ALGEBRAIC NUMBER THEORY

HOMEWORK 4

(1) Compute the ideal class group of $K = \mathbb{Q}(\sqrt{-17}\,)$.

Gauss bound $\mu_K \approx 4.7$; $(2) = \mathfrak{p}_2^2$, $(\frac{-17}{3}) = +1$, $(3) = \mathfrak{p}_3\mathfrak{p}_3'$ with $\mathfrak{p}_3 = (3, 1+\sqrt{-17}\,)$.

From $(1+\sqrt{-17}\,) = \mathfrak{p}_2\mathfrak{p}_3^2$ and $\mathfrak{p}_2^2 = (2) \sim (1)$ we get $\mathfrak{p}_3^4 \sim (1)$. Since there are no elements of norm $3, 9$ or 27, we conclude that $[\mathfrak{p}_3]$ generates a cyclic subgroup of order 4 in $\mathrm{Cl}(K)$. Since $[\mathfrak{p}]^2 = [\mathfrak{p}_2]^{-1} = [\mathfrak{p}_2]$ and $[\mathfrak{p}_3'] = [\mathfrak{p}_3]^{-1} = [\mathfrak{p}_3]^3$, every ideal class contains a power of $\mathfrak{p}_3$, and this shows that $\mathrm{Cl}(K) = \langle[\mathfrak{p}_3]\rangle \simeq \mathbb{Z}/4\mathbb{Z}$.

(2) Compute the ideal class group of $\mathbb{Q}(\sqrt{-47}\,)$.

The Gauss bound is < 4, and we have $(2) = \mathfrak{p}_2\mathfrak{p}_2'$ and $(3) = \mathfrak{p}_3\mathfrak{p}_3'$ with $\mathfrak{p}_2 = (2, \frac{1+\sqrt{-47}}{2})$ and $\mathfrak{p}_3 = (3, \frac{1+\sqrt{-47}}{2})$.

Now $(\frac{1+\sqrt{-47}}{2}) = \mathfrak{p}_2^2\mathfrak{p}_3$ shows that every ideal class must be a power of $[\mathfrak{p}_2]$. The smallest power of 2 that is a norm is easily seen to be $2^5 = N(\frac{9+\sqrt{-47}}{2})$, and $(\frac{9+\sqrt{-47}}{2}) = \mathfrak{p}_2^5$ shows that $[\mathfrak{p}_2]$ has order 5. Thus $\mathrm{Cl}(K) = \langle[\mathfrak{p}_2]\rangle \simeq \mathbb{Z}/5\mathbb{Z}$.

(3) Show that $\mathbb{Q}(\sqrt{-163}\,)$ has class number 1.

the Gauss bound is < 8; the ideals (2), (3), (5) and (7) are all inert, so all ideals with norm < 8 are principal. This implies the claim.

(4) Compute the class number of $\mathbb{Q}(\sqrt{65}\,)$.

Here we have to look at prime ideals with norms ≤ 3. We find $(2) = \mathfrak{p}_2\mathfrak{p}_2'$ and that (3) is inert. Is $\mathfrak{p}_2 = (2, \frac{1+\sqrt{65}}{2}) = (\alpha)$ principal? Write $\alpha = \frac{a+b\sqrt{65}}{2}$ and consider the equation $N\alpha = \pm 2$, that is, $a^2 - 65b^2 = \pm 8$. Reduction mod 5 gives the contradiction $(\frac{\pm 2}{5}) = +1$. Thus $\mathfrak{p}_2$ is not primcipal. On the other hand, $(\frac{9+\sqrt{65}}{2}) = \mathfrak{p}_2^2$ shows that $\mathfrak{p}_2^2 \sim (1)$, and since $[\mathfrak{p}_2'] = [\mathfrak{p}_2]^{-1} = [\mathfrak{p}_2]$, the class group has order 2 and is generated by $[\mathfrak{p}_2]$.

(5) Compute the class number of $\mathbb{Q}(\sqrt{221}\,)$.

The only prime ideals with norm less than the Gauss bound are (2), $\mathfrak{p}_5 = (5, 1+\sqrt{221}\,)$, and $\mathfrak{p}_5'$. Clearly $\mathfrak{p}_5^2 = (14 - \sqrt{221}\,)$ is principal. What about $\mathfrak{p}_5$? If $\mathfrak{p}_5 \sim (1)$, then $a^2 - 5b^2 = \pm 20$ must be solvable. Reduction mod 13 gives a contradiction. Thus $\mathrm{Cl}(K) = \langle[\mathfrak{p}_5]\rangle \simeq \mathbb{Z}/2\mathbb{Z}$.