ALGEBRAIC NUMBER THEORY

HOMEWORK 2

(1) Let $a \in \mathbb{N}$ be a natural number. Find a basis (as a $\mathbb{Z}$-module) for the ideal $\mathfrak{a} = (a)$ in $\mathcal{O}_K$, where $K = \mathbb{Q}(\sqrt{m}\,)$ is a quadratic number field. Hint: $\mathfrak{a} = [n, c + m\omega]$; make an educated guess what n, c, m might be, and prove your conjecture.

(2) Show directly that $(7, 1 + \sqrt{-6}\,) = [7, 1 + \sqrt{-6}\,]$ in $R = \mathbb{Z}[\sqrt{-6}\,]$, i.e., that every R-linear combination $3\alpha + (1 + \sqrt{-6}\,)\beta$ with $\alpha, \beta \in R$ can already be written in the form $7a + (1 + \sqrt{-6}\,)b$ with $a, b \in \mathbb{Z}$.

(3) Let $K = \mathbb{Q}(\sqrt{m}\,)$ be a quadratic number field, where m is squarefree. Prove the following:
   - If $m \equiv 2 \bmod 4$ then $2\mathcal{O}_K = (2, \sqrt{m}\,)^2$.
   - If $m \equiv 3 \bmod 4$ then $2\mathcal{O}_K = (2, 1 + \sqrt{m}\,)^2$.
   - If $m \equiv 1 \bmod 8$ then $2\mathcal{O}_K = \mathfrak{a}\mathfrak{a}'$, where $\mathfrak{a} = (2, \frac{1+\sqrt{m}}{2}\,)$ and $\mathfrak{a} \neq \mathfrak{a}'$.
   - If $m \equiv 5 \bmod 8$ then $2\mathcal{O}_K$ is prime.

(4) Let $R = \mathbb{Z}[X]$, and consider $\mathfrak{a} = (2, X)$. Show that there does not exist an ideal $\mathfrak{b} \neq (0)$ in R such that $\mathfrak{a}\mathfrak{b}$ is principal. (I haven't thought of a simple argument; I don't even know for sure that the result is true.)