

TRANSCENDENTAL LATTICE OF AN EXTREMAL ELLIPTIC SURFACE

ALEX DEGTYAREV

ABSTRACT. We develop an algorithm computing the transcendental lattice and the Mordell–Weil group of an extremal elliptic surface. As an example, we compute the lattices of four exponentially large series of surfaces

1. INTRODUCTION

1.1. Principal results. An *extremal elliptic surface* can be defined as a Jacobian elliptic surface X of maximal Picard number, $\mathrm{rk} \, NS(X) = h^{1,1}(X)$, and minimal Mordell–Weil rank, $\mathrm{rk} \, MW(X) = 0$. For alternative, more topological descriptions, see Definition 2.2.2 and Remark 3.4.4.

Extremal elliptic surfaces are rigid; they are defined over algebraic number fields. Up to isomorphism, such a surface X (without type $\tilde{\mathbf{E}}$ singular fibers) is determined by an oriented 3-regular ribbon graph Γ_X , called *skeleton* of X , see Subsection 2.3. This intuitive approach gives one a simple way to construct and classify extremal elliptic surfaces, see, *e.g.*, [2] or [3]; however, the relation between the invariants of X and the structure of Γ_X is not yet well understood. A few first attempts to compute the invariants of some surfaces were recently made in [1]. In slightly different terms, general properties of the (necessarily finite) Mordell–Weil group of an extremal elliptic surface and a few examples are found in [9]. (Due to [10] and Nikulin’s theory of lattice extensions [7], the Mordell–Weil group and the transcendental lattice are closely related, *cf.* 2.2.3.)

The principal results of this paper are Theorem 4.3.4 and Corollaries 4.3.8 and 4.3.9, computing the transcendental lattice \mathcal{T}_X and the Mordell–Weil group $MW(X)$ of an extremal elliptic surface X without type $\tilde{\mathbf{E}}$ singular fibers in terms of its skeleton Γ_X . (Some generalizations to wider classes of surfaces are discussed in Section 6, see Theorems 6.1.1 and 6.2.2.) It is important to notice that the algorithm uses a computer friendly presentation of the graph (by a pair of permutations, see Remark 4.1.3); combined with the known classification results (see, *e.g.*, [2]) and various lattice analyzing software, it can be used for computer experiments.

1.2. Examples. Originally, this paper was motivated by a construction in [3], producing exponentially large series of non-isomorphic extremal elliptic surfaces. Here, we compute the invariants of these surfaces.

2000 *Mathematics Subject Classification.* Primary: 14J27.

Key words and phrases. Elliptic surface, transcendental lattice, Mordell–Weil group, fundamental group, dessins d’enfants.

Given an integer $k \geq 1$, define the lattices (see Subsection 2.1) \mathcal{V}_{k-1} and \mathcal{W}_k as the orthogonal direct sums

$$(1.2.1) \quad \mathcal{V}_{k-1} = \bigoplus_{i=1}^{k-1} \mathbb{Z}\mathbf{v}_i, \quad \mathcal{W}_k = \bigoplus_{i=1}^{k-1} \mathbb{Z}\mathbf{v}_i \oplus \mathbb{Z}\mathbf{w},$$

where $\mathbf{v}_i^2 = 1$, $i = 1, \dots, k-1$, and $\mathbf{w}^2 = 0$.

1.2.2. Theorem. *Let X be an extremal elliptic surface with singular fibers*

$$\tilde{\mathbf{A}}_{10s-2} \oplus (2s+1)\tilde{\mathbf{A}}_0^*, \quad s \geq 1.$$

Then $\mathcal{T}_X \cong (3\mathbf{v}_1 + \dots + 3\mathbf{v}_s + \mathbf{v}_{s+1} + \dots + \mathbf{v}_{2s-1})^\perp \subset \mathcal{V}_{2s-1}$.

1.2.3. Theorem. *Let X be an extremal elliptic surface with singular fibers*

$$\tilde{\mathbf{D}}_{10s-2} \oplus (2s)\tilde{\mathbf{A}}_0^*, \quad s \geq 1.$$

Then $\mathcal{T}_X \cong \mathbf{D}_{2s-2}$ (where we let $\mathbf{D}_0 = 0$, $\mathbf{D}_1 = [4]$, $\mathbf{D}_2 = 2\mathbf{A}_1$, and $\mathbf{D}_3 = \mathbf{A}_3$).

Theorems 1.2.2 and 1.2.3 are proved in Subsection 5.4.

1.2.4. Theorem. *Let X be an extremal elliptic surface with singular fibers*

$$\tilde{\mathbf{D}}_{10s+3} \oplus \tilde{\mathbf{D}}_5 \oplus (2s)\tilde{\mathbf{A}}_0^*, \quad s \geq 1.$$

Then $\mathcal{T}_X \cong \mathbf{D}_{2s-1} \oplus \mathbb{Z}\mathbf{x}$, where $\mathbf{x}^2 = 4$.

Let $f_s = 3\mathbf{v}_1 + \dots + 3\mathbf{v}_{s-1} + \mathbf{v}_s + \dots + \mathbf{v}_{2s-2} \in \mathcal{V}_{2s-2}$, and denote by \mathcal{V}'_{2s-2} the group \mathcal{V}_{2s-2} with the bilinear form $x \otimes y \mapsto x \cdot y + \frac{1}{4}(f_s \cdot x)(f_s \cdot y)$, where \cdot stands for the original product in \mathcal{V}_{2s-2} . (Certainly, \mathcal{V}'_{2s-2} is *not* an integral lattice.)

1.2.5. Theorem. *Let X be an extremal elliptic surface with singular fibers*

$$\tilde{\mathbf{A}}_{10s-7} \oplus \tilde{\mathbf{D}}_5 \oplus (2s-1)\tilde{\mathbf{A}}_0^*, \quad s \geq 1.$$

Then \mathcal{T}_X is the index 4 sublattice $\{x \in \mathcal{V}'_{2s-2} \mid f_s \cdot x = 0 \pmod{4}\} \subset \mathcal{V}'_{2s-2}$.

Theorems 1.2.4 and 1.2.5 are proved in Subsection 5.6.

Note that, in Theorems 1.2.2–1.2.5, a simple count using the Riemann–Hurwitz formula for the j -invariant shows that the base of any extremal elliptic surface with one of the combinatorial types of singular fibers indicated in the statements is \mathbb{P}^1 .

The Jacobian elliptic surfaces as in Theorems 1.2.2–1.2.5 appeared in [3]; within each of the four series, the number of fiberwise equisingular deformation classes grows faster than a^{4s} for any $a < 2$, cf. 5.1.3, and the original goal of this project was to distinguish these surfaces topologically, hoping that the definite lattices \mathcal{T}_X would fall into distinct isomorphism classes. The four theorems above show that this approach fails. (Note that the theorems imply as well that, for each surface X in question, the Mordell–Weil group $MW(X)$ is trivial.) To add to the disappointment, one can also use [3] and some intermediate results of this paper and compute the fundamental groups $\pi_1(\Sigma \setminus (C \cup E))$ of the ramification loci of the double coverings $X \rightarrow \Sigma$, see 3.4.1 for the definition and Subsection 5.7 for the computation. Most groups turn out to be abelian; hence they also depend on s only (within each of the four series).

1.2.6. Theorem. *Let X be one of the surfaces as in Theorems 1.2.2–1.2.5, and assume that $s > 1$. Then the fundamental group $\pi_1(\Sigma \setminus (C \cup E))$ is cyclic.*

This theorem is proved in Subsection 5.7. In the four exceptional cases corresponding to the value $s = 1$, the groups can also be computed; they are listed in Remark 5.7.2. In two cases, the trigonal curve C is reducible.

Thus, neither \mathcal{T}_X nor $\pi_1(\Sigma \setminus (C \cup E))$ distinguish the surfaces, and the following problem, which motivated this paper, still stands.

1.2.7. Problem. Are surfaces X as in Theorems 1.2.2–1.2.5 fiberwise homeomorphic (for each given s and within each given series)? Are they Galois conjugate?

An answer to the first question should be given by the Hurwitz equivalence class of the braid monodromy of the ramification locus. The monodromies are given by (5.7.1); at present, I do not know whether they are Hurwitz equivalent.

1.3. Contents of the paper. In Section 2 we remind a few concepts related to integral lattices and elliptic surfaces. Section 3 deals with the topological part of the computation; it is used in the proof of the main theorem and its corollaries in Section 4. In Section 5, we consider a special class of skeletons, the so called *pseudo-trees*, and prove Theorems 1.2.2–1.2.6. Finally, in Section 6, we discuss a few generalizations of the principal results.

1.4. Acknowledgements. I am grateful to I. Itenberg, with whom I discussed this project at its early stages, and to I. Dolgachev and I. Shimada for their helpful remarks. I would also like to extend my gratitude to the referee of this paper, whose remarks helped me to improve clarity of the exposition.

2. PRELIMINARIES

2.1. Lattices. An (*integral*) *lattice* is a finitely generated free abelian group \mathcal{L} supplied with a symmetric bilinear form $\mathcal{L} \otimes \mathcal{L} \rightarrow \mathbb{Z}$ (which is usually referred to as *product* and denoted by $x \otimes y \mapsto x \cdot y$ and $x \otimes x \mapsto x^2$). A lattice is called *even* if $x^2 = 0 \pmod{2}$ for all $x \in \mathcal{L}$. Occasionally, we will also consider *rational lattices*, which are free abelian groups supplied with \mathbb{Q} -valued symmetric bilinear forms. A lattice structure on \mathcal{L} is uniquely determined by the function $x \mapsto x^2$: one has $x \cdot y = \frac{1}{2}[(x+y)^2 - x^2 - y^2]$.

Given a lattice \mathcal{L} , one can define the *associated homomorphism* $\varphi_{\mathcal{L}}: \mathcal{L} \rightarrow \mathcal{L}^* := \text{Hom}(\mathcal{L}, \mathbb{Z})$ via $x \mapsto [y \mapsto x \cdot y] \in \mathcal{L}^*$. The *kernel* $\ker \mathcal{L}$ is the kernel of $\varphi_{\mathcal{L}}$. (We use the notation $\ker \mathcal{L}$ for the kernel of a lattice as opposed to $\text{Ker } \alpha$ for the kernel of a homomorphism α .) A lattice \mathcal{L} is called *nondegenerate* if $\ker \mathcal{L} = 0$; it is called *unimodular* if $\varphi_{\mathcal{L}}$ is an isomorphism. For example, the *intersection lattice* $H_2(X)/\text{Tors}$ of an oriented closed 4-manifold X is unimodular (Poincaré duality).

We will fix the notation \mathcal{U} for the *hyperbolic plane*, which is the unimodular lattice generated by two elements $\mathbf{u}_1, \mathbf{u}_2$ with $\mathbf{u}_1^2 = \mathbf{u}_2^2 = 0, \mathbf{u}_1 \cdot \mathbf{u}_2 = 1$. We will also use the notation $\mathbf{A}_p, \mathbf{D}_q, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$ for the irreducible positive definite lattices generated by the root systems of the same name.

2.1.1. If \mathcal{L} is nondegenerate, the quotient $\text{discr } \mathcal{L} := \mathcal{L}^*/\mathcal{L}$ is a finite group; it is called the *discriminant group* of \mathcal{L} . Since $\varphi_{\mathcal{L}} \otimes \mathbb{Q}$ is an isomorphism, \mathcal{L}^* turns into a rational lattice and $\text{discr } \mathcal{L}$ inherits a (\mathbb{Q}/\mathbb{Z}) -valued symmetric bilinear form

$$(x \bmod \mathcal{L}) \otimes (y \bmod \mathcal{L}) \mapsto (x \cdot y) \bmod \mathbb{Z},$$

called the *discriminant form* of \mathcal{L} . In general, if \mathcal{L} is degenerate, we define $\text{discr } \mathcal{L}$ to be $\text{discr}(\mathcal{L}/\ker \mathcal{L})$. As a group, $\text{discr } \mathcal{L} = \text{Tors}(\mathcal{L}^*/\mathcal{L})$.

If \mathcal{L} is even, $\text{discr } \mathcal{L}$ inherits also a $(\mathbb{Q}/2\mathbb{Z})$ -valued quadratic extension of the discriminant form; it is given by $(x \bmod \mathcal{L}) \mapsto x^2 \bmod 2\mathbb{Z}$.

2.1.2. Let \mathcal{L} be a unimodular lattice, and let $\mathcal{S} \subset \mathcal{L}$ be a nondegenerate primitive sublattice. Denote $\mathcal{T} = \mathcal{S}^\perp$; it is also nondegenerate. According to Nikulin [7], the image of the restriction homomorphism $\mathcal{L}^* \rightarrow \mathcal{S}^* \oplus \mathcal{T}^* \rightarrow \text{discr } \mathcal{S} \oplus \text{discr } \mathcal{T}$ is the graph of a certain anti-isometry $q: \text{discr } \mathcal{S} \rightarrow \text{discr } \mathcal{T}$. (If \mathcal{L} is even, then so are \mathcal{S} and \mathcal{T} and q is also an anti-isometry of the quadratic extensions.) Furthermore, the pair (\mathcal{T}, q) , up to the action of $O(\mathcal{T})$ on $\text{discr } \mathcal{T}$, determines the isomorphism class of the extension $\mathcal{L} \supset \mathcal{S}$.

2.2. Elliptic surfaces. Here, we remind a few facts concerning elliptic surfaces. The references are [4] or the original paper [6].

A *Jacobian elliptic surface* is a compact complex surface X equipped with an elliptic fibration $\text{pr}: X \rightarrow B$ (*i.e.*, a fibration with all but finitely many fibers nonsingular elliptic curves) and a distinguished section $E \subset X$ of pr . (From the existence of a section it follows that X has no multiple fibers.) Throughout the paper we assume that surfaces are *relatively minimal*, *i.e.*, fibers of pr contain no (-1) -curves.

For the topological type of a singular fiber F , we use the notation $\tilde{\mathbf{A}}, \tilde{\mathbf{D}}, \tilde{\mathbf{E}}$ referring to the extended Dynkin graph representing the adjacencies of the components of F . The advantage of this approach is the fact that it reflects the type of the corresponding singular point of the ramification locus of X , *cf.* 3.4.1. For the relation to Kodaira's notation I–IV*, values of the j -invariant, and some other invariants, see Table 1 in [3].

2.2.1. Let $B^\circ \subset B$ be the set of regular values of pr , and define the (*functional*) j -invariant $j_X: B \rightarrow \mathbb{P}^1$ as the analytic continuation of the function $B^\circ \rightarrow \mathbb{C}^1$ sending each nonsingular fiber to its classical j -invariant (divided by 12^3). It is important to emphasize that the target \mathbb{P}^1 is the standard Riemann sphere $\mathbb{C} \cup \{\infty\}$.

The surface X is called *isotrivial* if $j_X = \text{const}$. In this paper, we mainly deal with non-isotrivial surfaces.

The monodromy $\mathfrak{h}_X: \pi_1(B^\circ) \rightarrow SL(2, \mathbb{Z})$ (in the 1-homology of the fiber) of the locally trivial fibration $\text{pr}^{-1} B^\circ \rightarrow B^\circ$ is called the *homological invariant* of X . Its reduction to $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm 1\}$ is determined by the j -invariant. Together, j_X and \mathfrak{h}_X determine X up to isomorphism; conversely, any pair (j, \mathfrak{h}) that agrees in the sense just described gives rise to a Jacobian elliptic surface.

In particular, the homological invariant determines the *type specification* of X , *i.e.*, a choice of type, $\tilde{\mathbf{A}}$ or $\tilde{\mathbf{D}}, \tilde{\mathbf{E}}$, of each singular fiber. If the base B is rational, then the type specification and j_X determine \mathfrak{h}_X .

2.2.2. Definition. A non-isotrivial Jacobian elliptic surface X is called *extremal* if it satisfies the following conditions:

- (1) j_X has no critical values other than 0, 1, and ∞ ;
- (2) each point in $j_X^{-1}(0)$ has ramification index at most 3, and each point in $j_X^{-1}(1)$ has ramification index at most 2;
- (3) X has no singular fibers of types $\tilde{\mathbf{D}}_4, \tilde{\mathbf{A}}_0^{**}, \tilde{\mathbf{A}}_1^*,$ or $\tilde{\mathbf{A}}_2^*$.

(In fact, this more topological definition is the contents of [8].)

2.2.3. Let X be a Jacobian elliptic surface. Denote by $\sigma_X \subset H_2(X)$ the set of classes realized by the components of the singular fibers of X . (We assume that X does have at least one singular fiber.) Let $\mathcal{S}_X \subset H_2(X)$ be the sublattice spanned by σ_X and $[E]$ (sometimes, \mathcal{S}_X is called the *simple lattice* of X), and let $\tilde{\mathcal{S}}_X := (\mathcal{S}_X \otimes \mathbb{Q}) \cap H_2(X)$ be its primitive hull. The quotient $\tilde{\mathcal{S}}_X/\mathcal{S}_X$ is equal to the torsion $\text{Tors } MW(X)$ of the Mordell–Weil group of X , see [10].

The orthogonal complement $\mathcal{T}_X := \mathcal{S}_X^\perp$ is called the (*stable*) *transcendental lattice* of X . Note that \mathcal{S}_X is nondegenerate; hence so is \mathcal{T}_X .

The collection $(H_2(X), \sigma_X, [E])$, considered up to auto-isometries of $H_2(X)$ preserving $[E]$ and σ_X as a set, is called the *homological type* of X . If \mathcal{S}_X is primitive, the homological type is determined by the combinatorial type of the singular fibers of X , the lattice \mathcal{T}_X , and the anti-isometry $q: \text{discr } \mathcal{S}_X \rightarrow \text{discr } \mathcal{T}_X$ defining the extension $H_2(X) \supset \mathcal{S}_X$, see 2.1.2.

2.3. The skeleton Γ_X . Let X be an extremal elliptic surface over a base B . Define its *skeleton* as the embedded bipartite graph $\Gamma_X := j_X^{-1}[0, 1] \subset B$. The pullbacks of 0 and 1 are called, respectively, \bullet - and \circ -vertices of Γ_X . (Thus, Γ_X is the *dessin d'enfants* of j_X in the sense of Grothendieck; however, we reserve the word ‘dessin’ for the more complicated graphs describing arbitrary, not necessarily extremal, surfaces, see [3].) Since X is extremal, Γ_X has the following properties:

- (1) each region of Γ_X (*i.e.*, component of $B \setminus \Gamma_X$) is a topological disk;
- (2) the valency of each \bullet -vertex is ≤ 3 , the valency of each \circ -vertex is ≤ 2 .

In particular, it follows that Γ_X is connected.

The skeleton Γ_X determines j_X ; hence the pair $(\Gamma_X, \mathfrak{h}_X)$ determines X . (Here, it is important that B is considered as a *topological* surface; its analytic structure is given by the Riemann existence theorem.)

2.3.1. From now on, we will speak about extremal surfaces *without $\tilde{\mathbf{E}}$ type singular fibers*. In this case, all \bullet -vertices of Γ_X are of valency 3 and all its \circ -vertices are of valency 2. Hence, the \circ -vertices can be disregarded (with the convention that a \circ -vertex is to be understood at the center of each edge connecting two \bullet -vertices). Furthermore, in view of condition (1) above, one can also disregard the underlying surface B and retain the ribbon graph structure of Γ_X only. For future references, we restate the definition:

- (*) Γ_X is a ribbon graph with all vertices of valency 3.

Under the assumptions, the surface B containing Γ is reconstructed from the ribbon graph structure. Its genus is called the *genus* of Γ .

In Subsection 4.2 below, we explain that the homological invariant \mathfrak{h}_X can be described in terms of an orientation of Γ_X , reducing an extremal elliptic surface to an oriented 3-regular ribbon graph.

3. THE TOPOLOGICAL ASPECTS

3.1. The notation. Consider a Jacobian elliptic surface $\text{pr}: X \rightarrow B$ over a base B of genus g . Let $E \subset X$ be the section of X , and denote by F_1, \dots, F_r its singular fibers. Let $S = \bigcup_i F_i$, $i = 1, \dots, r$.

Recall that *stable* are the singular fibers of X of type $\tilde{\mathbf{A}}_0^*$ or $\tilde{\mathbf{A}}_p$, $p \geq 1$. One has $H_1(F_i) = \mathbb{Z}$ if F_i is stable and $H_1(F_i) = 0$ otherwise.

For each $i = 1, \dots, r$, pick a regular neighborhood N_i of F_i of the form $\text{pr}^{-1}U_i$, where $U_i \subset B$ is a small disk about $\text{pr } F_i$. Let $N_S = \bigcup N_i$, $i = 1, \dots, r$. Let,

further, N_E be a tubular neighborhood of E . We assume N_E and all N_i so small that $N := N_E \cup N_S$ is a regular neighborhood of $E \cup S$. Thus, the spaces N , N_E , and N_i contain, respectively, $E \cup S$, E , and F_i as strict deformation retracts.

Denote by X° the closure of $X \setminus N$ and decompose the boundary ∂X° into the union $\partial_E X^\circ \cup \partial_S X^\circ$, $\partial_S X^\circ := \bigcup \partial_i X^\circ$, where $\partial_E X^\circ := \partial X^\circ \cap N_E$ and $\partial_i X^\circ := \partial X^\circ \cap N_i$, $i = 1, \dots, r$. Since $\partial X^\circ = \partial N$, we will use the same notation $\partial_\bullet N$ for the corresponding parts of the boundary of N , so that $\partial_\bullet N = \partial_\bullet X^\circ$.

We also use the notation \mathcal{S}_X , $\tilde{\mathcal{S}}_X$, and \mathcal{T}_X introduced in 2.2.3.

3.2. Tubular neighborhoods. First, recall that the inclusion $E \hookrightarrow X$ induces isomorphisms, see, e.g., [4],

$$(3.2.1) \quad H_1(E) \xrightarrow{\cong} H_1(X), \quad H^1(X) \xrightarrow{\cong} H^1(E).$$

The inverse isomorphisms are induced by the projection $\text{pr}: X \rightarrow B$ and the obvious identification $E = B$.

Consider a singular fiber F_i , $i = 1, \dots, r$. The boundary $\partial_i N = \partial N_i \setminus \text{interior } N$ is fibered over the circle ∂U_i , the fiber being a punctured torus F° . Denote by \mathfrak{m}_i and \mathfrak{m}_i^* the monodromy of this fibration in $H_1(F^\circ)$ and $H^1(F^\circ)$, respectively. One has

$$(3.2.2) \quad H_2(\partial_i N) = \text{Ker}[(\mathfrak{m}_i - \text{id}): H_1(F^\circ) \rightarrow H_1(F^\circ)],$$

$$(3.2.3) \quad H^2(\partial_i N) = \text{Coker}[(\mathfrak{m}_i^* - \text{id}): H^1(F^\circ) \rightarrow H^1(F^\circ)].$$

All monodromies \mathfrak{m}_i are known, see, e.g., [4] or Example 4.4.2 below. In particular, \mathfrak{m}_i has invariant vectors if and only if F_i is a stable singular fiber. Thus, $H_2(\partial_S N)$ is a free group and one has

$$(3.2.4) \quad \text{rk } H_2(\partial_S N) = \text{rk } H_1(S) = \text{number of stable singular fibers of } X.$$

3.2.5. Let Y be an oriented 4-manifold with boundary. Recall that, if $H_1(Y)$ is torsion free (or, equivalently, $H^2(Y)$ is torsion free), then $H_2(Y, \partial Y) = H^2(Y) = (H_2(Y))^*$ and the relativization homomorphism $\text{rel}: H_2(Y) \rightarrow H_2(Y, \partial Y)$ coincides with the homomorphism associated with the intersection index form, see Subsection 2.1. In particular, one has isomorphisms $\text{Tors Coker rel} = \text{Tors } H_1(\partial Y) = \text{discr } H_2(Y)$. (The resulting (\mathbb{Q}/\mathbb{Z}) -valued bilinear form on $\text{Tors } H_1(\partial Y)$ is called the *linking coefficient form*; it can be defined geometrically in terms of ∂Y only.)

Since $H_1(N) = H_1(S \cup E)$ is torsion free and $H_2(N) = \mathcal{S}_X / \ker$, one has

$$\text{discr } \mathcal{S}_X = \text{Tors } H_1(\partial N) = \text{Tors } H^2(\partial N).$$

3.2.6. Lemma. *The inclusion homomorphism $H^2(\partial N) \rightarrow H^2(\partial_S N)$ restricts to an isomorphism $\text{Tors } H^2(\partial N) = \text{Tors } H^2(\partial_S N)$.*

Proof. Denote $\partial' N_S = \partial N_S \cap N_E$ and consider the commutative diagram

$$\begin{array}{ccccc} H_2(N) & \xrightarrow{\text{rel}_1} & H_2(N, \partial N) & \xrightarrow{\partial_1} & H_1(\partial N) \\ \downarrow & & \downarrow & & \downarrow \\ H_2(N_S, \partial' N_S) & \xrightarrow{\text{rel}_2} & H_2(N_S, \partial N_S) & \xrightarrow{\partial_2} & H_1(\partial N_S, \partial' N_S), \end{array}$$

where the rows are fragments of exact sequences of pairs and vertical arrows are induced by appropriate inclusions, the rightmost arrow being Poincaré dual to the homomorphism in question. The cokernels $\text{Coker } \partial_i$, $i = 1, 2$, belong to the free groups $H_1(N)$ and $H_1(N_S, \partial' N_S)$, respectively; hence all torsion elements come from the cokernels Coker rel_i . It remains to observe that

$$\mathcal{S}_X = H_2(N)/\ker = \mathcal{U} \oplus (H_2(N_S)/\ker),$$

hence $\text{Tors Coker rel}_1 = \text{discr } H_2(N) = \text{discr } H_2(N_S) = \text{Coker rel}_2$. To establish the last equality, notice that, for each singular fiber F_i , there is a decomposition (not orthogonal) $H_2(N_i, \partial' N_i) = H_2(N_i) \oplus \mathbb{Z}[E_i, \partial E_i]$, where $E_i = E \cap N_i$; hence one can identify $H_2(N_i, \partial' N_i)$ with $(H_2(N_i)/\ker) \oplus \mathcal{U}$. \square

The advantage of Lemma 3.2.6 is the fact that the isomorphisms $\text{discr } H_2(N_i) = \text{Tors } H^2(\partial_i N)$ are local: they can be computed in terms of the topological types of the singular fibers of X .

3.3. The homology of X° . In this subsection, we compute the invariants \mathcal{T}_X and $\text{Tors } MW(X)$ of an arbitrary Jacobian elliptic surface X in terms of the (co-)homology of X° .

3.3.1. Lemma. *The group $H_2(X^\circ)$ is free and there is a short exact sequence*

$$0 \rightarrow \ker H_2(X^\circ) \rightarrow H_2(X^\circ) \rightarrow \mathcal{T}_X \rightarrow 0,$$

so that $\mathcal{T}_X = H_2(X^\circ)/\ker$. Furthermore, the homomorphism $H_2(\partial_S X^\circ) \rightarrow H_2(X^\circ)$ induced by the inclusion establishes an isomorphism $H_2(\partial_S X^\circ) = \ker H_2(X^\circ)$.

Proof. The first statement is an immediate consequence from the Poincaré duality $H_2(X^\circ) = H^2(X^\circ, \partial X^\circ)$ and the exact sequence

$$H^1(X) \rightarrow H^1(N) \xrightarrow{\partial} H^2(X^\circ, \partial X^\circ) \rightarrow H^2(X) \rightarrow H^2(N);$$

the kernel of the last homomorphism is $\mathcal{T}_X \subset H^2(X) = H_2(X)$, and the cokernel $H^1(N)/H^1(X) = H^1(S)$ is free, cf. (3.2.1). As another consequence, the rank $\text{rk Im } \partial$ equals the number of stable singular fibers of X .

The homomorphism ∂ above is Poincaré dual to ∂ in the following commutative diagram:

$$\begin{array}{ccc} H_3(X, X^\circ) & \xrightarrow{\partial} & H_2(X^\circ) \\ \parallel & & \uparrow \text{in}_* \\ H_3(N, \partial X^\circ) & \longrightarrow & H_2(\partial X^\circ). \end{array}$$

It follows that $\text{Im } \partial \subset \text{Im in}_* \subset \ker H_2(X^\circ)$. (Classes coming from the boundary are always in the kernel of the intersection index form.) Since \mathcal{T}_X is nondegenerate, both inclusions are equalities.

Finally, consider the exact sequence

$$H_2(\partial_S X^\circ) \rightarrow H_2(\partial X^\circ) \rightarrow H_2(\partial X^\circ, \partial_S X^\circ) \xrightarrow{\partial} H_1(\partial_S X^\circ).$$

One has $H_*(\partial X^\circ, \partial_S X^\circ) = H_*(E', \partial E') \otimes H_*(S^1)$, where $E' = E \setminus N_S$, and it is easy to see that $\text{Ker } \partial = H_1(E', \partial E') \otimes H_1(S^1)$ and that each element of this kernel lifts to a class in $H_2(\partial X^\circ)$ that vanishes in $H_2(X^\circ)$. (If α is a relative 1-cycle in $(E', \partial E')$, the lift is the boundary of $\text{pr}^{-1} \text{pr } \alpha \setminus N_E$.) Thus, the image of $H_2(\partial X^\circ)$ in $H_2(X^\circ)$ coincides with that of $H_2(\partial_S X^\circ)$. Since the ranks of $H_2(\partial_S X^\circ)$ and its image coincide (both equal to the number of stable singular fibers of X), the inclusion induces an isomorphism. \square

3.3.2. Lemma. *There is an exact sequence*

$$0 \rightarrow \mathcal{S}_X \rightarrow H_2(X) \rightarrow H^2(X^\circ) \rightarrow H_1(S) \rightarrow 0.$$

In particular, $\text{Tors } H^2(X^\circ) = \tilde{\mathcal{S}}_X / \mathcal{S}_X = \text{Tors } MW(X)$.

Proof. The statement follows from the Poincaré duality $H^2(X^\circ) = H_2(X^\circ, \partial X^\circ)$, the exact sequence

$$H_2(N) \rightarrow H_2(X) \rightarrow H_2(X^\circ, \partial X^\circ) \rightarrow H_1(N) \rightarrow H_1(X),$$

and the fact that $\text{Ker}[H_1(N) \rightarrow H_1(X)] = H_1(S)$, cf. (3.2.1). \square

Assume that \mathcal{S}_X is primitive in $H_2(X)$, i.e., $\tilde{\mathcal{S}}_X = \mathcal{S}_X$. Then, due to 3.2.5 and Lemma 3.3.2, there is an isomorphism $\text{discr } \mathcal{T}_X = \text{Tors } H^2(\partial X^\circ)$, which gives rise to an isomorphism $\text{discr } \mathcal{T}_X = \text{Tors } H^2(\partial_S X^\circ)$, see Lemma 3.2.6.

3.3.3. Lemma. *If \mathcal{S}_X is primitive in $H_2(X)$, then the anti-isometry $q: \text{discr } \mathcal{T}_X \rightarrow \text{discr } \mathcal{S}_X$ defining the homological type of X , see 2.2.3, can be identified with the composition $j^{-1} \circ i$ of the isomorphisms*

$$\text{discr } \mathcal{T}_X \xrightarrow{i} \text{Tors } H^2(\partial_S X^\circ) \xleftarrow{j} \text{discr } \mathcal{S}_X$$

induced by the inclusions $\partial_S X^\circ \hookrightarrow X^\circ$ and $\partial_S X^\circ \hookrightarrow N_S$.

Proof. Using Lemma 3.2.6, one can replace $\partial_S X^\circ$ with ∂X° . Then the statement follows from the Mayer–Vietoris exact sequence

$$H^2(X) \rightarrow H^2(N) \oplus H^2(X^\circ) \rightarrow H^2(\partial X^\circ)$$

and the definition of q . \square

3.4. The counts. We conclude this section with a few counts.

3.4.1. Let X be an extremal elliptic surface over a curve B of genus g , and let $\Gamma = \Gamma_X \subset B$ be the skeleton of X . Assume that all singular fibers of X are of type $\tilde{\mathbf{A}}_0^*$, $\tilde{\mathbf{A}}_p$, $p \geq 1$, or $\tilde{\mathbf{D}}_q$, $q \geq 5$, and denote by t the number of $\tilde{\mathbf{D}}$ type fibers. Let $\chi(X) = 6(k+t)$. (Recall that $12 \mid \chi(X)$.) Then the quotient $X/\pm 1$ blows down to a ruled surface Σ over B with an exceptional section E with $E^2 = -(k+t)$. The ramification locus of the projection $X \rightarrow \Sigma$ is the union $C \cup E$, where C is a certain *trigonal curve* (i.e., a curve disjoint from E and intersecting each generic fiber of the ruling at three points) with simple singularities only.

The surface X is diffeomorphic to the double covering $X' \rightarrow \Sigma$ ramified at E and a nonsingular trigonal curve C' . Using this fact and taking into account (3.2.1), one can easily compute the inertia indices σ_\pm of the intersection index form on $H_2(X)$:

$$\sigma_+(X) = k + t + 2g - 1, \quad \sigma_-(X) = 5k + 5t + 2g - 1.$$

3.4.2. Let $\Gamma = \Gamma_X \subset B$ be the skeleton of X . The numbers of vertices, edges, and regions of Γ are, respectively,

$$v = 2k, \quad e = 3k, \quad r = k + 2 - 2g.$$

The latter count r is also the number of singular fibers of X . The ‘total Milnor number’ of the singular fibers of X is given by $\mu = 2g + 5k + 5t - 2$. (Indeed, each n -gonal region R contributes $(n-1)$ or $(n+4)$ depending on whether R contains an $\tilde{\mathbf{A}}$ or $\tilde{\mathbf{D}}$ type fiber. The total number of corners of the regions is $6k$.) Taking into account Lemma 3.3.1 and (3.2.4), one arrives at the following statement.

3.4.3. Lemma. *In the notation above, \mathcal{T}_X is a positive definite lattice of rank $k + t + 2g - 2$. Furthermore, one has $\text{rk ker } H_2(X^\circ) = k - t + 2 - 2g$, and $H_2(X^\circ)$ is a positive semi-definite lattice of rank $2k$. \square*

3.4.4. Remark. The assertion that the lattice \mathcal{T}_X is positive definite still holds if X has type $\tilde{\mathbf{E}}$ singular fibers. In fact, this property can be taken for the definition of an extremal elliptic surface.

4. THE MAIN THEOREM

4.1. Skeletons. To ease the further exposition, we redefine a skeleton in the sense of 2.3.1(*) as a set of ends of its edges. However, we will make no distinction between a skeleton in the sense of Definition 4.1.1 below and its geometric realization.

4.1.1. Definition. A *skeleton* is a collection $\Gamma = (\mathcal{E}, \text{op}, \text{nx})$, where \mathcal{E} is a finite set, $\text{op}: \mathcal{E} \rightarrow \mathcal{E}$ is a free involution, and $\text{nx}: \mathcal{E} \rightarrow \mathcal{E}$ is a free automorphism of order 3. The elements of \mathcal{E} are called *ends*, the orbits of op are called the *edges* of Γ , and the orbits of nx are called its *vertices*. Informally, op assigns to an end the other end of the same edge, and nx assigns the next end at the same vertex with respect to its cyclic order (which is a part of the conventional ribbon graph structure; recall that it is the counterclockwise cyclic order with respect to the complex orientation of the original surface B).

4.1.2. According to this definition, the sets of edges and vertices of a skeleton Γ can be referred to as \mathcal{E}/op and \mathcal{E}/nx , respectively. An *orientation* of Γ is a section $+: \mathcal{E}/\text{op} \rightarrow \mathcal{E}$ of op , sending each edge e to its *head* e^+ . Given such a section, its composition with op sends each edge e to its *tail* e^- . It is worth mentioning that, from this point of view, a *marking* of Γ in the sense of [3] is merely a section $\bar{1}: \mathcal{E}/\text{nx} \rightarrow \mathcal{E}$ of nx , sending each vertex to the first edge end attached to it. Then the sections $\bar{2} := \text{nx} \circ \bar{1}$ and $\bar{3} := \text{nx}^2 \circ \bar{1}$ send a vertex to the second and third edge ends, respectively.

The elements op and nx of order 2 and 3, respectively, generate the modular group $PSL(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$, which acts on \mathcal{E} . A skeleton is *connected* if this action is transitive. Recall that each element $w \in PSL(2, \mathbb{Z})$ can be uniquely represented by a reduced word $w_1 w_2 w_3 \dots$ of the form $\text{op nx}^{\pm 1} \text{op} \dots$ or $\text{nx}^{\pm 1} \text{op nx}^{\pm 1} \dots$. The length of this word is called the *length* of w .

4.1.3. Remark. It is worth mentioning that Definition 4.1.1 results in a computer friendly presentation of Γ : it is given by two permutations op and nx , the former splitting into a product of cycles of length 2, the latter, into a product of cycles of length 3. Certainly, this description is equivalent to the presentation of the ramified covering $B \rightarrow \mathbb{P}^1$ defined by Γ by its Hurwitz system.

4.1.4. Definition. A *path* in a skeleton $\Gamma = (\mathcal{E}, \text{op}, \text{nx})$ can be defined as a pair $\gamma = (\alpha, w)$, where $\alpha \in \mathcal{E}$ and $w \in PSL(2, \mathbb{Z})$. If w is a positive power of $\text{nx}^{-1} \text{op}$, then γ is called a *left turn path* (cf. Figure 6, left, in Subsection 5.3 below). The *endpoint* of γ is the element $w(\alpha) \in \mathcal{E}$. If the length of w is even and $w(\alpha) = \alpha$, the path is called a *loop*.

4.1.5. Representing w by a reduced word $w_r \dots w_1$, one can identify a path (α, w) with a sequence $(\alpha_0, \dots, \alpha_r)$, where $\alpha_0 = \alpha$ and $\alpha_i = w_i(\alpha_{i-1})$ for $i \geq 1$.

4.1.6. A *region* of a skeleton Γ can be defined as an orbit of the cyclic subgroup of $PSL(2, \mathbb{Z})$ generated by $nx^{-1} \text{op}$. Given an n -gonal region R , $n \geq 1$, and an element $\alpha_0 \in R$, the *boundary* ∂R is the left turn path of length $2n$ starting at α_0 . It is a loop. In the sequence $(\alpha_0, \alpha_1, \dots, \alpha_{2n} = \alpha_0)$ representing ∂R , each even term α_{2i} is an element of R , and each odd term has the form $\alpha_{2i+1} = \text{op} \alpha_{2i}$.

Patching the boundary of each region of Γ with a disk, one obtains the surface B containing Γ . Hence, the genus $g(\Gamma)$ of Γ , see 2.3.1, is given by

$$2 - 2g(\Gamma) = \#(\mathcal{E}/nx) - \#(\mathcal{E}/\text{op}) + \#(\mathcal{E}/nx^{-1} \text{op}).$$

4.2. The homological invariant. Let $\mathcal{H} = \mathbb{Z}a \oplus \mathbb{Z}b$ with the skew-symmetric bilinear form $\wedge^2 \mathcal{H} \rightarrow \mathbb{Z}$ given by $a \cdot b = 1$. Introduce the isometries $\mathbb{X}, \mathbb{Y}: \mathcal{H} \rightarrow \mathcal{H}$ given (in the standard basis $\{a, b\}$) by the matrices

$$\mathbb{X} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbb{Y} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

One has $\mathbb{X}^3 = \text{id}$ and $\mathbb{Y}^2 = -\text{id}$. If $c = -a - b \in \mathcal{H}$, then \mathbb{X} acts *via*

$$(a, b) \xrightarrow{\mathbb{X}} (c, a) \xrightarrow{\mathbb{X}} (b, c) \xrightarrow{\mathbb{X}} (a, b).$$

It is well known that \mathbb{X} and \mathbb{Y} generate the group $SL(2, \mathbb{Z})$ of isometries of \mathcal{H} . We fix the notation \mathcal{H}, a, b, c and \mathbb{X}, \mathbb{Y} throughout the paper.

Let $\text{pr}: X \rightarrow B$ be an elliptic surface with singular fibers of type $\tilde{\mathbf{A}}_0^*, \tilde{\mathbf{A}}_p, p \geq 1$, or $\tilde{\mathbf{D}}_q, q \geq 5$, only. We use the results of [3] to describe the homological invariant of X in terms of the skeleton $\Gamma = \Gamma_X$. More precisely, we describe the monodromy in $H_1(\text{fiber})$ of the locally trivial fibration $\text{pr}: \text{pr}^{-1}\Gamma \rightarrow \Gamma$.

Consider the double covering $X \rightarrow \Sigma$ ramified at $C \cup E$, see 3.4.1. Pick a vertex v of Γ , let F_v be the fiber of X over v , and let \bar{F}_v be its projection to Σ . Then, F_v is the double covering of \bar{F}_v ramified at $\bar{F}_v \cap (C \cup E)$ (the three black points in Figure 1 and ∞).

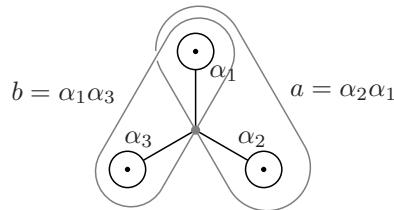


FIGURE 1. The basis in $H_1(F_v)$

In the presence of a trigonal curve, Σ has a well defined zero section (the fiberwise barycenter of the points of the curve with respect to the canonical \mathbb{C}^1 -affine structure in the open fibers $\bar{F} \setminus E$). Let $\bar{z}_v \in \bar{F}_v$ be the value of the zero section at a vertex v of Γ . For each vertex v , pick and fix one of the two pull-backs of \bar{z}_v in F_v ; denote it by z_v . The collection $\{z_v\}, v \in \mathcal{E}/nx$, is called a *reference set*.

Recall that a *marking* at a vertex v of Γ is defined in [3] as a counterclockwise ordering of the three edge ends at v ; in other words, a marking is a choice of one of

these ends for $\bar{1}(v)$, see 4.1.2. The three points of the intersection $\bar{F}_v \cap C$ form an equilateral triangle (with respect to the canonical affine structure in $\bar{F}_v \setminus E \cong \mathbb{C}^1$); this triangle turns into an isosceles triangle with the angle at the vertex $> \pi/3$ when the fiber slides along one of the edges. (Both are well known properties of the classical j -invariant.) This fact establishes a canonical one-to-one correspondence between the three edges at v and the three points of $\bar{F}_v \cap C$, an edge corresponding to the point that becomes the vertex over this edge. Thus, a marking at v gives rise to an ordering p_1, p_2, p_3 of the three points of $\bar{F}_v \cap C$; according to [3], this ordering is *clockwise*.

A marking at v defines a *canonical basis* $\{\alpha_1, \alpha_2, \alpha_3\}$ for the fundamental group $\pi_1(\bar{F}_v \setminus (C \cup E), \bar{z}_v)$, see [3] and Figure 1: one takes for $\alpha_i, i = 1, 2, 3$, the class of the loop formed by a small counterclockwise circle about p_i connected to \bar{z}_v (the barycenter of the triangle) by a straight segment. Note that, unlike [3], we take for the reference point the barycenter \bar{z}_v rather than a ‘point at infinity’; this choice simplifies the definition and removes the ambiguity found in [3].

Choose a marking at v and let $\{\alpha_1, \alpha_2, \alpha_3\}$ be the canonical basis, see above. Then the group $H_1(F_v) = \pi_1(F_v, z_v)$ is generated by the lifts $a_v = \alpha_2\alpha_1$ and $b_v = \alpha_1\alpha_3$ (the two grey cycles in Figure 1), and one can use the map $a_v \mapsto a, b_v \mapsto b$ to identify $H_1(F_v)$ with \mathcal{H} . (Note that one has $a_v \cdot b_v = 1$.)

In the sequel, we consider a separate copy F_α of F_v for each edge end $\alpha \in v$.

4.2.1. Definition. The *canonical identification* is the isomorphism $H_1(F_\alpha) \rightarrow \mathcal{H}$ constructed above using the marking at v defined *via* $\alpha = \bar{1}(v)$.

4.2.2. Lemma. *Under the canonical identification, the identity map $F_\alpha \rightarrow F_{\text{nx}\alpha}$, regarded as an automorphism of \mathcal{H} , is given by \mathbb{X}^{-1} .*

Proof. This map is the change of basis from $\{a, b\}$ to $\{c, a\}$. \square

4.2.3. Lemma. *Let u and v be two vertices (not necessarily distinct) connected by an edge e , and let $\alpha \in u$ and $\beta \in v$ be the respective ends of e . Under the canonical identifications over u and v , the monodromy $H_1(F_\alpha) \rightarrow H_1(F_\beta)$ along e , regarded as an automorphism of \mathcal{H} , is given by $\pm\mathbb{Y}$.*

Proof. This monodromy is a lift of monodromy $m_{1,1}$ in [3]; geometrically (in Σ), the black ramification point surrounded by α_1 crosses the segment connecting the ramification points surrounded by α_2 and α_3 . \square

The sign ± 1 in Lemma 4.2.3 depends on the homological invariant \mathfrak{h}_X and on the choice of a reference set. The monodromy from v to u is $(\pm\mathbb{Y})^{-1} = \mp\mathbb{Y}$.

4.2.4. Definition. Given an elliptic surface X as above and a reference set $\{z_v\}$, $v \in \mathcal{E}/\text{nx}$, we define an orientation of Γ as follows: an edge e is oriented so that the monodromy $H_1(F_{e-}) \rightarrow H_1(F_{e+})$ along e be given by $+\mathbb{Y}$.

Changing the lift z_v over a vertex v to the other one results in a change of sign of the canonical identification $H_1(F_\alpha) \rightarrow \mathcal{H}$ for each end $\alpha \in v$. As a consequence, each monodromy starting or ending at v changes sign. Thus, two orientations of Γ give rise to the same monodromy over Γ if and only if they are obtained from each other by the following operation: pick a subset V of the set of vertices of Γ and reverse the orientation of each edge that has exactly one end in V . Summarizing, one arrives at the following statement.

4.2.5. Lemma. *An extremal elliptic surface X without $\tilde{\mathbf{E}}$ type singular fibers is determined up to isomorphism by an oriented ribbon graph Γ_X as in 2.3.1(*). Conversely, oriented ribbon graph Γ_X is determined by X up to isomorphism and a change of orientation just described.*

Proof. If X is extremal and without $\tilde{\mathbf{E}}$ type fibers, then Γ_X is a strict deformation retract of B° and the monodromy over Γ_X determines \mathfrak{h}_X . \square

4.3. The tripod calculus. Let $\Gamma = (\mathcal{E}, \text{op}, \text{nx}, +)$ be a connected oriented skeleton. Place a copy \mathcal{H}_α of \mathcal{H} at each element $\alpha \in \mathcal{E}$, and let $\mathcal{H} \otimes \Gamma = \bigoplus \mathcal{H}_\alpha$, $\alpha \in \mathcal{E}$. For a vector $h \in \mathcal{H} \otimes \Gamma$, we denote by h_α its projection to \mathcal{H}_α , $\alpha \in \mathcal{E}$; for a vector $u \in \mathcal{H}$ and element $\alpha \in \mathcal{E}$, denote by $u \otimes \alpha \in \mathcal{H} \otimes \Gamma$ the vector whose only nontrivial projection is $(u \otimes \alpha)_\alpha = u$. Convert $\mathcal{H} \otimes \Gamma$ to a rational lattice by letting

$$(4.3.1) \quad h^2 = -\frac{1}{3} \sum_{\alpha \in \mathcal{E}} h_\alpha \cdot \mathbb{X}h_{\text{nx}\alpha}, \quad h \in \mathcal{H} \otimes \Gamma,$$

where \cdot stands for the product in \mathcal{H} . Let \mathcal{H}_Γ be the sublattice of $\mathcal{H} \otimes \Gamma$ subject to the following relations:

- (1) $h_\alpha + \mathbb{X}h_{\text{nx}\alpha} + \mathbb{X}^2h_{\text{nx}^2\alpha} = 0$ for each element $\alpha \in \mathcal{E}$;
- (2) $h_{e^+} + \mathbb{Y}h_{e^-} = 0$ for each edge $e \in \mathcal{E}/\text{op}$.

Similarly, consider the dual group $\mathcal{H}^* \otimes \Gamma = \bigoplus \mathcal{H}_\alpha^*$, $\alpha \in \mathcal{E}$, where \mathcal{H}_α^* is a copy of the dual group \mathcal{H}^* , and define \mathcal{H}_Γ^* as the quotient of $\mathcal{H}^* \otimes \Gamma$ by the subgroup spanned by the vectors of the form

- (3) $u \otimes \alpha + \mathbb{X}^*u \otimes (\text{nx}\alpha) + (\mathbb{X}^*)^2u \otimes (\text{nx}^2\alpha)$ for each $u \in \mathcal{H}^*$ and $\alpha \in \mathcal{E}$;
- (4) $u \otimes e^+ + \mathbb{Y}^*u \otimes e^-$ for each $u \in \mathcal{H}^*$ and $e \in \mathcal{E}/\text{op}$.

(Here, $\mathbb{X}^*, \mathbb{Y}^*: \mathcal{H}^* \rightarrow \mathcal{H}^*$ are the adjoint of \mathbb{X}, \mathbb{Y} .) It is easy to see that \mathcal{H}_Γ annihilates the subgroup spanned by (3), (4), inducing a pairing $\mathcal{H}_\Gamma \otimes \mathcal{H}_\Gamma^* \rightarrow \mathbb{Z}$. (Observe that the maps $h \mapsto h_\alpha \in \mathcal{H}$ and $u \mapsto u \otimes \alpha \in \mathcal{H}^* \otimes \Gamma$ are adjoint to each other.) Note that, in general, $\mathcal{H}_\Gamma^* \neq (\mathcal{H}_\Gamma)^*$, as \mathcal{H}_Γ^* may have torsion.

4.3.2. Remark. Since $\mathbb{X}^3 = \text{id}$, in relation (1) above it suffices to pick a marking $\bar{1}: \mathcal{E}/\text{nx} \rightarrow \mathcal{E}$, see 4.1.2, and consider one relation

- (5) $h_{\bar{1}(v)} + \mathbb{X}h_{\bar{2}(v)} + \mathbb{X}^2h_{\bar{3}(v)} = 0$ for each vertex $v \in \mathcal{E}/\text{nx}$.

Furthermore, since \mathbb{X} is an isometry, the restriction to \mathcal{H}_Γ of the quadratic form given by (4.3.1) can be simplified to

$$(4.3.3) \quad h^2 = - \sum_{v \in \mathcal{E}/\text{nx}} h_{\bar{1}(v)} \cdot \mathbb{X}h_{\bar{2}(v)}, \quad h \in \mathcal{H} \otimes \Gamma.$$

This expression (when restricted to \mathcal{H}_Γ) does not depend on the marking.

Now, let $X = X_\Gamma$ be the extremal elliptic surface defined by Γ , see Lemma 4.2.5. Next theorem computes the (co-)homology of X_Γ° , see 3.1, in terms of Γ .

4.3.4. Theorem. *There are isomorphisms $H_2(X_\Gamma^\circ) = \mathcal{H}_\Gamma$ and $H^2(X_\Gamma^\circ) = \mathcal{H}_\Gamma^*$. The former takes the intersection index form to the form given by (4.3.1); the latter takes the Kronecker product to the pairing $\mathcal{H}_\Gamma \otimes \mathcal{H}_\Gamma^* \rightarrow \mathbb{Z}$ defined above.*

Proof. Replace X° with its strict deformation retract $X' := \text{pr}^{-1}\Gamma \setminus N_E$; it fibers over Γ , the fiber being punctured torus. Subdivide Γ into cells by taking its \bullet - and

◦-vertices for 0-cells and half edges (*i.e.*, edges of the form $\bullet \text{---} \circ$) for 1-cells, and let X'_0 be the pull-back of the 0-skeleton of Γ . Then, in the exact sequence

$$H_2(X'_0) \rightarrow H_2(X') \rightarrow H_2(X', X'_0) \xrightarrow{\partial} H_1(X'_0)$$

of pair (X', X'_0) one has $H_2(X'_0) = 0$; hence $H_2(X^\circ) = H_2(X') = \text{Ker } \partial$.

Pick a marking of Γ , see 4.1.2, and a reference set $\{z_\alpha\}$, $\alpha \in \mathcal{E}/\text{nx}$, with respect to which \mathfrak{h}_X defines the given orientation of Γ , see Definition 4.2.4. Note that, for each fiber F , the inclusion $F^\circ \hookrightarrow F$ induces an isomorphism $H_1(F^\circ) = H_1(F)$.

The half edges of Γ are in a one-to-one correspondence with the elements of \mathcal{E} , and, under the canonical identifications, see Definition 4.2.1, the group $H_2(X', X'_0)$ splits into direct sum $\bigoplus_{\alpha \in \mathcal{E}} H_1(F_\alpha) \otimes H_1(I_\alpha, \partial I_\alpha) = \bigoplus_{\alpha \in \mathcal{E}} \mathcal{H} \otimes \mathbb{Z}$, where I_α is the half edge containing α . To establish an isomorphism $H_1(I_\alpha, \partial I_\alpha) = \mathbb{Z}$, we use the fundamental class $[I_\alpha, \partial I_\alpha]$ corresponding to the orientation of I_α towards its \bullet -vertex. In other words, for each $\alpha \in \mathcal{E}$, we consider a direct summand

$$(4.3.5) \quad \mathcal{H}_\alpha := H_1(F_\alpha) \otimes \mathbb{Z}[\bullet \leftarrow \circ], \quad H_1(F_\alpha) = \mathcal{H}.$$

Thus, there is a canonical isomorphism $H_2(X', X'_0) = \mathcal{H} \otimes \Gamma$.

For each \bullet -vertex v , identify $H_1(F_v)$ with \mathcal{H} using the chosen marking, so that $H_1(F_v) = H_1(F_{\bar{1}(v)})$. Then the composition $H_2(X', X'_0) \rightarrow H_1(X'_0) \rightarrow H_1(F_v) = \mathcal{H}$ of the boundary operator ∂ and the projection to $H_1(F_v)$ is given by the left hand side of 4.3.2(5) at v , see Lemma 4.2.2.

Finally, a \circ -vertex w of Γ is represented by the edge e containing this vertex, and we identify $H_1(F_w)$ with $H_1(F_{e^+})$ (and further with \mathcal{H}). Then the composition $H_2(X', X'_0) \rightarrow H_1(X'_0) \rightarrow H_1(F_w) = \mathcal{H}$ is given, up to sign (-1) , by the left hand side of 4.3(2) at e , see Lemma 4.2.3.

Thus, after appropriate identifications, ∂ is a map

$$(4.3.6) \quad \partial: \mathcal{H} \otimes \Gamma \rightarrow \bigoplus_{v \in \mathcal{E}/\text{nx}} \mathcal{H} \oplus \bigoplus_{e \in \mathcal{E}/\text{op}} \mathcal{H},$$

and its components are given by the left hand sides of the respective constraints 4.3.2(5) and 4.3(2) defining \mathcal{H}_Γ . Hence one has $H_2(X^\circ) = \text{Ker } \partial = \mathcal{H}_\Gamma$.

The proof for the cohomology is literally the same, and the interpretation of the Kronecker product is straightforward.



FIGURE 2. Shift of a marked skeleton

To compute the self-intersection in X° of a 2-cycle in X' , we mark Γ , shift it in B° as shown in Figure 2, left, and shift the cycle accordingly. Next to each \bullet -vertex v of Γ , an intersection point forms; it contributes one term to (4.3.3). (One needs to apply \mathbb{X} to $h_{\bar{2}(v)}$ in order to bring $\mathcal{H}_{\bar{1}(v)}$ and $\mathcal{H}_{\bar{2}(v)}$ to the same basis, see Lemma 4.2.2.) The shifts do not need to agree, as a possible intersection point at the middle of an edge of Γ , see Figure 2, right, would not contribute to the *self*-intersection of a cycle (since self-intersections in $H_1(\text{fiber}) \cong \mathcal{H}$ are trivial). \square

4.3.7. Corollary. *All equations 4.3(2) and 4.3.2(5) are linearly independent.*

Proof. This statement follows from Theorem 4.3.4 and a simple dimension count using Lemma 3.4.3. \square

4.3.8. Corollary. *There is an isomorphism $\mathcal{T}_X = \mathcal{H}_\Gamma / \ker$.*

Proof. The statement follows from Theorem 4.3.4 and Lemma 3.3.1. \square

4.3.9. Corollary. *There is an isomorphism $MW(X_\Gamma) = \text{Tors } \mathcal{H}_\Gamma^*$.*

Proof. The statement follows from Theorem 4.3.4, Lemma 3.3.1, and the fact that the Mordell–Weil group of an extremal surface has rank 0. \square

4.3.10. Remark. Alternatively, one can compute $MW(X_\Gamma)$ in terms of $\mathcal{H} \otimes \Gamma$ only, *via* $MW(X_\Gamma) = \text{Ext}(\text{Coker } \partial, \mathbb{Z})$, where ∂ is the map given by (4.3.6).

4.4. The monodromy. Definition 4.4.1 below is a combinatorial counterpart of the computation of the homological invariant \mathfrak{h}_X given by Lemmas 4.2.2 and 4.2.3. Unlike 4.2, here we are dealing with the groups \mathcal{H}_α of 2-chains, see (4.3.5), rather than the groups $H_1(F_\alpha)$ of 1-cycles, and we are interested in propagating a 2-chain along a path in Γ . When following a path, at each step the orientation in the base is reversed (compared to the convention $\bullet \leftarrow \circ$ set in (4.3.5)); it is this fact that explains the extra sign -1 in Definition 4.4.1. In other words, the sign is chosen so that the parallel transport $\|\gamma, h_0\|$ defined below be a cycle except over the endpoints of γ . Note that, since loops have even length, the monodromy along a loop would formally coincide with that given by Lemmas 4.2.2 and 4.2.3.

4.4.1. Definition. Let $\gamma = (\alpha, w)$ be a path in Γ . Represent w by a reduced word $w_r \dots w_1$, let $(\alpha_0, \dots, \alpha_r)$ be the sequence of edge ends that belong to γ , and lift w_i and w to $\mathfrak{m}_i, \mathfrak{m} = \mathfrak{m}_r \dots \mathfrak{m}_1 \in SL(2, \mathbb{Z})$ as follows:

- (1) if $w_i = \text{nx}^{\pm 1}$, let $\mathfrak{m}_i = -\mathbb{X}^{\mp 1}$,
- (2) if $w_i = \text{op}$, hence $[\alpha_{i-1}, \alpha_i]$ is an edge e and $\alpha_i = e^\pm$, let $\mathfrak{m}_i = -\mathbb{Y}^{\pm 1}$.

The map $\mathfrak{m} = \mathfrak{m}_\gamma: \mathcal{H}_{\alpha_0} \rightarrow \mathcal{H}_{\alpha_r}$ is called the *monodromy* along γ . Given a vector $h_0 \in \mathcal{H}$, we define the *parallel transport* $\|\gamma, h_0\| \in \mathcal{H} \otimes \Gamma$ to be $\sum_{i=0}^r h_i \otimes \alpha_i$, where $h_i = \mathfrak{m}_i(h_{i-1})$, $i = 1, \dots, r$.

4.4.2. Example. The monodromy along the boundary of an n -gonal region R of Γ , see 4.1.6, is

$$\pm(\mathbb{X}\mathbb{Y})^n = \pm \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

Thus, the orientation of Γ determines its type specification in a simple way: the fiber inside R is of type $\tilde{\mathbf{A}}$ or $\tilde{\mathbf{D}}$ if the sign above is $+$ or $-$, respectively.

4.4.3. Definition. Let $\gamma = (\alpha, w)$ be a loop, and assume that the monodromy \mathfrak{m}_γ has an invariant vector $h \in \mathcal{H}_\alpha$. Then one can define the *fundamental cycle*

$$[\gamma, h] := \|\gamma, h\| - h \otimes \alpha = \sum_{i=0}^{r-1} h_i \otimes \alpha_i = \sum_{i=1}^r h_i \otimes \alpha_i$$

(where $h_i = \mathfrak{m}_i(h_{i-1})$, see Definition 4.4.1); it is an element of $\overline{\mathcal{H}}_\Gamma$.

4.4.4. Example. If R is an n -gonal region of Γ , see 4.1.6, containing an $\tilde{\mathbf{A}}$ type singular fiber, then a is invariant under the monodromy $\mathfrak{m}_{\partial R}$, see Example 4.4.2; hence $[\partial R, a]$ is a well defined element of $\mathcal{H}_\Gamma = H_2(X_\Gamma^\circ)$. (Up to sign, this element does not depend on the choice of the initial point of ∂R .) Shifting the cycle realizing this element inside R , one can see that $[\partial R, a] \in \ker H_2(X_\Gamma^\circ)$.

4.4.5. Proposition. Let R_1, \dots, R_{f-t} be the regions of Γ containing the stable singular fibers of X_Γ . Then the elements $[\partial R_i, a]$, $i = 1, \dots, f-t$, see Example 4.4.4, form a basis for the kernel $\ker \mathcal{H}_\Gamma$. (Here, t is the number of \mathbf{D} type fibers of X_Γ , see 3.4.1.)

Proof. Due to (3.2.2) and Example 4.4.2, the elements in question form a basis for $H_2(\partial_S X_\Gamma^\circ)$, and the statement follows from Lemma 3.3.1 and Theorem 4.3.4. \square

4.4.6. Let R be an n -gonal region of Γ . Represent the boundary path ∂R by a sequence $(\alpha_0, \alpha_1, \dots, \alpha_{n-1})$, see 4.1.5, omitting $\alpha_n = \alpha_0$. Let $\mathcal{H} \otimes \partial R = \bigoplus \mathcal{H}_i^*$ be the direct sum of n copies of \mathcal{H}^* , one copy for each vertex α_i , and define the restriction homomorphism $\text{res}: \mathcal{H}^* \otimes \Gamma \rightarrow \mathcal{H}^* \otimes \partial R$ via $u \otimes \alpha \mapsto \sum u \otimes \alpha_i$, the summation running over all vertices α_i that are equal to α . (Note that the chain representing ∂R may have repetitions.)

Let $\mathfrak{m}_i^*: \mathcal{H}_i^* \rightarrow \mathcal{H}_{i-1}^*$ be the map adjoint to \mathfrak{m}_i , see Definition 4.4.1. For \mathfrak{m}_n , we identify \mathcal{H}_n^* with \mathcal{H}_0^* . The following statement is straightforward, cf. the proof of Theorem 4.3.4; if \mathcal{S}_X is primitive in $H_2(X_\Gamma)$, it describes the lattice extension $H_2(X_\Gamma) \supset \mathcal{S}_X$, cf. Lemma 3.3.3.

4.4.7. Proposition. Let R be an n -gonal region of Γ containing a singular fiber F_j of X_Γ . Then there is an isomorphism $H^2(\partial_j X_\Gamma^\circ) = \mathcal{H}^* \otimes \partial R / \langle u = \mathfrak{m}_i u \rangle$, $u \in \mathcal{H}_i^*$, $i = 1, \dots, n$, and the inclusion homomorphism $H^2(X_\Gamma^\circ) \rightarrow H^2(\partial_j X_\Gamma^\circ)$ is induced by the restriction res defined above. \square

5. EXAMPLE: PSEUDO-TREES

5.1. Admissible trees and pseudo-trees. An embedded tree $\Xi \subset S^2$ is called *admissible* if all its vertices have valency 3 (*nodes*) or 1 (*leaves*). Each admissible tree Ξ gives rise to a skeleton Γ_Ξ : one attaches a small loop to each leaf of Ξ , see Figure 3, left. A skeleton obtained in this way is called a *pseudo-tree*. Clearly, each pseudo-tree is a skeleton of genus 0.

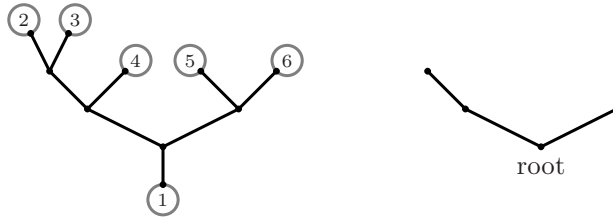


FIGURE 3. An admissible tree Ξ (black) and skeleton Γ_Ξ (left); the related binary tree (right)

5.1.1. A nonempty admissible tree Ξ has an even number $2k \geq 2$ of vertices, of which $(k-1)$ are nodes and $(k+1)$ are leaves. Unless $k=1$, each leaf is adjacent to a unique node. A *loose end* is a leaf sharing the same node with an even number of other leaves. (If $k > 2$, a loose end is the only leaf adjacent to a node.) One has

$$(5.1.2) \quad \#\{\text{loose ends of } \Xi\} = (k+1) \bmod 2.$$

As a consequence, an admissible tree with $2k = 0 \bmod 4$ vertices has a loose end.

5.1.3. A *marking* of an admissible tree Ξ is a choice of one of its leaves v_1 . Given a marking, one can number all leaves of Ξ consecutively, starting from v_1 and moving in the clockwise direction (see Figure 3, where the indices of the leaves are shown inside the loops). Declaring the node adjacent to v_1 the root and removing all leaves, one obtains an oriented rooted binary tree with $(k-1)$ vertices, see Figure 3, right; conversely, an oriented rooted binary tree B gives rise to a unique marked admissible tree: one attaches a leaf v_1 at the root of B and an extra leaf instead of each missing branch of B . As a consequence, the number of marked admissible trees with $2k$ vertices is given by the Catalan number $C(k-1)$. (Hence, the number of unmarked admissible trees is bounded from below by $C(k-1)/(k-1)$.)

5.1.4. The *vertex distance* m_i between two consecutive leaves v_i, v_{i+1} of a marked admissible tree Ξ is the vertex length of the shortest path in Ξ from v_i to v_{i+1} . (Note that, due to the numbering convention, this shortest path is a left turn path.) For example, in Figure 3 one has $(m_1, m_2, m_3, m_4, m_5) = (5, 3, 4, 5, 3)$. The vertex distance between two leaves $v_i, v_j, j > i$, is defined to be $\sum_{k=i}^{j-1} m_k$; it is the vertex length of the shortest left turn path connecting v_i to v_j in the associated skeleton Γ_Ξ , cf. Figure 6, left, in Subsection 5.3 below.

5.1.5. Given a marked admissible tree Ξ with $2k$ vertices, define an integral lattice \mathcal{Q}_Ξ as follows: as a group, \mathcal{Q}_Ξ is freely generated by k vectors $\mathbf{q}_i, i = 1, \dots, k$ (informally corresponding to pairs (v_i, v_{i+1}) of consecutive leaves), and the products are given by

$$\mathbf{q}_i^2 = m_i - 2, \quad \mathbf{q}_i \cdot \mathbf{q}_j = 1 \text{ if } |i-j| = 1, \quad \mathbf{q}_i \cdot \mathbf{q}_j = 0 \text{ if } |i-j| \geq 2,$$

where $m_i, i = 1, \dots, k$, is the vertex distance from v_i to v_{i+1} . Next, define the *characteristic functional*

$$(5.1.6) \quad \chi_\Xi := \sum_{i=1}^k m_i \mathbf{q}_i^* \in \mathcal{Q}_\Xi^*.$$

5.2. Contractions. An *elementary contraction* of an admissible tree Ξ is a new admissible tree Ξ' obtained from Ξ by removing two leaves adjacent to the same node (and thus converting this node to a leaf), see Figure 4. If Ξ is marked, we require in addition that the two leaves removed should be consecutive. (In other words, we do not allow the removal of the pair v_{k+1}, v_1 .) The contraction retains a marking: if the leaves removed are v_1, v_2 , we assign index 1 to their common node, becoming a leaf; otherwise, v_1 remains the first leaf in Ξ' .



FIGURE 4. A tree Ξ and its elementary contraction Ξ'

By a sequence of elementary contractions any (marked) admissible tree Ξ can be reduced to a simplest tree Ξ_0 with two vertices. (For proof, it suffices to consider an extremal node of the associated binary tree: it is adjacent to two consecutive leaves.) The resulting tree Ξ_0 can be identified with an induced subtree of Ξ , and the reduction procedure is called a *contraction of Ξ towards Ξ_0* . If Ξ_0 contains a leaf w of the original tree Ξ , we will also speak about a contraction of Ξ towards w . The argument above shows that any marked admissible tree Ξ can be contracted towards its first leaf v_1 ; similarly, Ξ can be contracted towards its last leaf v_{k+1} . (In general, a contraction is *not* uniquely determined by its terminal subtree $\Xi_0 \subset \Xi$.)

5.2.1. Lemma. *Any contraction of a marked admissible tree Ξ with $2k$ vertices gives rise to an isomorphism $\mathcal{Q}_\Xi \cong \mathcal{W}_k$, see (1.2.1).*

Proof. First, change the sign of each even generator \mathbf{q}_{2i} so that the nontrivial exdiagonal entries of the Gram matrix of \mathcal{Q}_Ξ become -1 rather than 1 . The new form is represented by the graph

$$(5.2.2) \quad \bullet \xrightarrow{m_1-2} \bullet \xrightarrow{m_2-2} \dots \xrightarrow{m_k-2} \bullet$$

where, as usual, generators are represented by the vertices (their squares being the weights indicated) and the product of two generator connected by an edge is -1 , whereas the generators not connected are orthogonal. Whenever a graph as above has a vertex of weight 1 , it can be ‘contracted’ as follows:

$$\dots \xrightarrow{m} \bullet \xrightarrow{1} \bullet \xrightarrow{n} \dots \quad \mapsto \quad \dots \xrightarrow{m-1} \bullet \xrightarrow{n-1} \bullet \xrightarrow{\dots} \dots$$

Arithmetically, this procedure corresponds to splitting the corresponding generator of square 1 as a direct summand (passing from $\mathbf{q}_{i-1}, \mathbf{q}_i, \mathbf{q}_{i+1}$ to $\mathbf{q}_{i-1} + \mathbf{q}_i, \mathbf{q}_i, \mathbf{q}_{i+1} + \mathbf{q}_i$, disregarding \mathbf{q}_i , and leaving other generators unchanged). On the other hand, two leaves v_i, v_{i+1} at a vertex distance $m_i = 3$ are adjacent to the same node, and the procedure just described establishes an isomorphism $\mathcal{Q}_\Xi \cong \mathbb{Z}\mathbf{q}_i \oplus \mathcal{Q}_{\Xi'}$, $\mathbf{q}_i^2 = 1$, where Ξ' is the corresponding elementary contraction of Ξ . (In Ξ' , the vertex distances just next to m_i decrease by 1 .) Contracting Ξ to a two vertex tree $\Xi_0 \subset \Xi$ and observing that $\mathcal{Q}_{\Xi_0} = \mathbb{Z}\mathbf{q}_1, \mathbf{q}_1^2 = 0$, one obtains an isomorphism as in the statement. \square

5.2.3. Remark. Analyzing the proof, one can easily conclude that the converse of Lemma 5.2.1 also holds: the lattice represented by a linear tree (5.2.2) is isomorphic to \mathcal{W}_k if and only if, up to the signs of the generators, it has the form \mathcal{Q}_Ξ for some marked admissible tree Ξ .

According to Lemma 5.2.1, a contraction $\Xi \rightsquigarrow \Xi_0$ sends each linear functional $\varphi \in \mathcal{Q}_\Xi^*$ to a functional $\bar{\varphi} \in \mathcal{W}_k^*$; we will say that φ *contracts* to $\bar{\varphi}$. The following statement is straightforward.

5.2.4. Lemma. *If a marked admissible tree Ξ with $2k$ vertices is contracted towards its first leaf v_1 , the functional \mathbf{q}_1^* contracts to \mathbf{w}^* . If Ξ is contracted towards its last leaf v_{k+1} , the functional \mathbf{q}_k^* contracts to $(-1)^{k+1}\mathbf{w}^*$. \square*

5.2.5. Lemma. *Up to isomorphism, the lattice $\text{Ker } \chi_\Xi \subset \mathcal{Q}_\Xi$ does not depend on the choice of a marking of Ξ .*

We postpone the proof of this lemma till next subsection, see 5.3.5, where a simple geometric argument is given.

5.2.6. Lemma. *If $k = 2s$ is even, the characteristic functional χ_{Ξ} , see (5.1.6), of a marked admissible tree Ξ with $2k$ vertices contracts to*

$$\bar{\chi} = 3\mathbf{v}_1^* + \dots + 3\mathbf{v}_s^* + \mathbf{v}_{s+1}^* + \dots + \mathbf{v}_{k-1}^*$$

(up to reordering and changing the signs of the generators \mathbf{v}_i).

Proof. *A priori*, the result of contraction may depend on the choice of a marking of Ξ and on the contraction used (*cf.* Remark 5.2.8 below). However, we assert that, if one set of choices results in the functional $\bar{\chi}$ given in the statement, then so does any other set (up to reordering and changing the signs). Indeed, the divisibility of $\bar{\chi}$ (the maximal integer $r \in \mathbb{Z}_{>0}$ such that $\bar{\chi}/r$ still takes values in \mathbb{Z}) is the same as that of χ_{Ξ} , and one can easily see that, up to a scalar multiple, $\bar{\chi}$ is the only functional with the following properties:

- (1) $\ker \text{Ker } \bar{\chi} \neq 0$,
- (2) $\det(\text{Ker } \bar{\chi} / \ker) = 5k - 1$, and
- (3) the maximal root system contained in $\text{Ker } \bar{\chi} / \ker$ is $\mathbf{A}_{s-1} \oplus \mathbf{A}_{s-2}$,

and it remains to apply Lemma 5.2.5. (Indeed, if $\bar{\chi} = \sum_i r_i \mathbf{v}_i^* + t \mathbf{w}^*$ with t and all r_i coprime, then (1) means that $t = 0$, (2) is equivalent to $\sum_i r_i^2 = 5k - 1$, and (3) means that the absolute values $|r_i|$ assume exactly two distinct values, one s -fold and one $(s - 1)$ -fold.)

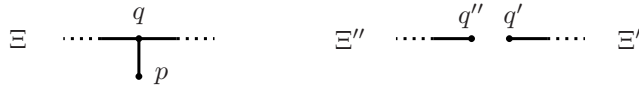


FIGURE 5. Cutting a tree Ξ at a loose end p

Now, we prove the statement by induction in k . For the only tree with 4 vertices (the case $k = 2$) it is straightforward. Consider a tree Ξ with $4s \geq 8$ vertices. In view of (5.1.2), Ξ has a loose end p , which is the only leaf adjacent to a certain node q . Remove p and double q , cutting Ξ into two trees Ξ' and Ξ'' containing the copies q' and q'' of q , respectively, see Figure 5. We may assume that Ξ' contains no loose ends of the original tree Ξ , as otherwise we could use that extra loose end instead of p . Then, q' is the only loose end of Ξ' and, due to (5.1.2), the number of vertices in Ξ' is $4s' = 0 \pmod{4}$. By additivity, the number of vertices in Ξ'' is $4(s - s') = 4s'' = 0 \pmod{4}$. If necessary, interchange Ξ' and Ξ'' so that Ξ' is to the right from p , as in Figure 5, and mark the trees so that $q' = v'_{2s'+1}$ is the last leaf of Ξ' and $q'' = v''_1$ is the first leaf of Ξ'' . Then, mark Ξ so that $v_1 = v'_1$.

Contract Ξ' and Ξ'' towards q' and q'' , respectively. This procedure contracts Ξ to a tree with a single node q . Disregarding the generators \mathbf{v}'_i and \mathbf{v}''_j that are split off during the contraction (in the obvious sense, they are the same for Ξ and Ξ', Ξ''), one arrives at the quadratic form $\mathbb{Z}\mathbf{w}' \oplus \mathbb{Z}\mathbf{w}''$, $(\mathbf{w}')^2 = (\mathbf{w}'')^2 = 1$, $\mathbf{w}' \cdot \mathbf{w}'' = -1$. Here, the squares of the generators resulting from Ξ differ by 1 from those resulting from Ξ' and Ξ'' , as so do the corresponding vertex distances. For the same reason, the characteristic functional χ_{Ξ} can be identified with $\chi_{\Xi'} + (\mathbf{q}'_s)^* + \chi_{\Xi''} + (\mathbf{q}''_1)^*$. Due to the induction hypothesis and Lemma 5.2.4, it contracts (in the obvious notation) to $\bar{\chi}' - (\mathbf{w}')^* + \bar{\chi}'' + (\mathbf{w}'')^*$, and one last contraction gives the statement for Ξ . \square

As a corollary, we get a partial result for the case of k odd.

5.2.7. Lemma. *If $k = 2s - 1$ is odd and a marked admissible tree Ξ with $2k$ vertices is contracted towards its last leaf v_{k+1} , the functional χ_Ξ contracts to*

$$\bar{\chi} = 3\mathbf{v}_1^* + \dots + 3\mathbf{v}_{s-1}^* + \mathbf{v}_s^* + \dots + \mathbf{v}_{k-1}^* + 2\mathbf{w}^*$$

(up to reordering and changing the signs of the generators \mathbf{v}_i).

Proof. Convert v_{k+1} to a node by attaching two extra leaves, contract the resulting tree Ξ' with $4s$ vertices towards its last leaf, apply Lemma 5.2.6, and use Lemma 5.2.4 to compensate for the difference between Ξ and Ξ' . \square

5.2.8. Remark. In the case of $k = 2s - 1$ odd, the resulting functional $\bar{\chi}$ does depend on the choice of a contraction used.

5.2.9. Corollary. *If Γ is a marked pseudo-tree with $2k \geq 6$ vertices, then the vertex distances m_i are coprime: $\text{g. c. d.}(m_1, \dots, m_k) = 1$. \square*

5.3. The case of all loops of type $\tilde{\mathbf{A}}_0^*$. Consider a pseudo-tree $\Gamma = \Gamma_\Xi$ and choose the homological invariant so that the singular fibers inside the loops attached to Ξ are all of type $\tilde{\mathbf{A}}_0^*$. (This choice corresponds to the boundary orientation of each edge bounding a loop: if v_i is a leaf and $\bar{1}(v_i)$ belongs to the original tree Ξ , then $\bar{2}(v_i)$ is the tail of the new edge attached at v_i . The orientations of the edges of the original tree are irrelevant.) Then the fiber inside the outer region of Γ is of type $\tilde{\mathbf{A}}_{5k-2}$ if k is even or $\tilde{\mathbf{D}}_{5k+3}$ if k is odd.

Pick a marking of Ξ , see 5.1.3, and let $n_i = \sum_{j=i}^k m_j$, $i = 1, \dots, k$, be the vertex distance from v_i to v_{k+1} , see 5.1.4. In the computation below, we retain the notation $a, b, c \in \mathcal{H}$ for the three special elements of \mathcal{H} introduced in 4.3.

Mark Γ at each leaf v_i so that $\bar{1}(v_i)$ belongs to the original tree Ξ , see 4.1.2. Let ξ_i be the boundary of the loop attached at v_i , and denote by \mathcal{H}_Γ^0 the subgroup spanned by the classes $[\xi_i, a]$, $i = 1, \dots, k+1$. One has $\mathcal{H}_\Gamma^0 \subset \ker \mathcal{H}_\Gamma$, cf. Example 4.4.4. Taking into account constraints 4.3.2(5) at v_i and 4.3(2) at ξ_i , one concludes that the restriction of each element $h \in \mathcal{H}_\Gamma$ to the three ends constituting v_i is a linear combination of $a \otimes \bar{2}(v_i) - b \otimes \bar{3}(v_i) = [\xi_i, a] \in \mathcal{H}_\Gamma^0$ and the element

$$(5.3.1) \quad c \otimes \bar{1}(v_i) + b \otimes \bar{2}(v_i) + a \otimes \bar{3}(v_i).$$

Hence, modulo \mathcal{H}_Γ^0 this restriction is a multiple of (5.3.1). To find a simpler basis, consider the subgroup $\mathcal{H}'_\Gamma \subset \mathcal{H} \otimes \Gamma$ consisting of the vectors satisfying all but one conditions 4.3(2) and 4.3.2(5): namely, relax 4.3.2(5) at v_{k+1} to

$$(5.3.2) \quad h_{\bar{1}(v_{k+1})} + \mathbb{X}h_{\bar{2}(v_{k+1})} + \mathbb{X}^2h_{\bar{3}(v_{k+1})} = 0 \text{ mod } b.$$

Contracting the tree towards its last vertex v_{k+1} and using the relations, one can easily see that an element of \mathcal{H}'_Γ whose restriction to each $\mathcal{H} \otimes \bar{1}(v_i)$, $i = 1, \dots, k$, is zero belongs to \mathcal{H}_Γ^0 . Thus, each restriction (5.3.1), $i = 1, \dots, k$, admits at most one, hence exactly one, see below, lift to the quotient $\mathcal{H}''_\Gamma := \mathcal{H}'_\Gamma / \mathcal{H}_\Gamma^0$, and \mathcal{H}''_Γ is freely generated by these lifts. Up to sign, the lifts are

$$\mathbf{e}_i := \varepsilon_i b \otimes \bar{2}(v_i) + \varepsilon_i a \otimes \bar{3}(v_i) + \|\gamma_i, \varepsilon_i c\| + b \otimes \bar{2}(v_{k+1}) + a \otimes \bar{3}(v_{k+1}),$$

$i = 1, \dots, k$, where γ_i is the shortest left turn path from v_i to v_{k+1} and the signs $\varepsilon_i = \pm 1$ are chosen so that the monodromy

$$\mathbf{m}_{\gamma_i} = \pm \mathbb{Y}(\mathbb{X}\mathbb{Y})^{n_i-2} = \pm \begin{bmatrix} 0 & -1 \\ 1 & n_i - 2 \end{bmatrix}$$

take $\varepsilon_i c$ to the element $u_i := c + n_i b$; these signs depend on the orientations of the edges of the original tree Ξ . Informally, \mathbf{e}_i is obtained by extending (5.3.1) along γ_i , see Definition 4.4.1, and ‘closing’ it at v_{k+1} to satisfy the relaxed set of conditions; condition (5.3.2) was chosen so that the latter closure exists and is unique modulo \mathcal{H}_Γ^0 : one merely disregards the term $n_i b$ in u_i above and completes $c \otimes \bar{1}(v_{k+1})$ to (5.3.1). The supports of \mathbf{e}_i are shown in shades of grey in Figure 6, left; after a shift, they can be made pairwise disjoint except in a neighborhood of the last vertex v_{k+1} .



FIGURE 6. Supports of \mathbf{e}_i (left) and their shifts (right)

Bringing back the last relation 4.3.2(5) at v_{k+1} , one can see that the subgroup $\mathcal{H}_\Gamma/\mathcal{H}_\Gamma^0 \subset \mathcal{H}_\Gamma''$ is the kernel $\text{Ker } \varphi$, where

$$\varphi = \sum_i n_i \mathbf{e}_i^*.$$

(The multiples of $n_i b_i$ disregarded in the construction of \mathbf{e}_i must sum up to zero.) The self-intersection of a cycle $\sum_i r_i \mathbf{e}_i \in \text{Ker } \varphi$ (assuming that it *is* a cycle) can be computed geometrically, by shifting all paths ‘to the left’; it is given by

$$(5.3.3) \quad \left(\sum_i r_i \mathbf{e}_i \right)^2 = - \sum_i r_i^2 - \left(\sum_i r_i \right)^2 - \sum_{1 \leq i < j \leq k} r_i r_j (u_i \cdot u_j),$$

where $u_i \cdot u_j = n_i - n_j$. During the shift, the supports can be kept pairwise disjoint except in a small neighborhood U of v_{k+1} ; the shift inside U is shown in light solid lines in Figure 6, right. The i -th term of the first sum in (5.3.3) is the contribution of the self-intersection of $r_i \mathbf{e}_i$ in a neighborhood of v_i , *cf.* (4.3.3). The last two terms are contributed by U . To compute this contribution, one should bring all 1-cycles in the fibers to the same basis (*cf.* the proof of Theorem 4.3.4); we choose the basis in $\mathcal{H}_{\bar{1}(v_{k+1})}$. Then the 1-cycle over the i -th vertical segment in Figure 6, right, is u_i . The 1-cycle over the left arced segment is $w_0 := \mathbb{X}^2(ra) = rb$, where $r = \sum_i r_i$, and the 1-cycles over the consecutive (left to right) horizontal segments, concluding with the right arced segment, are $w_i := w_0 + \sum_{j=1}^i r_j u_j$, $i = 1, \dots, k$. (Recall that $\sum_i r_i \mathbf{e}_i$ is *assumed* a cycle. The orientation of the arced and horizontal segments is left to right in the figure, *i.e.*, the one induced from the orientation of the loop of Γ .) The intersection points are all seen in the figure, and the total contribution from U is $-\sum_{i=1}^k w_i \cdot u_i$, which simplifies to the last two terms in (5.3.3).

Since we are only interested in the values of (5.3.3) on the kernel $\text{Ker } \varphi$, we can add to (5.3.3) the quadratic expression

$$\left(\sum_i r_i \right) \left(\sum_i n_i r_i \right).$$

Now, extend the new quadratic form to the whole group \mathcal{H}_Γ'' and consider the corresponding symmetric bilinear form; in the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ it is given by the matrix $E = [e_{ij}]$, where $e_{ii} = n_i - 2$ and $e_{ij} = n_{\max\{i,j\}} - 1$ for $i \neq j$. It is straightforward that, for $i < j < k$, one has

$$(\mathbf{e}_i - \mathbf{e}_j) \cdot \mathbf{e}_k = 0, \quad (\mathbf{e}_i - \mathbf{e}_j) \cdot \mathbf{e}_j = 1, \quad (\mathbf{e}_i - \mathbf{e}_j) \cdot \mathbf{e}_i = n_i - n_j.$$

Hence, in the new basis $\mathbf{q}_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, $i = 1, \dots, k-1$, $\mathbf{q}_k = \mathbf{e}_k$, the form turns into \mathcal{Q}_Ξ , see 5.1.5, and the functional φ above turns into χ_Ξ . Finally, there is an isomorphism

$$(5.3.4) \quad \mathcal{H}_\Gamma / \ker = \text{Ker } \chi_\Xi / \ker.$$

5.3.5. Proof of Lemma 5.2.5. The statement follows from (5.3.4) and the fact that the left hand side does not depend on the choice of a marking of Ξ . \square

5.4. Proof of Theorems 1.2.2 and 1.2.3. The skeleton Γ of an extremal elliptic surface X as in the theorems is necessarily a pseudo-tree, $\Gamma = \Gamma_\Xi$, and the singular fibers of X inside the loops of Γ are all of type \mathbf{A}_0^* . (One has $k = 2s$, $t = 0$ in Theorem 1.2.2 and $k = 2s - 1$, $t = 1$ in Theorem 1.2.3.) Hence, in view of Corollary 4.3.8, the lattice \mathcal{T}_X is given by (5.3.4), and its structure is described by Lemmas 5.2.1, 5.2.6, and 5.2.7. (In the case of $k = 2s - 1$ odd, the quotient map $\mathcal{W}_k \rightarrow \mathcal{W}_k / \ker = \mathcal{V}_{k-1}$ projects \mathcal{T}_X to an even index 2 sublattice of \mathcal{V}_{k-1} ; by definition, it is \mathbf{D}_{k-1} .) \square

5.5. The case of one type $\tilde{\mathbf{D}}_5$ fiber. Now, choose the homological invariant so that one of the loops contain a type $\tilde{\mathbf{D}}_5$ fiber and mark Ξ so that this loop is attached to the last leaf v_{k+1} . Let ξ_i and γ_i be as in Subsection 5.3, denote by \mathcal{H}_Γ^0 the subgroup spanned by $[\xi_i, a]$, $i = 1, \dots, k$ (note that the index runs to k rather than $k+1$), and let \mathcal{H}_Γ'' be the subgroup of $(\mathcal{H}_\Gamma / \mathcal{H}_\Gamma^0) \otimes \mathbb{Q}$ generated over \mathbb{Z} by the rational cycles

$$\begin{aligned} \mathbf{e}_i &:= \varepsilon_i b \otimes \bar{2}(v_i) + \varepsilon_i a \otimes \bar{3}(v_i) + \|\gamma_i, \varepsilon_i c\| + v(n_i) \otimes \bar{2}(v_{k+1}) + w(n_i) \otimes \bar{3}(v_{k+1}), \\ \text{where } v(n) &= \frac{n}{2}a + \frac{2-n}{4}b \quad \text{and} \quad w(n) = \frac{2-n}{4}a - \frac{n}{2}b. \end{aligned}$$

(The vectors $v(n)$, $w(n)$ are chosen to ‘close’ the chain over v_{k+1} , as solutions to the system $(c + nb) + \mathbb{X}v + \mathbb{X}^2w = v + \mathbb{Y}w = 0$.) Then, $\mathcal{H}_\Gamma / \mathcal{H}_\Gamma^0$ is the index 4 subgroup of \mathcal{H}_Γ'' defined by the parity condition

$$\psi(x) = 0 \pmod{4}, \quad \text{where } \psi = \sum_i (n_i - 2)\mathbf{e}_i^*.$$

The intersection indices $\mathbf{e}_i \cdot \mathbf{e}_j$ can easily be computed either using Theorem 4.3.4 or as in Subsection 5.3. One has

$$\mathbf{e}_i^2 = \frac{1}{4}(n_i + 2)(n_i - 2), \quad \mathbf{e}_i \cdot \mathbf{e}_j = \frac{1}{4}(n_i + 2)(n_j - 2) - 1 \quad \text{for } i < j.$$

In the new basis $\mathbf{q}_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, $i = 1, \dots, k-1$, $\mathbf{q}_k = \mathbf{e}_k$ the functional ψ above takes the form

$$(5.5.1) \quad \psi = \sum_i m'_i \mathbf{q}_i^*, \quad \text{where } m'_i = m_i \text{ for } i = 1, \dots, k-1 \text{ and } m'_k = m_k - 2,$$

and the intersection indices are

$$\mathbf{q}_i^2 = \frac{(m'_i)^2}{4} + m_i - 2, \quad \mathbf{q}_i \cdot \mathbf{q}_{i+1} = \frac{m'_i m'_{i+1}}{4} + 1, \quad \mathbf{q}_i \cdot \mathbf{q}_j = \frac{m'_i m'_j}{4}, \quad j > i + 1.$$

In other words, one can identify \mathcal{H}'_Γ with the group \mathcal{Q}_Ξ supplied with the modified bilinear form

$$x \otimes y \mapsto x \cdot y + \frac{1}{4}\psi(x)\psi(y),$$

where \cdot is the original form on \mathcal{Q}_Ξ ; under this identification, $\psi = \chi_\Xi - 2\mathbf{q}_k^*$.

5.6. Proof of Theorems 1.2.4 and 1.2.5. The skeleton Γ of an extremal elliptic surface X as in the theorems is necessarily a pseudo-tree, $\Gamma = \Gamma_\Xi$, and the singular fibers of X inside the loops of Γ are one copy of $\tilde{\mathbf{D}}_5$ and k copies of $\tilde{\mathbf{A}}_0^*$. (One has $k = 2s$, $t = 2$ in Theorem 1.2.4 and $k = 2s - 1$, $t = 1$ in Theorem 1.2.5.) Mark Ξ as in Subsection 5.5 and contract it towards its last leaf v_{k+1} , establishing an isomorphism $\mathcal{Q}_\Xi = \mathcal{W}_k$, see Lemma 5.2.1. Due to Lemmas 5.2.6 and 5.2.7, the functional $\psi = \chi_\Xi - 2\mathbf{q}_k^*$ contracts to

$$\bar{\psi} = \begin{cases} 3\mathbf{v}_1^* + \dots + 3\mathbf{v}_s^* + \mathbf{v}_{s+1}^* + \dots + \mathbf{v}_{k-1}^* - 2\mathbf{w}^*, & \text{if } k = 2s \text{ is even,} \\ 3\mathbf{v}_1^* + \dots + 3\mathbf{v}_{s-1}^* + \mathbf{v}_s^* + \dots + \mathbf{v}_{k-1}^*, & \text{if } k = 2s - 1 \text{ is odd.} \end{cases}$$

(The correction term $-2\mathbf{w}^*$ is given by Lemma 5.2.4.) Thus, due to Corollary 4.3.8 and the results of Subsection 5.5, one has $\mathcal{T}_X = \{x \in \mathcal{W}'_k \mid \bar{\psi}(x) = 0 \pmod{4}\} / \ker$, where \mathcal{W}'_k is \mathcal{W}_k with the modified bilinear form $x \otimes y \mapsto x \cdot y + \frac{1}{4}\bar{\psi}(x)\bar{\psi}(y)$.

If k is odd, the kernel $\ker \mathcal{W}'_k$ is generated by \mathbf{w} , and passing to the quotient $\mathcal{T}_X \subset \mathcal{W}'_k / \mathbf{w}$ one obtains the description given in Theorem 1.2.5.

If k is even, one has an orthogonal decomposition $\mathcal{T}_X = \text{Ker } \bar{\psi} \oplus \mathbb{Z}\mathbf{x}$, where $\mathbf{x} = 2\mathbf{w}$, and $\text{Ker } \bar{\psi}$ is generated by the vectors $\mathbf{v}_s - \mathbf{v}_{s+1} + \mathbf{w}$, $\mathbf{v}_1 + \mathbf{v}_2 + 3\mathbf{w}$, and $\mathbf{v}_i - \mathbf{v}_{i+1}$, $i = 1, \dots, s-1, s+1, \dots, k-2$. It is immediate that $\text{Ker } \bar{\psi} \cong \mathbf{D}_{k-1}$. \square

5.7. Proof of Theorem 1.2.6. We use Zariski–van Kampen’s method [5] applied to the ruling of Σ . The braid monodromy is computed using [3].

If there is a type $\tilde{\mathbf{D}}_5$ fiber, mark Ξ as explained in Subsection 5.5; otherwise, mark it arbitrarily. Mark Γ at v_{k+1} as shown in Figure 7, so that $\bar{1}(v_{k+1})$ belongs to the original tree Ξ . Take the fiber F over v_{k+1} for the reference fiber, and let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a canonical basis in F defined by the chosen marking, see [3]. Let δ_i be the path in the base composed of the loop of Γ at v_i , $i = 1, \dots, k+1$, connected to v_{k+1} by the shortest left turn path *ending at* $\bar{2}(v_{k+1})$, see Figure 7. According to [3], the braid monodromy \mathbf{m}_i along δ_i is given by

$$(5.7.1) \quad \mathbf{m}_i = \sigma_1^{n_i} \sigma_2 \sigma_1^{-n_i}, \quad i = 1, \dots, k, \quad \mathbf{m}_{k+1} = \sigma_2 (\sigma_1 \sigma_2)^{3\epsilon},$$

where parameters n_i are the vertex distances introduced in Subsection 5.3, $\epsilon = 0$ or 1 if the singular fiber next to v_{k+1} is of type $\tilde{\mathbf{A}}_0^*$ or $\tilde{\mathbf{D}}_5$, respectively, and σ_1, σ_2 are the Artin generators of the braid group \mathbb{B}_3 acting on the free group $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ *via*

$$\sigma_1: (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_1 \alpha_2 \alpha_1^{-1}, \alpha_1, \alpha_3), \quad \sigma_2: (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_1, \alpha_2 \alpha_3 \alpha_2^{-1}, \alpha_2).$$

Then one has

$$\pi_1(\Sigma \setminus (C \cup E)) = \langle \alpha_1, \alpha_2, \alpha_3 \mid \mathbf{m}_i = \text{id}, i = 1, \dots, k+1, (\alpha_1 \alpha_2 \alpha_3)^{k+t} = 1 \rangle,$$

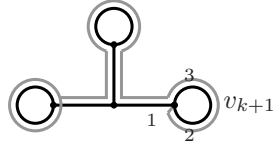


FIGURE 7. A loop δ_i (grey)

where k and t are as introduced in 3.4.1. Here, each *braid relation* $\mathfrak{m}_i = \text{id}$ is understood as the triple of relations $\mathfrak{m}_i(\alpha_j) = \alpha_j$, $j = 1, 2, 3$; as a consequence, for each $\alpha \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ one has a relation $\mathfrak{m}_i(\alpha) = \alpha$. The last relation in the above presentation is called the *relation at infinity*; in its presence, the braid relation about the remaining singular fiber in the outer region of Γ can be ignored.

The braid relation $\mathfrak{m}_i(\alpha_3) = \alpha_3$, $i = 1, \dots, k$ implies $\alpha_3 = \sigma_1^{n_i} \alpha_2$. Hence one has $\sigma_1^{n_i} \alpha_2 = \sigma_1^{n_j} \alpha_2$ for $1 \leq i, j \leq k$. Since σ_1 preserves α_3 and the product $\rho := \alpha_1 \alpha_2 \alpha_3$ (and α_2, α_3 , and ρ generate $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$), for each $\alpha \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ one has a relation $\sigma_1^{n_i} \alpha = \sigma_1^{n_j} \alpha$. Replacing α with $\sigma_1^{-n_j} \alpha$, one can rewrite this relation in the form $\sigma_1^{n_i - n_j} \alpha = \alpha$.

If $k > 2$, the differences $n_i - n_j$, $1 \leq i, j \leq k$, are coprime, see Corollary 5.2.9. Hence, an appropriate iteration of the relations $\sigma_1^{n_i - n_j} \alpha = \alpha$ obtained above results in $\sigma_1 \alpha = \alpha$, $\alpha \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle$. In particular, $\sigma_1 \alpha_2 = \alpha_2$, *i.e.*, $\alpha_1 = \alpha_2$. Then, the original braid relation $\alpha_3 = \sigma_1^{n_i} \alpha_2$ simplifies to $\alpha_3 = \alpha_1$ or $\alpha_3 = \alpha_2$, depending on the parity of n_i . In any case, one has $\alpha_1 = \alpha_2 = \alpha_3$, and the group is cyclic. \square

5.7.2. Remark. In the exceptional cases $k = 1, 2$, the fundamental groups are also easily computed. We skip details and merely indicate the result:

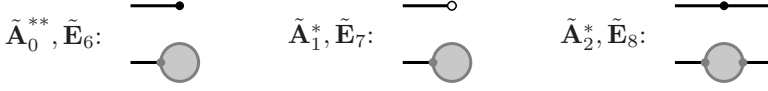
$$\begin{aligned} k = 2, t = 0 : & \quad \mathbb{B}_3 / (\sigma_1 \sigma_2)^3, \\ k = 2, t = 2 : & \quad \mathbb{Z}_3 \rtimes \mathbb{Z}_{12}, \\ k = 1, \tilde{\mathbf{D}}_8 \text{ type fiber} : & \quad \mathbb{Z} \times \mathbb{Z}_2, \\ k = 1, \tilde{\mathbf{D}}_5 \text{ type fiber} : & \quad \mathbb{Z}[t] / (t^2 - 1) \rtimes \mathbb{Z}_2; \end{aligned}$$

in the last case, the generator of \mathbb{Z}_2 act on the kernel *via* multiplication by t . It follows that, for $k = 1$, the trigonal curve C is reducible.

6. GENERALIZATIONS

In this section, we briefly outline two generalizations of Theorem 4.3.4, one to surfaces with type $\tilde{\mathbf{E}}$ singular fibers, and one to non-extremal surfaces.

6.1. Extremal surfaces with $\tilde{\mathbf{E}}$ type fibers. Let X be an extremal elliptic surface *with* type $\tilde{\mathbf{E}}$ singular fibers. (Accidentally, at this point we can also admit singular fibers of types $\tilde{\mathbf{A}}_0^{**}$, $\tilde{\mathbf{A}}_1^*$, or $\tilde{\mathbf{A}}_2^*$, provided that X satisfies conditions 2.2.2(1) and (2) and has no fibers of type $\tilde{\mathbf{D}}_4$.) Let $\Gamma = \Gamma_X$ be the skeleton of X ; it may have \bullet -vertices of valency ≤ 2 or \circ -vertices of valency 1. Replace these irregular vertices with the boundaries of small disks, see Figure 8, bottom row, converting Γ to a regular 3-graph Γ' . Unlike Γ , the new graph Γ' is a skeleton in the sense of Definition 4.1.1.

FIGURE 8. The modification Γ' of Γ

Orient the new edges of Γ' as the boundary of the shaded regions in Figure 8. Assign label $\mathbb{Y} \in PSL(2, \mathbb{Z})$ to each edge of the original skeleton Γ , and label the new edges (grey in the figure) as follows:

- type $\tilde{\mathbf{A}}_0^{**}$ ($\tilde{\mathbf{E}}_6$): the label is \mathbb{X} (respectively, $-\mathbb{X}$);
- type $\tilde{\mathbf{A}}_1^*$ ($\tilde{\mathbf{E}}_7$): the label is $-\mathbb{X}\mathbb{Y}\mathbb{X}$ (respectively, $\mathbb{X}\mathbb{Y}\mathbb{X}$);
- type $\tilde{\mathbf{A}}_2^*$ ($\tilde{\mathbf{E}}_8$): the two labels are either both \mathbb{X} or both $-\mathbb{X}$ (respectively, one label is \mathbb{X} and one is $-\mathbb{X}$).

(In the last case, when two new edges are inserted, there are two choices of the labelling; they result in distinct homological invariants of X , cf. 6.1.3 below.)

Define $\mathcal{H} \otimes \Gamma$ to be $\mathcal{H} \otimes \Gamma'$, see Subsection 4.3, and let \mathcal{H}_Γ be the subgroup of $\mathcal{H} \otimes \Gamma$ subject to the following relations:

- (1) $h_\alpha + \mathbb{X}h_{\text{nx}\alpha} + \mathbb{X}^2h_{\text{nx}^2\alpha} = 0$ for each element $\alpha \in \mathcal{E}$;
- (2) $h_{e^+} + Lh_{e^-} = 0$ for each edge $e \in \mathcal{E}/\text{op}$ labelled L .

Similarly, let $\mathcal{H}^* \otimes \Gamma = \mathcal{H}^* \otimes \Gamma'$ and define \mathcal{H}_Γ^* as the quotient of $\mathcal{H}^* \otimes \Gamma$ by the subgroup spanned by the vectors of the form

- (3) $u \otimes \alpha + \mathbb{X}^*u \otimes (\text{nx}\alpha) + (\mathbb{X}^*)^2u \otimes (\text{nx}^2\alpha)$ for each $u \in \mathcal{H}^*$ and $\alpha \in \mathcal{E}$;
- (4) $u \otimes e^+ + L^*u \otimes e^-$ for each $u \in \mathcal{H}^*$ and each edge $e \in \mathcal{E}/\text{op}$ labelled L .

There is a natural pairing $\mathcal{H}_\Gamma \otimes \mathcal{H}_\Gamma^* \rightarrow \mathbb{Z}$.

6.1.1. Theorem. *There are isomorphisms $H_2(X^\circ) = \mathcal{H}_\Gamma$ and $H^2(X^\circ) = \mathcal{H}_\Gamma^*$. The former takes the intersection index form to the form given by (4.3.1); the latter takes the Kronecker product to the pairing $\mathcal{H}_\Gamma \otimes \mathcal{H}_\Gamma^* \rightarrow \mathbb{Z}$ defined above.*

Proof. The proof repeats literally that of Theorem 4.3.4: the space X° has a strict deformation retract X' which fibers over the new graph Γ' . The braid monodromy along Γ' (hence the homological invariant, cf. Subsection 4.2) is computed in [3], and the edge labels above are defined to produce the result that would agree with this computation. \square

6.1.2. Corollary. *One has $\mathcal{T}_X = \mathcal{H}_\Gamma / \ker$ and $\text{Tors } MW(X) = \text{Tors } \mathcal{H}_\Gamma^*$. \square*

6.1.3. One can also mimic Definition 4.4.1 and define the *monodromy* \mathbf{m}_γ and the *parallel transport* $\|\gamma, h_0\| \in \mathcal{H} \otimes \Gamma$ along a path γ in the new graph Γ' . Part 4.4.1(2) of the definition should be replaced with $\mathbf{m}_i = -L^{\pm 1}$ for an edge $e = [\alpha_{i-1}, \alpha_i]$ labelled L and $\alpha_i = e^\pm$. Under this definition, the monodromy along the boundary of each region R other than the shaded disks in Figure 8 is still of the form $\pm(\mathbb{X}\mathbb{Y})^n$, where n is the number of corners (\bullet -vertices) in the boundary ∂R in the original graph Γ . The monodromy along the boundary of a shaded disk R is of the form $\pm\mathbb{X}^{\pm 1}$ or $\pm\mathbb{X}\mathbb{Y}\mathbb{X}^{-1}$, depending on the type of the singular fiber inside R . It follows that the monodromies $\mathbf{m}_{\partial R}$ determine the type specification of X and that $\mathbf{m}_{\partial R}$ has an invariant vector if and only if the singular fiber inside R is stable. In particular, one still has analogues of Propositions 4.4.5 and 4.4.7.

6.2. Non-extremal surfaces. Now, consider a non-isotrivial Jacobian elliptic surface X , not necessarily extremal, satisfying the following conditions (*cf.* Definition 2.2.2):

- (1) j_X has no critical values other than 0, 1, and ∞ ;
- (2) each point in $j_X^{-1}(0)$ has ramification index $(0 \bmod 3)$, and each point in $j_X^{-1}(1)$ has ramification index 2;
- (3) X has no singular fibers of type $\tilde{\mathbf{D}}_4$.

(We do not discuss whether any elliptic surface can be deformed to one satisfying (1)–(3). For each particular surface X , this can be decided in terms of equisingular degenerations of the dessin of X , see [3].)

As in Subsection 2.3, define the *skeleton* $\Gamma_X = j_X^{-1}[0, 1]$; it is a ribbon graph with all vertices of valency $(0 \bmod 3)$. (The idea of considering skeletons with multiple vertices rather than dessins in the sense of [3] was suggested to me by I. Shimada.) To accommodate $\Gamma = \Gamma_X$, modify Definition 4.1.1 by replacing the condition $nx^3 = \text{id}$ with the requirement that each orbit of nx should have length divisible by 3. Then, as in Subsection 4.3, define $\mathcal{H} \otimes \Gamma = \bigoplus \mathcal{H}_\alpha$, $\alpha \in \mathcal{E}$, and let $\mathcal{H}_\Gamma \subset \mathcal{H} \otimes \Gamma$ be the subgroup subject to the following conditions:

- (1) $\sum_{i=0}^{n-1} \mathbb{X}^i h_{\alpha_i} = 0$ for each vertex $(\alpha_0, \dots, \alpha_{n-1}) \in \mathcal{E}/nx$ of valency n ;
- (2) $h_{e^+} + \mathbb{Y} h_{e^-} = 0$ for each edge $e \in \mathcal{E}/\text{op}$.

Also, define \mathcal{H}_Γ^* as the quotient of $\mathcal{H}^* \otimes \Gamma$ by the image of the maps adjoint to the left hand sides of (1), (2). There is a pairing $\mathcal{H}_\Gamma \otimes \mathcal{H}_\Gamma^* \rightarrow \mathbb{Z}$.

Convert $\mathcal{H} \otimes \Gamma$ to a rational lattice, defining the square h^2 of $h \in \mathcal{H} \otimes \Gamma$ to be

$$(6.2.1) \quad h^2 = - \sum_{\alpha \in \mathcal{E}} \sum_{d=1}^{n(\alpha)-2} \frac{n(\alpha) - d - 1}{n(\alpha)} h_\alpha \cdot \mathbb{X}^d h_{nx^d \alpha},$$

where $n(\alpha)$ is the valency of the vertex represented by α , *i.e.*, the length of the orbit of nx containing α . (An alternative expression for the restriction of this form to \mathcal{H}_Γ is given by (6.2.3) below.)

6.2.2. Theorem. *There are isomorphisms $H_2(X_\Gamma^\circ) = \mathcal{H}_\Gamma$ and $H^2(X_\Gamma^\circ) = \mathcal{H}_\Gamma^*$. The former takes the intersection index form to the form given by (6.2.1); the latter takes the Kronecker product to the pairing $\mathcal{H}_\Gamma \otimes \mathcal{H}_\Gamma^* \rightarrow \mathbb{Z}$ defined above.*

Proof. Again, the proof repeats literally that of Theorem 4.3.4. (For a $3k$ -fold vertex v of Γ , the correspondence between the $3k$ edges at v and the three points of $\bar{F}_v \cap C$ is k -to-one, *cf.* the discussion of markings in Subsection 4.2; otherwise, the computation of the homological invariant is the same as in 4.2.) To compute the contribution to the self intersection h^2 of a cycle $h \in \mathcal{H}_\Gamma$ by a marked n -valent vertex $(\alpha_0, \dots, \alpha_{n-1}) \in \mathcal{E}/nx$, ‘spread out’ the vertex and shift it off as shown in Figure 9. The resulting expression is

$$(6.2.3) \quad - \sum_{i=1}^{n-2} \sum_{j=0}^{i-1} \mathbb{X}^j h_j \cdot \mathbb{X}^i h_i = - \sum_{i=1}^{n-2} \sum_{j=0}^{i-1} h_j \cdot \mathbb{X}^{i-j} h_i,$$

cf. the proof of (5.3.3) and Figure 6. (We abbreviate $h_i = h_{\alpha_i}$ and use the fact that \mathbb{X} is an isometry.) Averaging over all n markings of the vertex, one arrives at (6.2.1). \square



FIGURE 9. Spreading out and shifting a vertex

6.2.4. Corollary. *One has $\mathcal{T}_X = \mathcal{H}_\Gamma / \ker$ and $\text{Tors } MW(X) = \text{Tors } \mathcal{H}_\Gamma^*$. \square*

With the obvious modifications, the material of Subsection 4.4 extends to the general case. One can also combine the constructions of this and the previous subsections and consider non-extremal surfaces with \mathbf{E} type singular fibers. (For the sake of simplicity, it is better to consider a skeleton Γ with all \circ -vertices of valency ≤ 2 and all \bullet -vertices of valency either $(0 \bmod 3)$ or ≤ 2 .) We leave details to the reader.

REFERENCES

1. K. Arima, I. Shimada, *Zariski–van Kampen method and transcendental lattices of certain singular K3 surfaces*, [arXiv:0806.3311](https://arxiv.org/abs/0806.3311) (to appear).
2. F. Beukers, H. Montanus, *Explicit calculation of elliptic fibrations of K3-surfaces and their Belyi-maps*, Number theory and polynomials, London Math. Soc. Lecture Note Ser., vol. 352, Cambridge Univ. Press, Cambridge, 2008, pp. 33–51.
3. A. Degtyarev, *Zariski k-plets via dessins d’enfants*, Comment. Math. Helv. **84** (2009), no. 3, 639–671.
4. R. Friedman, J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin-New York, 1994.
5. E. R. van Kampen, *On the fundamental group of an algebraic curve*, Amer. J. Math. **55** (1933), 255–260.
6. K. Kodaira, *On compact analytic surfaces, II–III*, Annals of Math. vol 77–78 (1963), 563–626, 1–40.
7. V. V. Nikulin, *Integer quadratic forms and some of their geometrical applications*, Izv. Akad. Nauk SSSR, Ser. Mat **43** (1979), 111–177 (Russian); English transl. in Math. USSR–Izv. **14** (1980), 103–167.
8. M. Nori, *On certain elliptic surfaces with maximal Picard number*, Topology **24** (1985), no. 2, 175–186.
9. T. Shioda, *On elliptic modular surfaces.*, J. Math. Soc. Japan **24** (1972), 20–59.
10. T. Shioda, *On the Mordell–Weil lattices*, Comment. Math. Univ. St. Paul **39** (1990), no. 2, 211–240.

DEPARTMENT OF MATHEMATICS, BILKENT UNIVERSITY, 06800 ANKARA, TURKEY
E-mail address: degt@fen.bilkent.edu.tr