is $(+2)(-1)(-1)^2 = -2$, and $12 \cdot (-2) = -24$.

(2) How many 10-digit numbers are there which use each digit exactly once, where no odd digit ever follows an even digit, and where the 4 and the 5 are next to each other?

If no odd digit ever follows an even digit, then the odd digits must make up the first five digits. If the 4 and the 5 are next to each other, then the fifth digit is 5 and the sixth is 4. The other four odd digits can be placed in 4! ways, and the same holds for the even digits. Thus the answer is $(4!)^2 = 24^2$.

(3) Assume you are given 15 points in the plane, no three of which are collinear (lie on the same line). How many lines do these points determine?

Every line is determined by two points P and Q, where the order of P and Q is not important since Q and P determines the same line. Since no three points are collinear, every pair of points determines exactly one line. There are $\binom{15}{2} = 105$ ways of selecting two points out of 15 if the order is not important. Thus there are 105 lines.

(4) Use induction to show that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Can you also give a direct proof?

For $n = 1$ the statement is obviously true. Assume it holds for n; then

$$\begin{aligned}
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} \\
&= \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} \\
&= \frac{1}{n+1}\Big(n + \frac{1}{n+2}\Big) = \frac{1}{n+1}\frac{n^2+2n+1}{n+2} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2},
\end{aligned}$$

hence the claim is true for $n + 1$ as well.

A direct proof would go like this:

$$\begin{aligned}\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} \\ &= \Big(\frac{1}{1} - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \ldots + \Big(\frac{1}{n} - \frac{1}{n+1}\Big) \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1}.\end{aligned}$$

(5) Use induction to show that

$$1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1.$$

Can you also give a direct proof?

The statement holds for $n = 0$ since $1 = 2^1 - 1$. Assume it holds for n; then

$$1 + 2 + 4 + \ldots + 2^n + 2^{n+1} = 2^{n+1} - 1 + 2^{n+1} = 2^{n+2} - 1.$$

For a direct proof, let $S = 1 + 2 + \ldots + 2^n$; then

$$\begin{aligned}S + 1 &= 2 + 2 + 4 + \ldots + 2^n \\ &= 4 + 4 + 8 + \ldots + 2^n \\ &= \ldots = 2^n + 2^n = 2^{n+1}.\end{aligned}$$

Here's another one: $2S = 2 + 4 + \ldots + 2^{n+1}$, hence $S = 2S - S = 2^{n+1} - 1$.