TOPOLOGY OF PLANE ALGEBRAIC CURVES: THE ALGEBRAIC APPROACH

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Abstract. We discuss the principle tools and results and state a few open problems concerning the classification and topology of plane sextics and trigonal curves in ruled surfaces.

1. Introduction

1.1. Contents. The purpose of this survey is to discuss the algebro-geometric aspects of plane algebraic curves of small degree or 'codegree' and state a few open problems concerning topology of these curves. More precisely, we will concentrate on two classes of curves: sextics in the projective plane and trigonal curves in ruled surfaces, the latter being closely related to plane curves with a singular point of multiplicity (degree $-$ 3).

In the last 25 years, topology of singular plane algebraic curves has been an area of active research; the modern state of affairs and important open problems are thoroughly presented in recent surveys [7], [54], [55], and [71]. Analyzing recent achievements, one cannot help noticing that, with relatively few exceptions, algebraic objects are studied by purely topological means, making very little use of the analytic structure. These methods, largely based on the braid monodromy, apply equally well to pseudo-holomorphic curves and to the so called *Hurwitz curves*, see [48], which roughly are smooth surfaces mapped to $\mathbb{P}^2$ so that they behave like algebraic curves with respect to one chosen pencil of lines. (Remarkably, a similar phenomenon was observed in topology of real algebraic curves, leading Rokhlin and Viro to the concept of *flexible curves*. This idea was later developed by Orevkov, who initiated the study of real pseudo-holomorphic curves.) Although well justified in general, this approach seems to fail when curves of small degree are concerned and more precision is required. For this reason, I chose to restrict this survey to plane sextics and to the more algebraic methods that have recently been used in their study. As part of these methods, trigonal curves appear; since they seem to be of interest in their own right, I dedicated a separate section to their theory and related problems.

Although this paper is of a purely expository nature, I do announce several new results, among them being Theorem 5.6.4, computing the transcendental lattices of a series of elliptic surfaces, and updated lists of fundamental groups of sextics, see comments to Problems 4.2.2 and 5.5.1.

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2. Preliminaries

Throughout the paper, all varieties are either complex analytic (when speaking about the moduli of $K3$-surfaces) or algebraic over $\mathbb{C}$. Unless stated otherwise, all curves considered are reduced.

2.1. Equivalence relations. The principal subject of the paper is the topology of a reduced plane algebraic curve $C \subset \mathbb{P}^2$ or, more generally, a curve C in a fixed algebraic surface. Typically, plane curves are considered up to one of the following major equivalence relations:

(1) $\sim_{\text{def}}$ – (*equisingular*) *deformation equivalence* (or *rigid isotopy*): $C_1 \sim_{\text{def}} C_2$ if C_1 and C_2 can be connected by a path in an equisingular stratum of the space $\mathcal{C}_d$ of plane algebraic curves of a given degree d;

(2) $\sim_{\text{PL}}$ – *PL-equivalence*: $C_1 \sim_{\text{PL}} C_2$ if the pairs $(\mathbb{P}^2, \mathbb{C}_1)$ and $(\mathbb{P}^2, C_2)$ are *PL*-homeomorphic;

(3) $\sim_{\text{top}}$ – *topological equivalence*: $C_1 \sim_{\text{top}} C_2$ if the complements $\mathbb{P}^2 \smallsetminus C_1$ and $\mathbb{P}^2 \smallsetminus C_2$ are homeomorphic;

(4) $\sim_{\text{cfg}}$ – *combinatorial equivalence*: $C_1 \sim_{\text{cfg}} C_2$ if C_1 and C_2 have the same combinatorial type of singularities (or *configuration type*); formally, this means that pairs (T_1, C_1) and (T_2, C_2) are *PL*-homeomorphic, where T_i is a regular neighborhood of C_i in $\mathbb{P}^2$, $i = 1, 2$.

Clearly, one has (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) and (4).

2.1.1. Remark. In order to avoid moduli in the case of non-simple singular points, we use *PL*-homeomorphisms rather than diffeomorphisms in Items (2) and (4). For the same reason, 'equisingular' in Item (1) is understood in the *PL*-category. (Alternatively, two singular points are considered equivalent if they have combinatorially isomorphic resolutions.)

2.1.2. Remark. The configuration type, see Item (4), of an irreducible curve is determined by its degree and the set of *PL*-types of its singularities. In general, one should take into account the degrees of the components of the curve and the distribution of the branches at the singular points among the components. For a precise combinatorial definition, see [8].

2.1.3. Remark. In Item (3), one could as well consider homeomorphisms of the pairs $(\mathbb{P}^2, C_i)$, *cf.* Item (2). However, most invariants currently used to distinguish curves take into account the complement $\mathbb{P}^2 \smallsetminus C_i$ only.

2.2. Classical problems and results. The two principal problems of topology of plane algebraic curves are the classification of curves (of a given degree, with a given set of singularities, *etc.*) up to one of the equivalence relations discussed in Subsection 2.1 and the study of the topological type of the pair $(\mathbb{P}^2, C)$ or the complement $\mathbb{P}^2 \smallsetminus C$. In the latter case, of special interest is the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus C)$, which is commonly referred to as the *fundamental group of* C; this group controls the existence of algebraic surfaces ramified at C, reducing in a sense the study of surfaces to that of curves. For the particular groups appearing in this paper, we use the following notation:

- $\mathbb{Z}_n$ is the cyclic group of order n;
- $\mathbb{D}_{2n}$ is the dihedral group of order $2n$;
- $\mathbb{S}_n$ is the symmetric group of degree n;
- $\mathbb{B}_n$ is the braid group on n strands, and $\sigma_1, \ldots, \sigma_{n-1}$ is its Artin basis;
- $Q_8 := \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$ is the *quaternion group* of order 8.

In general, the group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ is very difficult to compute. Following Zariski, Libgober [50] suggested an intermediate invariant, the *Alexander polynomial* $\Delta_C(t)$: if C is irreducible, $\deg C = d$, one can consider a resolution of singularities $\tilde{X}_d$ of the d-fold cyclic covering $X_d \to \mathbb{P}^2$ ramified at C and define $\Delta_C(t)$ as the characteristic polynomial of the deck translation action on $H_1(\tilde{X}_d; \mathbb{C})$.

As mentioned in the Introduction, most tools used in the subject are of a purely topological nature and apply to arbitrary Hurwitz curves. The two truly algebraic results that I am aware of are Nori's theorem [65] (which is the best known generalization of Zariski's conjecture on nodal curves) and a formula computing the Alexander polynomial, see Theorem 2.2.2 below.

For Nori's theorem, introduce the invariant $n(S)$ of a (type of) singularity S as follows: $n(\mathbf{A}_1) = 2$, and for any other type, $n(S)$ is the decrement of the self-intersection of the curve when S is resolved to a divisor with normal intersections. Then the following statement holds, see [65].

2.2.1. Theorem. *Given an irreducible plane curve C, if $C^2 > \sum n(S)$, the summation running over all singular points S of C, then the group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ is abelian.*

For the Alexander polynomial, given a singular germ S and a rational $r \in \mathbb{Q}$, one can define the so called *ideal of quasiadjunction* $\mathcal{J}_{S,r} \subset \mathcal{O}_S$, depending only on r and the (analytic) type of S. (For the precise definition and computation of $\mathcal{J}_{S,r}$, see the citations at the end of this paragraph.) Now, given an algebraic curve $C \subset \mathbb{P}^2$ of degree d and an integer $i \in \{0, \ldots, d-1\}$, define the *quasiadjunction sheaf of ideals* $\mathcal{J}_i \subset \mathcal{O}$ as follows: the stalk $\mathcal{J}_i|_S$ equals $\mathcal{J}_{S,i/d}$ at each singular point S of C, and $\mathcal{J}_i|_P = \mathcal{O}_P$ at any other point P. Finally, define the linear system $\mathcal{L}_i$ as the space of sections of $\mathcal{J}_i \otimes \mathcal{O}(i-3)$. Next theorem, generalizing Zariski's computation [92] for the six cuspidal sextic, appeared under various disguises in [15], [33], [51], [56], and [70].

2.2.2. Theorem. *In the notation above, one has*

$$\Delta_C(t) = \prod_{i=0}^{r} [(t - \xi^i)(t - \xi^{-i})]^{\dim H^1(\mathbb{P}^2; \mathcal{L}_i)},$$

where $\xi_i = \exp(2\pi\sqrt{-1}/d)$ is the d-th primitive root of unity.

In classical terms, the number $\dim H^1(\mathbb{P}^2; \mathcal{L}_i)$ is called the *superabundance* of $\mathcal{L}_i$; it is equal to the difference between the actual dimension of $\mathcal{L}_i$ and its virtual (or expected) dimension. (Note that the expected dimension depends on the degree and the set of singularities of C only.) As an interesting consequence, the dimensions of certain linear systems turn out to be topological invariants. A similar phenomenon is discussed in the comments to Problems 4.3.6 and 4.3.7 below.

Theorem 2.2.2 admits generalizations to curves/divisors in arbitrary surfaces; there are versions of this theorem for the multivariable Alexander polynomials and characteristic varieties. For further references, see [54] and [71].

3. Plane sextics: tools and results

Simple sextics are related to $K3$-surfaces; we briefly outline this relation and its implications to topology and geometry of sextics in Subsections 3.3 and 3.4. Non-simple sextics are dealt with in Subsection 3.2.

3.1. Classes of sextics. First, we divide reduced plane sextics into *simple* and *non-simple*, the former having simple (*i.e.*, **ADE** type) singularities only and the latter having at least one non-simple singular point. Simple sextics are related to $K3$-surfaces, whereas non-simple ones are related to rational and, in few cases, irrational ruled surfaces. As a consequence, simple sextics turn out to be much more complicated; they are the subject of most problems below.

Another important class of sextics, first discovered by Zariski [92], is formed by the so called sextics of torus type.

3.1.1. Definition. A sextic $C \subset \mathbb{P}^2$ is said to be of *torus type* if one of the following two equivalent conditions holds:

(1) C is given by an equation of the form $f_2^3 + f_3^2 = 0$, where f_2 and f_3 are homogeneous polynomials of degree 2 and 3, respectively;
(2) C is the ramification locus (assuming that it is reduced) of a projection to $\mathbb{P}^2$ of a cubic surface in $\mathbb{P}^3$.

If C is irreducible, conditions (1), (2) are also equivalent to any of the following three conditions (see [18] and [22]):

(3) the group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ factors to the reduced braid group $\mathbb{B}_3/(\sigma_1\sigma_2)^3$;
(4) the group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ factors to the symmetric group $\mathbb{S}_3 = \mathbb{D}_6$;
(5) the Alexander polynomial $\Delta_C(t)$ is nontrivial, $\Delta_C(t) \not\equiv 1$.

The fact that condition (5) is equivalent to (1) was first conjectured by Oka [34].

Remarkably, conditions (1) and (2) above remain invariant under equisingular deformations. The simplest sextic of torus type, discovered in [92], has six cusps; it is the ramification locus of a generic projection of a nonsingular cubic, and its fundamental group is $\mathbb{B}_3/(\sigma_1\sigma_2)^3 \cong \mathbb{Z}_2 * \mathbb{Z}_3$. This curve and its properties can be generalized in at least two directions. First, one can consider the so called *curves of* (p, q)*-torus type*, *i.e.*, curves of degree pq given by an equation $f_q^p + f_p^q = 0$, see [44], [66], [68], and [47] for a further generalization. Second, one can study the ramification locus of a generic projection to $\mathbb{P}^2$ of a hypersurface in $\mathbb{P}^3$ (see [60]) or, more generally, a smooth surface $X \subset \mathbb{P}^N$, see [1] for further references. Both approaches provide interesting examples of plane curves.

3.2. Sextics with a non-simple singular point. Non-simple sextics turn out to be 'simple', and most classical problems related to such curves can be solved by fairly elementary methods. The principal results and tools used are outlined here.

3.2.1. Theorem. *The equisingular deformation type of an irreducible sextic with a non-simple singular point is determined by its set of singularities*; *in other words, the relations* $\sim_{\text{def}}$ *and* $\sim_{\text{cfg}}$ *restricted to such sextics are equivalent.*

A complete proof of this theorem, although rather straightforward, has never been published. Various partial statements, as well as a complete description of the combinatorial types realized by irreducible sextics, are scattered through [14], [18], and [22]. An independent treatment of sextics of torus type, including non-simple ones, is found in [72], [73], and [76].

The fundamental groups of non-simple sextics are also all known (in the sense that their presentations can easily be written down); for irreducible curves, the groups are listed in [18] and [22]. Apart from the 21 deformation families of sextics of torus type (all but two groups being $\mathbb{B}_3/(\sigma_1\sigma_2)^3$), there are five families with $\pi_1(\mathbb{P}^2 \smallsetminus C) = \mathbb{D}_{10} \times \mathbb{Z}_3$; all other groups are abelian.

A non-simple singular point has multiplicity at least three, and the projection from this point restricts to a degree at most 3 map $C \to \mathbb{P}^1$. The braid monodromy of this projection is easily computable and gives one a fairly good understanding of the topology of the pair $(\mathbb{P}^2, C)$. The most involved (and relatively new) is the case of multiplicity three, *i.e.*, a singular point adjacent to $\mathbf{J}_{10}$ in Arnol′d's notation (simple tangency of three smooth branches). The corresponding sextics are so called *trigonal curves*; they can be treated as described in Section 5 below.

These techniques should work for reducible non-simple sextics as well. However, in view of the number of details to be taken into account, I would refrain from a precise statement and pose it as a problem. (For other applications, see also comments to Problem 3.4.4.)

3.2.2. Problem. Classify reducible non-simple sextics. Are the relations $\sim_{\text{def}}$ and $\sim_{\text{cfg}}$ equivalent for such sextics?

3.3. Classification of simple sextics. Recall that a *lattice* is a free abelian group L of finite rank supplied with a symmetric bilinear form $L \otimes L \to \mathbb{Z}$ (which is usually referred to as *product* and denoted by $x \otimes y \mapsto x \cdot y$ and $x \otimes x \mapsto x^2$). A *root* in an even lattice is a vector of square (-2); a *root system* is a negative definite lattice generated by its roots.

It is convenient to identify a set of simple singularities with a root system, replacing each singular point with the irreducible root system of the same name ($\mathbf{A}_p$, $\mathbf{D}_q$, $\mathbf{E}_6$, $\mathbf{E}_7$, or $\mathbf{E}_8$) and taking a direct sum over all singular points in the set. A *marking* of a set of simple singularities Σ is a choice of a fundamental system of roots $\sigma \subset \Sigma$. (Recall that σ can be defined as the set of walls of a fixed Weyl chamber of Σ; the elements of σ form a basis for Σ, so that $\Sigma = \mathbb{Z}\sigma$.)

3.3.1. Definition. A *configuration* (a set of *lattice data* in [83]) extending a given marked set of simple singularities (Σ, σ) is a finite index lattice extension $\tilde{S} \supset S := \Sigma \oplus \mathbb{Z}h$, $h^2 = 2$, satisfying the following conditions:

(1) each root $r \in \tilde{S} \cap (\Sigma \otimes \mathbb{Q})$ belongs to Σ;
(2) there is no root $r \in \Sigma$ such that $\frac{1}{2}(r + h) \in \tilde{S}$.

An *isomorphism* of two configurations $\tilde{S}' \supset \mathbb{Z}\sigma' \oplus \mathbb{Z}h'$ and $\tilde{S}'' \supset \mathbb{Z}\sigma'' \oplus \mathbb{Z}h''$ is a lattice isomorphism $\tilde{S}' \to \tilde{S}''$ taking h' to h'' and σ' to σ''.

From now on, we fix a lattice $\mathbb{L} \cong 2\mathbf{E}_8 \oplus 3\mathbf{U}$, where $\mathbf{U}$ is the *hyperbolic plane*, *i.e.*, the lattice spanned by two generators u, v with $u^2 = v^2 = 0$ and $u \cdot v = 1$.

3.3.2. Definition. An *abstract homological type* extending a given configuration $\tilde{S} \supset S := \Sigma \oplus \mathbb{Z}h$ is an isometry $\mathcal{H}\colon \tilde{S} \to \mathbb{L}$ taking $\tilde{S}$ to a primitive sublattice of $\mathbb{L}$. An *orientation* of $\mathcal{H}$ is an orientation $\mathfrak{o}$ of maximal positive definite subspaces in the orthogonal complement $\mathcal{H}(\tilde{S})^\perp$. An *isomorphism* of two (oriented) abstract homological types $\mathcal{H}'\colon \tilde{S}' \to \mathbb{L}$ and $\mathcal{H}''\colon \tilde{S}'' \to \mathbb{L}$ is an automorphism of $\mathbb{L}$ inducing an isomorphism $\tilde{S}' \to \tilde{S}''$ of configurations (and taking the orientation of $\mathcal{H}'$ to the orientation of $\mathcal{H}''$).

3.3.3. Remark. Any two fundamental systems of roots in Σ can be interchanged by a sequence of reflections of Σ, which would extend trivially to any larger lattice containing Σ. Hence, for the simple enumeration tasks (*cf.* Theorem 3.3.4 below), the marking can be ignored. It is, however, important in some more subtle problems, *e.g.*, in the study of automorphisms of an abstract homological type.

Let $C \subset \mathbb{P}^2$ be a simple sextic. Consider the double covering $X \to \mathbb{P}^2$ ramified at C and its minimal resolution $\tilde{X} = \tilde{X}_C \to X$. It is a $K3$-surface. Introduce the following objects:

- $\sigma_C \subset H_2(\tilde{X})$, the set of classes of the exceptional divisors contracted in X;
- $\Sigma_C \subset H_2(\tilde{X})$, the sublattice spanned by σ_C;
- $h_C \in H_2(\tilde{X})$, the pull-back of the class $[\mathbb{P}^1] \in H_2(\mathbb{P}^2)$; one has $h_C^2 = 2$;
- $\tilde{S}_C \subset H_2(\tilde{X})$, the primitive hull of $S_C := \Sigma_C \oplus \mathbb{Z}h_C$ in $H_2(\tilde{X})$;
- $T_C = S_C^\perp$, the stable *transcendental lattice* of $\tilde{X}$;
- $\omega_C \in T_C \otimes \mathbb{C}$, the class of a holomorphic form on $\tilde{X}$ (the *period* of $\tilde{X}$);
- $\mathfrak{o}_C$, the orientation of the plane $\omega_C^{\mathbb{R}} := \mathbb{R}\operatorname{Re}\omega_C \oplus \mathbb{R}\operatorname{Im}\omega_C \subset T_C \otimes \mathbb{R}$.

Then Σ_C is isomorphic to the set of singularities of C (regarded as a lattice), σ_C is a marking of Σ_C, and the extension $\tilde{S} \supset S$ is a configuration. Hence, fixing an isomorphism $\phi\colon H_2(\tilde{X}) \to \mathbb{L}$ (called a *marking* of $\tilde{X}$) and combining it with the inclusion $\tilde{S}_C \hookrightarrow H_2(\tilde{X})$, one obtains an abstract homological type $\mathcal{H}_C\colon \tilde{S}_C \to \mathbb{L}$, called the *homological type* of C. The orientation $\mathfrak{o}_C$ extends to an orientation of $\mathcal{H}_C$. The pair $(\mathcal{H}_C, \mathfrak{o}_C)$ is well defined up to the choice of a marking ϕ, *i.e.*, up to isomorphism.

Next statement is found in [17]; the existence part was first proved in [88] and then exploited in [90].

3.3.4. Theorem. *The map $C \mapsto (\mathcal{H}_C, \mathfrak{o}_C)$ establishes a bijection between the set of equisingular deformation classes of simple sextics and the set of isomorphism classes of oriented abstract homological types.*

The proof of Theorem 3.3.4, as well as of most other statements related to simple sextics, is based on the so called global Torelli theorem and surjectivity of the period map for $K3$-surfaces, see *e.g.* [9]. A convenient restatement of these two facts and a description of a fine moduli space of marked polarized $K3$-surfaces is found in [10]. As another consequence, one obtains a description of the equisingular moduli spaces. Denote by $\mathcal{C}_6(\mathcal{H}, \mathfrak{o})$ the equisingular stratum of the space of sextics defined by an oriented abstract homological type $(\mathcal{H}, \mathfrak{o})$, and let $\mathcal{M}_6(\mathcal{H}, \mathfrak{o}) = \mathcal{C}_6(\mathcal{H}, \mathfrak{o})/PGL_3$ be the corresponding moduli space. Let $T = \mathcal{H}(\tilde{S})^\perp$, and let $\Omega_{\mathcal{H}}$ be the projectivization of the cone $\{\omega \in T \otimes \mathbb{C} \,|\, \omega^2 = 0,\ \omega \cdot \bar{\omega} > 0\}$. This space has two complex conjugate connected components, which differ by the orientation of the positive definite 2-subspace $\omega^{\mathbb{R}} := \mathbb{R}\operatorname{Re}\omega \oplus \mathbb{R}\operatorname{Im}\omega \subset T \otimes \mathbb{R}$; denote by $\Omega_{\mathcal{H}}^+$ the one that matches $\mathfrak{o}$. Then the following statement holds.

3.3.5. Theorem. *The equisingular moduli space $\mathcal{M}_6(\mathcal{H}, \mathfrak{o})$ can be identified with a Zariski open subset of the quotient $\Omega_{\mathcal{H}}^+/\operatorname{Aut}(\mathcal{H}, \mathfrak{o})$. In particular, $\mathcal{M}_6(\mathcal{H}, \mathfrak{o})$ is irreducible and one has $\dim \mathcal{M}_6(\mathcal{H}, \mathfrak{o}) = 19 - \operatorname{rk}\Sigma$.*

Note that, for a sextic C, the rank $\operatorname{rk}\Sigma_C$ equals the *total Milnor number* $\mu(C)$; roughly, it is the number of algebraic conditions selecting curves with the prescribed set of singularities. It follows that for simple sextics these conditions are always independent (*cf.* Theorem 3.4.3 below).

A sextic C with $\mu(C) = 19$ is called *maximizing*. The moduli space of maximizing sextics is discrete, *i.e.*, two maximizing sextics are deformation equivalent if and only if they are related by a projective transformation. Any maximizing sextic is defined over an algebraic number field (see comments to Problem 4.3.9 below).

3.4. Further applications of theory of $K3$-surfaces. As another consequence of the description of the moduli space of $K3$-surfaces, one obtains the following statement, concerning the generic (within an equisingular family) Picard group.

3.4.1. Theorem. *There is a dense Zariski open subset $\mathcal{C}' \subset \mathcal{C}_6(\mathcal{H}, \mathfrak{o})$ such that, for a sextic $C \in \mathcal{C}'$, the Picard group* $\operatorname{Pic}\tilde{X}_C$ *equals* $\tilde{S}_C$.

In [83], a sextic C with $\operatorname{Pic}\tilde{X}_C = \tilde{S}_C$ is called *lattice generic*. In fact, for a lattice generic sextic C one can even describe the Kähler cone of $\tilde{X}_C$ (hence the semigroup of numerically effective divisors) and the set of (the classes of) irreducible rational curves on $\tilde{X}_C$, see *e.g.* [9]; this description makes use of the fundamental polyhedron of the group generated by reflections of the hyperbolic lattice $\tilde{S}$, see [89]. The key rôle is played by the Riemann–Roch theorem, which, in the case of $K3$-surfaces, takes an especially simple form and asserts that, for any divisor D with $D^2 \geqslant -2$, either D or $-D$ is effective.

Theorem 3.4.1 means that the configuration $\tilde{S}_C \supset S_C$ captures the essential algebro-geometric information invariant under equisingular deformations. This fact lead Shimada [83] to the introduction of another equivalence relation, the so called lattice equivalence.

3.4.2. Definition. Two simple sextics C_1, C_2 are *lattice equivalent*, $C_1 \sim_{\mathrm{lat}} C_2$, if their configurations $\tilde{S}_{C_1} \supset S_{C_1}$ and $\tilde{S}_{C_2} \supset S_{C_2}$ are isomorphic.

This new relation is strictly between $\sim_{\mathrm{def}}$ and $\sim_{\mathrm{cfg}}$ (see [90]) and at present it seems somewhat easier to handle. The following geometric properties of simple sextics are known to be lattice invariant: the existence of certain dihedral covering and, in particular, being of torus type (see [18] and [41]), the existence of stable symmetries (in the irreducible case, see [19] and Problem 4.3.1 below), the existence of Z-splitting lines, conics, and cubics (see [83] and comments to Problems 4.3.6 and 4.3.7 below). Most of these statements (except the existence of dihedral coverings) are based on the Riemann–Roch theorem and do not extend to higher degrees.

Another remarkable result is a clear picture of the adjacencies of the equisingular strata of the space of simple sextics. According to [57], the isomorphism classes of perturbations of a simple singularity are enumerated by the induced subgraphs of its Dynkin diagram (up to a certain equivalence). Given an abstract homological type $\mathcal{H}\colon \tilde{S} \to \mathbb{L}$, $\tilde{S} \supset S \supset \Sigma \supset \sigma$, define its *combined Dynkin diagram* $D_{\mathcal{H}}$ as the Dynkin diagram of the fundamental system of roots σ. Any induced subgraph $D' \subset D_{\mathcal{H}}$ gives rise to a fundamental system of roots $\sigma' \subset \sigma$ and hence to a configuration $\tilde{S}' := \tilde{S} \cap (S' \otimes \mathbb{Q}) \supset S'$, where $S' = \mathbb{Z}\sigma' \oplus \mathbb{Z}h$. Restricting $\mathcal{H}$ to $\tilde{S}'$, one obtains a new abstract homological type $\mathcal{H}'$, and any orientation $\mathfrak{o}$ of $\mathcal{H}$ extends to an orientation $\mathfrak{o}'$ of $\mathcal{H}'$ in the obvious way. We call the pair $(\mathcal{H}', \mathfrak{o}')$ the *restriction* of $(\mathcal{H}, \mathfrak{o})$ to the subgraph $D' \subset D_{\mathcal{H}}$. The following statement is contained in [24] (see also [83] for a more formal proof).

3.4.3. Theorem. *A stratum* $\mathcal{C}_6(\mathcal{H}, \mathfrak{o})$ *is in the closure of* $\mathcal{C}_6(\mathcal{H}', \mathfrak{o}')$ *if and only if* $(\mathcal{H}', \mathfrak{o}')$ *is isomorphic to the restriction of* $(\mathcal{H}, \mathfrak{o})$ *to an induced subgraph* $D' \subset D_{\mathcal{H}}$.

In other words, the singular points of a simple sextic can be perturbed arbitrarily and independently. This fact is in sharp contrast to the case of higher degrees. For example, for each of the following curves

– the Pappus configuration of nine lines,
– union of three cubics passing through nine common points,
– union of a three cuspidal quartic and the tangents at its three cusps,

one cannot perturb a triple point to three nodes while keeping the other singular points intact. (These examples were communicated to me by E. Shustin; note that all curves have simple singularities only.) One can construct a great deal of other examples in degrees 7 and 8 by combining a simple sextic with a Z-splitting line or conic (see [83] and comments to Problems 4.3.6 and 4.3.7 below). At present, no example if irreducible curve with restricted perturbation is known, but there is no reason to believe that such examples do not exist. Some general conditions sufficient for the singular points of a plane curve to be perturbed independently are found in [40]; however, these conditions are too weak to cover all simple sextics.

It is not difficult to prove an analogue of Theorem 3.4.3 for non-simple sextics, *provided that the non-simple points are kept non-simple and their multiplicity does not change* (so that the projection mentioned in Subsection 3.2 is preserved during the perturbation; this projection can be used to describe the degenerations). This observation gives rise to the following problem.

3.4.4. Problem. Describe the degenerations of simple sextics to non-simple ones, as well as the degenerations increasing the multiplicity of a non-simple point.

A partial answer to this question is given in [72] and [73], where degenerations are used in the study of sextics of torus type.

4. Simple sextics: problems

One can anticipate that the two principal problems concerning sextics, namely their classification and the computation of their fundamental groups, will be completely solved within a few years. However, there still remain more subtle geometric questions, see Subsection 4.3, and attempting to understand the extent to which the reach experimental material gathered for sextics reflects properties of algebraic curves in general would be of utmost interest.

4.1. The classification. We start with a few classification questions.

4.1.1. Problem. Complete the deformation classification of simple sextics.

Theorem 3.3.4 reduces this problem to a purely arithmetical question, which can be solved using Nukulin's theory [62] of discriminant forms. As a preliminary step, Yang [90] has compiled a complete list of combinatorial classes and Shimada [83] has compiled a complete list of lattice equivalence classes; the latter contains about 11,500 items. Besides, Shimada [82] has also listed all deformation classes with the maximal total Milnor number $\mu = 19$. In the case $\mu \leqslant 18$, one expects to have very few configurations extending to more than one oriented abstract homological type. (At present, only one such example is known: irreducible sextics of torus type with the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_1$ form two complex conjugate deformation families, see [75].) However, in this case one needs to deal with the automorphism groups of indefinite lattices, which are not very well known. One can hope that most such lattices can be handled using the results of [58] and [59], which give a precise description of the cokernel of the natural homomorphism $O(L) \to \operatorname{Aut} \operatorname{discr} L$ for an indefinite lattice L of rank at least three.

It is expected that the final list will contain 11.5 to 12 thousand classes. Once the list is completed, next task would be understanding the result and deriving geometric consequences. For example, as a rule, maximizing sextics are much easier to construct explicitly, and having computed their topological invariants (such as the braid monodromy and fundamental group, see comments to Problem 4.2.2 below), one can obtain the invariants of all perturbed curves. Hence, the following question is of utmost importance.

4.1.2. Problem. Find a complete list of (irreducible) simple sextics that do *not* admit a degeneration to an (irreducible) maximizing one.

This question can probably be answered using the information already at hand, namely Shimada's list [83] of lattice types. One example is an irreducible sextic with the set of singularities $9\mathbf{A}_2$: it does not admit any further degeneration (in the class of simple sextics).

4.2. The topology. Next group of problems concerns the topology and homotopy type (in particular, the fundamental group) of the complement of a simple sextic.

4.2.1. Problem. Do all pairs of not deformation equivalent (irreducible) sextics sharing the same configuration type form Zariski pairs in the sense of Artal [3], *i.e.*, do they differ topologically?

It is worth mentioning that, in the case of simple singularities only, one can replace *PL*-homeomorphisms with diffeomorphisms in the definitions of equivalence relations in Subsection 2.1; thus, it follows from [17] that the relations $\sim_{\mathrm{def}}$ and $\sim_{\mathrm{PL}}$ (see 2.1(1) and (2), respectively) are equivalent when restricted to simple sextics. Hence, the existence of a pair C_1, C_2 with $C_1 \not\sim_{\mathrm{def}} C_2$ but $C_1 \sim_{\mathrm{top}} C_2$ would reflect some subtle phenomena of topology of 4-manifolds.

Several dozens of candidates to be tried are readily found in Shimada's list [82] of maximizing sextics: one should consider the pairs C_1, C_2 that share the same configuration and transcendental lattice T_{C_i} but are not complex conjugate. Some of these pairs have been intensively studied, see [4], [5], [6], [20], [26], [27], [28], [35], and the survey [7] for further references. Most pairs are either known or expected to be Galois conjugate, see Problem 4.3.9 below. In many cases, the fundamental groups $\pi_1(\mathbb{P}^2 \smallsetminus C_i)$ have been computed; they are either equal (when finite) are are not known to be distinct (irreducible sextics of torus type with the sets of singularities $2\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2$, $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2$, $\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$, $\mathbf{E}_6 \oplus 2\mathbf{A}_5 \oplus \mathbf{A}_3$; the groups must have isomorphic profinite completions, which makes it difficult to distinguish them).

Within each pair C_1, C_2 as above, the two curves differ by a rather subtle lattice theoretic invariant, which is responsible for one of the homomorphisms in the Mayer–Vietoris exact sequence of the triad $(\tilde{X}_i \smallsetminus C_i, N_i; \partial N_i)$, where N_i is a regular neighborhood of C_i in $\tilde{X}_i = \tilde{X}_{C_i}$, $i = 1, 2$. As a first attempt, one can try to assume that $\pi_1(\mathbb{P}^2 \smallsetminus C_i) = \mathbb{Z}_6$ and draw conclusions about the homotopy types of the complements $\mathbb{P}^2 \smallsetminus C_i$ or the 6-fold cyclic coverings $Y_i \to \mathbb{P}^2 \smallsetminus C_i$ by computing the homology of Y_i (preferably as $\pi_1(\mathbb{P}^2 \smallsetminus C_i)$-modules).

4.2.2. Problem. What are the fundamental groups of irreducible simple sextics?

There is a vast literature on the subject, which calls for a separate survey. The current conjecture, suggested in [23], is the following.

4.2.3. Conjecture. *The fundamental group of an irreducible sextic that is not of torus type is finite.*

Note that the group of a sextic of torus type is never finite, *cf.* condition 3.1.1(3). Conjecture 4.2.3 replaces original Oka's conjecture [34] (its part concerning the fundamental group) that was disproved in [18]. The conjecture has been verified for about 1,500 deformation families not covered by Nori's theorem [65]. In particular, it is known to be true for the following classes:

- all non-simple sextics ([18], [22]),
- all maximizing sextics with an **E** type singular point ([5], [26], [27], [28]),
- all but one sextics whose group admits a dihedral quotient ([23], [32], [36]).

A number of sporadic examples is contained in [4], [5], [6], [34], [37], [38], and a huge number of other curves can be obtained from those already mentioned by using Theorem 3.4.3. So far, very few nonabelian fundamental groups have been found, their commutants being

$$\mathbb{Z}_5,\quad \mathbb{Z}_7,\quad SL(2,\mathbb{Z}_9)\rtimes\mathbb{Z}_5,\quad ((\mathbb{Z}_2\times Q_8)\rtimes\mathbb{Z}_2)\rtimes\mathbb{Z}_5,\quad SL(2,\mathbb{F}_5),\quad SL(2,\mathbb{F}_{19}).$$

With one possible exception (the set of singularities $3\mathbf{A}_6\oplus\mathbf{A}_1$), only three soluble groups can appear:

$$\mathbb{Z}_3\times\mathbb{D}_{10},\quad \mathbb{Z}_3\times\mathbb{D}_{14},\quad (((\mathbb{Z}_2\times Q_8)\rtimes\mathbb{Z}_2)\rtimes\mathbb{Z}_5)\rtimes\mathbb{Z}_6.$$

(Note that, being soluble, $\pi_1(\mathbb{P}^2\smallsetminus C)$ must admit a dihedral quotient.)

The fundamental groups of most (all but ten sets of singularities) irreducible sextics of torus type are also known, see [25] for a 'map' of the results and [28] for a recent update. With few exceptions, they are minimal possible, *i.e.*, $\mathbb{B}_3/(\sigma_1\sigma_2)^3$, *cf.* condition 3.1.1(3).

In order to compute the group $\pi_1(\mathbb{P}^2\smallsetminus C)$, one needs an explicit construction of C (Theorem 3.3.4 merely states the existence); then, an appropriate version of the Zariski–van Kampen theorem [43] can be used. Sextics with a triple point can be reduced to trigonal curves (see [26], [27], [28]; the case of **D** type singularities is pending), and their braid monodromy, hence fundamental group, can be computed in a purely combinatorial way (see [21] and Section 5 below for a further discussion). Sextics with an involutive symmetry can also be reduced to trigonal curves (see [23], [24], [20], [25], [67]); the computation becomes slightly more involved as one needs to keep track of an extra section. One may hope to produce a few more deformation families using reducible curves and/or unstable symmetries. For the remaining curves, those with **A** type singularities only, most promising seems the approach of [5], producing sextics with at least eight double points, possibly infinitely near. This approach is also based on a certain involutive symmetry (see comments to Problem 4.3.5), reducing the computation of the fundamental group to curves of smaller degree.

The following problem, although not dealing with plane sextics directly, is closely related to 4.2.2.

4.2.4. Problem. What are the groups $\pi_1(X \smallsetminus E)$, where X is a $K3$-surface and E is an **ADE**-configuration, *i.e.*, a configuration of (-2)-curves spanning a negative definite sublattice?

If $X = \tilde{X}_C$ for a simple sextic $C \subset \mathbb{P}^2$ and E are the exceptional divisors contracted in $\mathbb{P}^2$, then the group in question is an index 2 subgroup of $\pi_1(\mathbb{P}^2 \smallsetminus C)/\delta^2$, where δ is the class of a meridian. Problem 4.2.4 was posed in [84], where certain conditions for $\pi_1(X \smallsetminus E)$ to be trivial were found. Numerous examples of nontrivial groups are given by Xiao's approach [91] to the classification of finite groups acting symplectically on $K3$-surfaces.

4.3. Geometry of simple sextics. The developed theory of $K3$-surfaces (the global Torelli theorem, the existence of fine moduli spaces, the Riemann–Roch theorem) lets one study more subtle geometric properties of simple sextics, such as symmetries, splitting curves, minimal fields of definition, *etc.* It seems to be a common belief that, given time, any particular question concerning $K3$-surfaces can be answered (although Problems 4.3.3 and 4.3.4 involving anti-holomorphic automorphisms seem more difficult than the purely holomorphic ones). Thus, in my opinion, the most interesting part is an attempt to understand the extent to which the properties of (simple) plane sextics generalize to curves of higher degree, see *e.g.* Problem 4.3.7.

4.3.1. Problem. Describe the groups of stable (under equisingular deformations) symmetries of reducible simple sextics. Is the quotient of such a sextic by a stable involutive symmetry always a maximal trigonal/hyperelliptic curve?

By a *symmetry* of a plane curve $C \subset \mathbb{P}^2$ we mean a projective transformation of $\mathbb{P}^2$ preserving C as a set. The answer to the corresponding question for irreducible simple sextics is given in [19].

If $\mu(C) \leqslant 18$, the group of stable symmetries of C depends on the configuration $\tilde{S}_C \supset S_C$ of C only: it is the group of automorphisms of the configuration acting identically on its discriminant. If $\mu(C) = 19$, then C is rigid and all its symmetries are stable. Hence, the problem can be solved by a careful analysis of Shimada's lists of lattice types [83] and maximizing sextics [82].

4.3.2. Problem. Describe the equisingular deformation classes of simple sextics with a prescribed finite group of symmetries.

4.3.3. Problem. Describe the equivariant equisingular deformation classes of real simple sextics or, more generally, those of simple sextics with a prescribed finite Klein group of symmetries.

Recall that a *real structure* on an algebraic variety X is an anti-holomorphic involution $c\colon X \to X$. More generally, a *Klein action* of a group G on X is an action of G by both holomorphic and anti-holomorphic maps. Thus, Problem 4.3.3 is a generalization of 4.3.2.

A characterization of finite groups that may act on $K3$-surfaces is known, see [45], [46], [61], [63], and [91], and one can reduce both problems to the study of equivariant isomorphism classes of oriented abstract homological types $(\mathcal{H}, \mathfrak{o})$ with a prescribed group G of (anti-)automorphisms. (Here, an anti-automorphism is an isometry induced by an anti-holomorphic map; it should reverse h and $\mathfrak{o}$ and take σ to $-\sigma$ as a set). Note, however, that in the presence of anti-holomorphic maps the corresponding moduli space $\mathcal{M}_6(\mathcal{H}, \mathfrak{o}; G)$ does not need to be connected (see [30] for an example, with G as simple as $\mathbb{D}_6$; this example is not related to plane sextics, but it does illustrate the phenomenon). Hence, an analog of Theorem 3.3.4 does not hold in this case and Problem 4.3.3 remains meaningful even in the following simplified version (and even for nonsingular sextics).

4.3.4. Problem. Given a simple sextic $C \subset \mathbb{P}^2$ with a finite group Klein action, is its equivariant equisingular deformation type determined by the diffeomorphism type of the action on $(\mathbb{P}^2, C)$?

In the example of [30], the answer to this question is in the affirmative: the three classes differ by the topology of the real point set. The only other results concerning real sextics that I know are the classification of nonsingular sextics [62] and the classification of sextics with a single node [42].

4.3.5. Problem. Describe stable birational symmetries of simple sextics. Find a lattice theoretic description of such symmetries.

A large number of examples is found in [5], where it is shown that any sextic with at least eight double points, possibly infinitely near, admits an involutive birational symmetry. The eight double points are used to define a pencil of elliptic curves, and the involution acts on the corresponding rational elliptic surface.

4.3.6. Problem. Extend Shimada's classification [83] of Z-splitting curves. For example, are there sextics with pairs, triples, *etc.* of Z-splitting curves? (As an alternative, but probably less interesting problem, one can study simple sextics with unstable splitting curves.)

4.3.7. Problem. What is the correct generalization of the theory of Z-splitting curves to curves of higher degree? As a wild guess, are Z-splitting curves related to the (integral) Alexander modules [52] or twisted Alexander polynomials [13]?

Roughly, a nonsingular curve B is called *Z-splitting* for a simple sextic C if

(1) the pull-back $\tilde{B}$ of B in the double covering $\tilde{X}_C$ splits, and
(2) B is stable, *i.e.*, it follows equisingular deformations of C retaining the splitting property.

(If $g(B) > 0$, it is required in addition that the two components of $\tilde{B}$ should realize distinct classes in $H_2(\tilde{X}_C)$; this seems to be a purely technical assumption, as well as the requirement that B should be nonsingular or irreducible.) The existence of Z-splitting curves of small degrees is a lattice invariant; it is due to the torsion $\tilde{S}_C/S_C$, hence to the existence of certain dihedral coverings (see [18] or [83]). Thus, in the case of simple sextics, Z-splitting curves *are* related to the Alexander module. However, whereas the direct implication (Z-splitting curves $\Longrightarrow$ dihedral coverings) is of a purely topological nature, the converse relies upon the Riemann–Roch theorem for $K3$-surfaces and does not generalize directly to higher degrees. (A similar relation between the existence of Z-splitting sections and the Alexander module for trigonal curves is mentioned in Remark 5.6.2 below.)

One can also speculate that the situation with Z-splitting curves is very similar to Theorem 2.2.2, where often the conclusion that $\Delta_C(t) \neq 1$ follows from the fact that one of the linear systems $\mathcal{L}_i$ is nonempty while having negative virtual dimension, *i.e.*, from the *existence* of certain auxiliary curves. Furthermore, these auxiliary curves are stable under deformations, and often they are also splitting in an appropriate covering (*e.g.*, this is always the case for simple sextics of torus type [83], as well as for many curves considered in [66] and [44]).

In general, Z-splitting curves seem to be more complicated than those appearing in Theorem 2.2.2: the conditions are not always linear in terms of the original sextic C. (In some of the examples found in [83], Z-splitting curves are tangent to C at its smooth points.)

Another example illustrating the relation between (unstable) splitting curves and dihedral coverings was recently discovered in [87], where the rôle of plane sextics is played by trigonal curves in the Hirzebruch surface Σ_2. The splitting sections used in [87] are triple tangent to the curve at its smooth points.

4.3.8. Problem. What are the geometric properties of the equisingular moduli spaces $\mathcal{M}_6(\mathcal{H}, \mathfrak{o})$?

In principle, the spaces $\mathcal{M}_6(\mathcal{H}, \mathfrak{o})$ are given by Theorem 3.3.5. For example, it follows immediately that the moduli spaces of maximizing sextics are one point sets. The two other statements that I know are the following:

- the moduli space of sextics with the set of singularities $3\mathbf{E}_6$ splits into two rational curves (see [69]);
- each moduli space containing an irreducible sextic with a stable symmetry is unirational (see [25], [23], [24], [20], and [32] for a case by case analysis).

These results are proved geometrically rather than using Theorem 3.3.5 directly.

4.3.9. Problem. What are minimal fields of definition of maximizing sextics? Are maximizing simple sextics sharing the same configuration and transcendental lattice always Galois conjugate?

In all examples where explicit equations are known (*e.g.*, [4], [5], [6], [35], [80]; see [7] for a more complete list), the curves *are* Galois conjugate and they can be defined over an algebraic number field of minimal degree (equal to the number of conjugate curves). General upper and lower bounds to the degree of the field of definition of a singular $K3$-surface are found in [78] and [81]; however, the upper bound seems too weak to provide the minimal degree for simple sextics.

4.4. Other problems related to $K3$-surfaces. There is another class of plane curves related to $K3$-surfaces: the ramification loci of a generic projection to $\mathbb{P}^2$ of a $K3$-surface $X \subset \mathbb{P}^N$. Each deformation family is uniquely determined by a pair n, k of positive integers, so that the hyperplane section h has square $2nk^2$ (hence $N = nk^2 + 1$) and h/k is primitive in $\operatorname{Pic} X$. (There are a few exceptions corresponding to the hyperelliptic case, see [77] for details.)

4.4.1. Problem. What is the topology of the ramification locus $C \subset \mathbb{P}^2$ of a generic projection $X \to \mathbb{P}^2$ of a $K3$-surface $X \subset \mathbb{P}^N$? What is the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus C)$? The group $\pi_1(X^G)$ of the Galois closure $X^G \to X \to \mathbb{P}^2$?

A few results in this direction were recently obtained in [86], where the ramification loci were studied by degenerating $K3$-surfaces to unions of planes. As a common generalization, one can also state the following problem.

4.4.2. Problem. What is the topology of the ramification locus $C \subset \mathbb{P}^2$ of a generic projection $X \to \mathbb{P}^2$ of a *singular* $K3$-surface $X \subset \mathbb{P}^N$?

In what concerns the classification, one can prove a theorem similar to 3.3.4, replacing h with $h^2 = 2$ in Definition 3.3.1 with a primitive (in $\tilde{S}$) element h' with

$(h')^2 = 2n$ and adjusting condition 3.3.1(2) appropriately, see [77]. However, it is not quite clear what the ramification locus is, especially when the surface is rigid.

5. Trigonal curves and elliptic surfaces

Let $C \subset \mathbb{P}^2$ be a plane curve of degree d with a distinguished singular point S of multiplicity $(d-3)$ (*e.g.*, a sextic with a triple point). Blow S up to obtain the ruled surface $\mathbb{P}^2(S) \cong \Sigma_1$. Then, the proper transform of C is a curve $C' \subset \Sigma_1$ intersecting each fiber of the ruling at three points, *i.e.*, a generalized trigonal curve. In this section, we outline a combinatorial approach to such curves and covering elliptic surfaces (which play the rôle of covering $K3$-surfaces for simple sextics).

For the sake of simplicity, in this paper we consider the case of *rational base only.* Most results cited below extend more or less directly to trigonal curves and elliptic surfaces over an arbitrary base B provided that it is considered as a *topological surface*, *i.e.*, the analytic structure on B is allowed to vary during the deformations.

5.1. Trigonal curves. The *Hirzebruch surface* is a geometrically ruled rational surface $\Sigma_k \to \mathbb{P}^1$ with an *exceptional section* E of square $-k$, $k \geqslant 0$. A *generalized trigonal curve* is a curve $C \subset \Sigma_k$ intersecting each generic fiber at three points. A *trigonal curve* is a generalized trigonal curve disjoint from E.

In what follows, *we assume that the curves do not contain as components fibers of the ruling.* Furthermore, given a generalized trigonal curve $C \subset \Sigma_k$, the points of intersection $C \cap E$ can be removed by a sequence of elementary transformations $\Sigma_k \dashrightarrow \Sigma_{k+1}$. Similarly, all non-simple singular points of C can be converted to simple ones by inverse elementary transformations $\Sigma_k \dashrightarrow \Sigma_{k-1}$. For this reason, *we will consider simple trigonal curves only.* (In fact, it would even suffice to deal with trigonal curves with double singular points only. Certainly, in the general case one would need to keep track of the vertical components and the elementary transformations used.)

If k is even, the minimal resolution of singularities $\tilde{X}_C$ of the double covering of Σ_k ramified at C and E is a relatively minimal Jacobian elliptic surface. Conversely, given a Jacobian elliptic surface X, the quotient $X/\pm 1$ contracts to Σ_k and is ramified at E and a trigonal curve C. With an abuse of the language, we call $\tilde{X}_C$ the elliptic surface ramified at C and C the ramification locus of X. If k is odd, the surface $\tilde{X}_C$ is only defined locally with respect to the base of the ruling. A *singular fiber* of C is the projection of a singular fiber of $\tilde{X}_C$. For the topological types of singular fibers, we use the common notation referring to the extended Dynkin graphs of exceptional divisors (see Table 1), the advantage being the fact that it reflects the types of the singular points of C. The fibers of type $\tilde{\mathbf{A}}_0^{**}$, $\tilde{\mathbf{A}}_1^{*}$, and $\tilde{\mathbf{A}}_2^{*}$ are called *unstable*; their types are not necessarily preserved under equisingular deformations of C.

The *j-invariant* $j_C\colon \mathbb{P}^1 \to \mathbb{P}^1$ is the analytic continuation of the function sending a nonsingular fiber of $\tilde{X}_C$ to its j-invariant (divided by 12^3). By definition, j_C is a birational invariant; hence it extends to generalized/non-simple trigonal curves. If $j_C = \mathrm{const}$, the curve is called *isotrivial*; such curves form a very restricted class. From now on, we assume all curves non-isotrivial.

5.1.1. Definition. A non-isotrivial trigonal curve C is called *maximal* if it has the following properties:

(1) C has no singular fibers of type $\tilde{\mathbf{D}}_4$;

TABLE 1. Types of singular fibers

Type of F		$j(F)$	Vertex	Valency
$\tilde{\mathbf{A}}_p$ ($\tilde{\mathbf{D}}_{p+5}$), $p \geqslant 1$	I_{p+1} (I^*_{p+1})	∞	$\times$	$2(p+1)$
$\tilde{\mathbf{A}}^*_0$ ($\tilde{\mathbf{D}}_5$)	I_1 (I^*_1)	∞	$\times$	2
$\tilde{\mathbf{A}}^{**}_0$ ($\tilde{\mathbf{E}}_6$)	II (II*)	0	$\bullet$	2 mod 6
$\tilde{\mathbf{A}}^*_1$ ($\tilde{\mathbf{E}}_7$)	III (III*)	1	$\circ$	2 mod 4
$\tilde{\mathbf{A}}^*_2$ ($\tilde{\mathbf{E}}_8$)	IV (IV*)	0	$\bullet$	4 mod 6

Comments. Fibers of type $\tilde{\mathbf{A}}_0$ are not singular; fibers of type $\tilde{\mathbf{D}}_4$ are not detected by the j-invariant. Fibers of type $\tilde{\mathbf{A}}_0$ or $\tilde{\mathbf{D}}_4$ with complex multiplication of order 2 (respectively, 3) are over the $\circ$-vertices of valency 0 mod 4 (respectively, over the $\bullet$-vertices of valency 0 mod 6). The types shown parenthetically are obtained from the corresponding $\tilde{\mathbf{A}}$-types by an elementary transformation.

(2) $j = j_C$ has no critical values other than 0, 1, and ∞;
(3) each point in the pull-back $j^{-1}(0)$ has ramification index at most 3;
(4) each point in the pull-back $j^{-1}(1)$ has ramification index at most 2.

A Jacobian elliptic surface is *maximal* if its ramification locus is maximal. A maximal surface is *extremal* if it has no unstable fibers.

The more topological definition of extremal elliptic surfaces given above fits better into the framework of this survey; in fact, it is the content of [64].

Maximal trigonal curves and maximal elliptic surfaces are defined over algebraic number fields. Maximal curves are indeed maximal in the sense of the total Milnor number, see [26]: for a non-isotrivial curve $C \subset \Sigma_k$ one has

$$\mu(C) \leqslant 5k - 2 - \#\{\text{unstable fibers of } C\},$$

the equality holding if and only if C is maximal. (Thus, truly maximal are curves without unstable fibers, *i.e.*, those corresponding to extremal elliptic surfaces.)

5.2. Dessins. A *trichotomic graph* is a directed graph $\Gamma \subset S^2$ decorated with the following additional structures (called the *colorings* of the edges and vertices of Γ):

– each edge of Γ is of one of the three kinds: solid, bold, or dotted;
– each vertex of Γ is of one of the four kinds: $\bullet$, $\circ$, $\times$, or monochrome (the vertices of the first three kinds being called *essential*)

and satisfying the following conditions:

(1) the valency of each essential vertex is at least 2, and the valency of each monochrome vertex is at least 3;
(2) the orientations of the edges of Γ form an orientation of $\partial(S^2 \smallsetminus \Gamma)$;
(3) all edges incident to a monochrome vertex are of the same kind;
(4) $\times$-vertices are incident to incoming dotted edges and outgoing solid edges;
(5) $\bullet$-vertices are incident to incoming solid edges and outgoing bold edges;
(6) $\circ$-vertices are incident to incoming bold edges and outgoing dotted edges.

In (4)–(6) the lists are complete, *i.e.*, vertices cannot be incident to edges of other kinds or with different orientation. $\bullet$-vertices of valency 0 mod 6 and $\circ$-vertices of valency 0 mod 4 are called *nonsingular*; all other essential vertices are *singular*.

A path in a trichotomic graph Γ is *monochrome* if all its vertices are monochrome. The graph Γ is said to be *admissible* if it has no oriented monochrome cycles. A *dessin* is an admissible trichotomic graph.

Let $\Gamma \subset S^2$ be a trichotomic graph, and let v be a vertex of Γ. Pick a regular neighborhood $U \ni v$ and replace the intersection $\Gamma \cap U$ with another decorated graph, so that the result Γ' is again a trichotomic graph. If $\Gamma' \cap U$ contains essential vertices of at most one kind and contains no monochrome vertices, then Γ' is called a *perturbation* of Γ (at v), and the original graph Γ is called a *degeneration* of Γ'. The perturbation Γ' as above is called *equisingular* if the intersection $\Gamma' \cap U$ has at most one singular vertex. (In this case, Γ is an *equisingular degeneration* of Γ'.) Two dessins $\Gamma', \Gamma'' \subset S^2$ are said to be *equivalent* if they can be connected by a chain $\Gamma' = \Gamma_0, \Gamma_1, \ldots, \Gamma_n = \Gamma''$ of dessins so that each Γ_i, $1 \leqslant i \leqslant n$, either is isotopic to Γ_{i-1} or is an equisingular perturbation or degeneration of Γ_{i-1}. Clearly, equivalence of dessins is an equivalence relation.

Following Orevkov [74], define the dessin Γ_C of a trigonal curve C as follows. As a set, $\Gamma_C = j_C^{-1}(\mathbb{P}^1_{\mathbb{R}})$; the $\bullet$-, $\circ$-, and $\times$-vertices are the pull-backs of 0, 1, and ∞, respectively (monochrome vertices being the critical points of j_C with other real critical values), the edges are solid, bold, or dotted provided that their images belong to $[\infty, 0]$, $[0, 1]$, or $[1, \infty]$, respectively, and the orientation of the edges is that induced from the positive orientation of $\mathbb{P}^1_{\mathbb{R}}$ (*i.e.*, order of $\mathbb{R}$).

The relation between the vertices of Γ_C and singular fibers of C is outlined in Table 1; it is this relation that motivates the above definition of *equisingular* perturbation of dessins.

Next theorem, based on the Riemann existence theorem, is motivated by [74]. Its proof is essentially contained on [31], see also [21]. A similar statement holds for generalized trigonal curves and those with triple or even non-simple singular points: one merely needs to consider dessins with appropriately marked vertices in order to keep track of the elementary transformations, *cf.* Subsection 5.1.

5.2.1. Theorem. *The map $C \mapsto \Gamma_C$ sending a trigonal curve C to its dessin establishes a bijection between the set of fiberwise deformation classes of trigonal curves* with double singular points only *and the set of equivalence classes of dessins.*

As a consequence, one obtains the following important property of trigonal curves that makes them similar to plane sextics (*cf.* Theorem 3.4.3): *the singular fibers of a trigonal curve can be perturbed arbitrarily and independently*. This statement can be proved by the standard 'cut-and-paste' techniques used to show that any elliptic surface deforms to a generic one. Alternatively, it can easily be proved using dessins. Note though that, when a triple point or an unstable fiber is perturbed, the j-invariant may change discontinuously, increasing its degree, so that a single vertex of the dessin (the one representing the fiber perturbed) is removed and replaced with a fragment containing essential vertices of all three kinds.

5.3. Skeletons. Dessins of maximal trigonal curves are called *maximal*. Such dessins do not admit nontrivial degenerations. Therefore, two maximal dessins are equivalent if and only if they are isotopic. A convenient way to encode maximal dessins is using *skeletons*, which are in fact the *dessins d'enfants* in Grothendieck's original sense. By definition, the skeleton Sk_C is obtained from the dessin Γ_C by removing all $\times$-vertices, solid and dotted edges, and $\circ$-vertices of valency 2. The skeleton Sk_C of a maximal trigonal curve C has the following properties:

(1) Sk_C is connected;
(2) the valency of each $\bullet$-vertex is at most 3;
(3) the valency of each $\circ$-vertex is 1;
(4) there is at least one $\bullet$-vertex.

Conversely, any graph $\mathrm{Sk} \subset S^2$ satisfying conditions (1)–(4) above extends to a unique maximal dessin: place a $\circ$-vertex at the center of each edge connecting two $\bullet$-vertices, place a $\times$-vertex v_R at the center of each region R of the complement $S^2 \smallsetminus \mathrm{Sk}$, and connect v_R to the vertices in ∂R by appropriate edges in the star like manner. (Note that $R \cong \mathrm{Cone}\,\partial R$ is an open topological disk due to (1).) Thus, next statement is an immediate consequence of Theorem 5.2.1.

5.3.1. Theorem. *The map $C \mapsto \mathrm{Sk}_C$ establishes a one-to-one correspondence between the set of fiberwise deformation classes of maximal trigonal curves with double singular points only and the set of orientation preserving homeomorphism classes of graphs $\mathrm{Sk} \subset S^2$ satisfying conditions (1)–(4) above.*

5.4. Classification of dessins. As in the case of plane sextics, we start with a few classification problems, some easy, some hopeless.

5.4.1. Problem. Find combinatorial invariants of dessins.

Since the equivalence of dessins defined in Subsection 2.1 involves perturbations and degenerations, it is difficult to decide whether two non-maximal dessins are equivalent. One approach would be to try to compute the fundamental groups, Alexander modules/polynomials, transcendental lattices, and similar invariants of the related trigonal curves and elliptic surfaces, see Subsections 5.5 and 5.6. However, known invariants do not always distinguish dessins, even maximal, see 5.6.

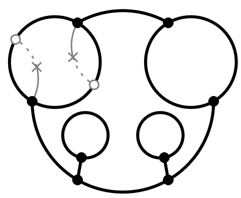

FIGURE 1. Skeleton Sk (black) for Problem 5.4.2

As another example, start with the skeleton Sk shown in black in Figure 1 and convert it to a dessin Γ by placing *two* monovalent $\times$-vertices inside one of the two bigons (as shown in grey in Figure 1) and a single $\times$-vertex inside each other region (*cf.* Subsection 5.3). Let $C \subset \Sigma_4$ be a trigonal curve whose dessin is Γ. It is birationally equivalent (the transformation being uniquely determined by the curve) to an irreducible plane sextic C' not of torus type and with the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_1$. Such sextics are known to form two complex conjugate equisingular strata, see [75].

5.4.2. Problem. Find a direct combinatorial proof of the fact that the dessin Γ described above is not equivalent to its mirror image.

5.4.3. Problem. Classify maximal dessins.

This problem is too vague and too hopeless. Even the simple counting of the number of skeletons with given sizes of the regions is a very difficult task: it is a special case of the so called *triple Hurwitz numbers.* The study of the Hurwitz numbers is a separate intensively developing area of mathematics: they are related to characters of symmetric groups, moduli spaces of stable curves, Gromov–Witten invariants, and more. For further references on Hurwitz numbers, see [39] or [79]. Even the asymptotic behavior of these numbers is of a certain interest: for example, one can construct exponentially large Zariski k-plets of trigonal curves (see [21] and comments to Theorem 5.6.4 below).

The complete classification of skeletons with $2k$ vertices for $k \leqslant 2$ is easy and well known (see, *e.g.*, [19]). A partial classification for $k = 3$ or 4 is found in [26], [27], and [28], where only the dessins arising from sextics with an $\mathbf{E}$ type singular point are considered. Next, and probably last, case would be that of $K3$-surfaces.

5.4.4. Problem. Classify extremal elliptic $K3$-surfaces.

This is equivalent to the study of maximal trigonal curves $C \subset \Sigma_k$ with t triple singular points such that $k+t = 4$. The case of $t = 0$ and stable singular fibers only is contained in [11], and the case $t \geqslant 2$ follows from the known results for $k \leqslant 2$. The remaining case $k = 3$, $t = 1$ is still pending.

5.5. Topology of trigonal curves. The class realized by a trigonal curve $C \subset \Sigma_k$ is $[C] = 3[E] + 3k[F]$, where E and F are, respectively, the exceptional section and a generic fiber of Σ_k. Hence, there is a unique cyclic covering of Σ_k ramified at C, and this covering is 3-fold. Although the latter might be well worth studying, it is nevertheless customary, when speaking about the fundamental group, Alexander module, *PL*-equivalence, *etc.* of a (generalized) trigonal curve $C \subset \Sigma_k$, to *refer to the corresponding invariants of the complement* $\Sigma_k \smallsetminus (C \cup E)$ *or pair* $(\Sigma_k, C \cup E)$.

5.5.1. Problem. What finite groups G can appear as the fundamental groups of (generalized) trigonal curves? What finite commutants $[G, G]$ can appear?

The corresponding problem for hyperelliptic curves in Hirzebruch surfaces (hence for plane curves of degree d with a singular point of multiplicity $(d - 2)$) has been solved in [16]. Apart from the cyclic groups $\mathbb{Z}_r$, only three finite groups can appear as the commutant $[G, G]$: Q_8, $SL(2, \mathbb{F}_3)$, and $SL(2, \mathbb{F}_5)$.

In the case of trigonal curves, the braid monodromy, hence a presentation of G, are determined by the dessin in a simple combinatorial way, see [21]. (In the case of generalized trigonal curves, additional care should be taken about the points at the exceptional section, *cf.* [26].) However, the situation is more complicated as, first, there is a huge number of non-equivalent dessins (see comments to Problem 5.4.3) and, second, the monodromy takes values in the non-abelian group $\mathbb{B}_3$, which makes the further analysis of the presentations rather difficult. As far as I know, only very few special cases (mainly those arising from plane sextics, see [26], [27], and [28]) have been tried so far. The finite commutants discovered are $\mathbb{Z}_5$, $SL(2, \mathbb{F}_5)$, and $SL(2, \mathbb{F}_{19})$. (For reducible curves, one can also encounter $\mathbb{Z}_3$ and Q_8.) Among other examples is a generalized trigonal curve in Σ_1 that blows down to an irreducible quintic with the set of singularities $3\mathbf{A}_4$; the commutant of its fundamental group is a non-split central extension of $(\mathbb{Z}_2)^2$ by $(\mathbb{Z}_2)^4$, see [16].

An answer to Problem 5.5.1 would shed a new light on the general problem of realizability of finite groups as fundamental groups of plane curves, see [54], [55]. Certainly, this approach has its limitations; for example, all groups admit a presentation with at most three generators. (Note though that the natural basis for G given by the Zariski–van Kampen method using the ruling of Σ_k is not necessarily a C-basis in the sense of Kulikov [48].)

5.5.2. Problem. Compute the Alexander like invariants, such as the Alexander polynomial [50], Alexander module [52], other invariants based on representations of the braid group [53], *etc.* of a generalized trigonal curve in terms of its dessin.

Of course, one can compute the fundamental group (or even braid monodromy) and use Fox calculus or other appropriate tools of combinatorial group theory. However, for simple invariants, one would expect a simple combinatorial expression.

It is worth mentioning that there is an analog of Theorem 2.2.2 for trigonal curves (as well as for curves on any surface). At present, it is not quite clear how the linear systems $\mathcal{L}_i$ fit into the description of curves in terms of dessins.

5.5.3. Problem. Find a relation between the linear systems $\mathcal{L}_i$ and dessins.

5.5.4. Problem. Do maximal trigonal curves sharing the same combinatorial type of singular fibers but not related by a fiberwise equisingular deformation differ topologically, *i.e.*, do they form Zariski pairs in the sense of Artal [3]?

As mentioned above, there are large collections of not deformation equivalent maximal trigonal curves sharing the same combinatorial type. A few examples, growing exponentially in the degree, are found in [21]. In Subsection 5.6 below it is explained that these curves share the same fundamental group and homological type; I do not know whether they differ topologically.

5.6. Topology and geometry of elliptic surfaces. We conclude with a few problems related to the elliptic surfaces ramified at trigonal curves (which can be regarded as analogues of the $K3$-surfaces ramified at simple sextics). In addition to a few general questions, we announce a new result (Theorem 5.6.4) and discuss related problems concerning a special class of extremal elliptic surfaces, namely those defined by the so called pseudo-trees.

5.6.1. Problem. Do extremal elliptic surfaces sharing the same combinatorial type of singular fibers but not related by a fiberwise equisingular deformation differ topologically?

This problem is closely related to 5.5.4: one can try to use double covering to distinguish curves. In analogy to simple sextics, define the *homological type* of a relatively minimal Jacobian elliptic surface X as the pair of lattice extensions

$$H_2(X) \supset \tilde{S}_X \supset S_X := \mathbb{Z}\sigma_X + \mathbb{Z}[E_X] + \mathbb{Z}[F_X],$$

where

- σ_X is the set of classes realized by the (-2)-curves in the fibers of X,
- E_X and F_X are, respectively, the section and a generic fiber of X, and
- $\tilde{S}_X = H_2(X) \cap (S_X \otimes \mathbb{Q})$ is the primitive hull of S_X.

(It is no longer required that each root of $\tilde{S}_X \cap \mathbb{Q}\sigma_X$ should belong to $\mathbb{Z}\sigma_X$.) An *isomorphism* of homological types is a bijective isometry of $H_2(X)$ (regarded as an abstract lattice) respecting σ_X (as a set), $[E_X]$, and $[F_X]$.

5.6.2. Remark. Another similarity between trigonal curves and plane sextics is a relation between the existence of splitting (Z-splitting?) sections and the Alexander module, *cf.* comments to Problem 4.3.7. (Here, by a *section* we mean a section of the Hirzebruch surface Σ_k containing the curve.) Indeed, according to [85], the quotient $\tilde{S}_X/S_X$ equals the torsion of the Mordell–Weil group $MW(X)$, and each nontrivial element of $MW(X)$, unless it is a component of the ramification locus C of X, projects to a splitting section of C. On the other hand, a simple homological computation shows that the same quotient $\tilde{S}_X/S_X$ controls the existence of certain dihedral coverings ramified at $C + E$, *cf.* [18], [22], or [83].

The orthogonal complement $T_X := \tilde{S}_X^\perp$ is called the *transcendental lattice* of X; it is positive definite if and only if X is extremal. The subset $\sigma'_X := \sigma_X \cap [E_X]^\perp$ is a root system; it encodes the combinatorial type of singular fibers of X (assuming that X has no unstable fibers). According to Nikulin's theory of discriminant forms [62], the isomorphism class of a homological type is determined by σ'_X, T_X (which should 'match' σ'_X), and a certain set of finite data that determines the finite index lattice extension $H_2(X) \supset S_X \oplus T_X$.

5.6.3. Problem. Describe the homological types of extremal elliptic surfaces; in particular, describe their transcendental lattices.

A few attempts to attack this problem were recently made in [2] and [29]. Some of the results of [29] are rather discouraging: one has the following theorem.

5.6.4. Theorem. *Let X be a Jacobian elliptic surface with one of the following combinatorial types of singular fibers*:

(1) $\tilde{\mathbf{A}}_{10s-2} \oplus (2s+1)\tilde{\mathbf{A}}_0^*$, $s \geqslant 1$;
(2) $\tilde{\mathbf{D}}_{10s-2} \oplus (2s)\tilde{\mathbf{A}}_0^*$, $s \geqslant 1$;
(3) $\tilde{\mathbf{D}}_{10s+3} \oplus \tilde{\mathbf{D}}_5 \oplus (2s)\tilde{\mathbf{A}}_0^*$, $s \geqslant 1$;
(4) $\tilde{\mathbf{A}}_{10s-7} \oplus \tilde{\mathbf{D}}_5 \oplus (2s-1)\tilde{\mathbf{A}}_0^*$, $s \geqslant 1$.

Then, within each of the four series, the isomorphism class of the transcendental lattice T_X is determined by s only.

In Theorem 5.6.4, one has $T_X \cong -\mathbf{D}_{2s-2}$ in case (2) and $T_X \cong -\mathbf{D}_{2s-1} \oplus \mathbb{Z}w$, $w^2 = 4$, in case (3) (where, as usual, we let $\mathbf{D}_0 = 0$, $\mathbf{D}_1 = [-4]$, $\mathbf{D}_2 = 2\mathbf{A}_1$, and $\mathbf{D}_3 = \mathbf{A}_3$). A precise description of T_X in the other two cases is also known, but it is more complicated.

The surfaces in Theorem 5.6.4 are extremal. The skeleton of each surface is a *pseudo-tree*, *i.e.*, it is obtained from a plane tree $\Xi \subset S^2$ with all vertices of valency 3 (*nodes*) or 1 (*leaves*) by patching each leaf with a small loop (see Figure 2, where the original tree Ξ is shown in black and the loops attached, in grey). The four series in the theorem differ by the number of vertices in Ξ ($4s$ in cases (1) and (3) or $4s-2$ in cases (2) and (4)) and the number of type $\tilde{\mathbf{D}}_5$ singular fibers over the $\times$-vertices in the regions inside the loops attached to the tree (*cf.* the passage from a skeleton to a dessin in Subsection 5.3; the number in question is none in cases (1) and (2) or one in cases (3) and (4)).

The number of pseudo-trees grows exponentially in s; for example, in cases 5.6.4(3) and (4), where the possible automorphisms of the tree are eliminated by distinguishing one of the leaves, the count is given by the Catalan numbers $C(2s-1)$ and $C(2s-2)$, respectively, see [21] for details. On the other hand, the number of automorphisms of the discriminant form $\operatorname{discr} T_X$ (which may lead to distinct

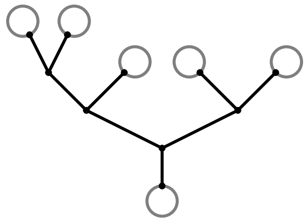

FIGURE 2. Converting a tree Ξ (black) to a skeleton

homological types) grows at most linearly. (This number is given by 2^m, where $m \leqslant \log|\operatorname{discr} T_X|$ is the number of distinct primes dividing $|\operatorname{discr} T_X|$.) Thus, we obtain exponentially large collections of extremal elliptic surfaces sharing the same homological type.

We conclude with three problems specific to pseudo-trees.

5.6.5. Problem. What are the transcendental lattices of extremal elliptic surfaces defined by pseudo-trees and with more than one singular fiber of type $\tilde{\mathbf{D}}_5$? Are they determined by the number of vertices and $\tilde{\mathbf{D}}_5$ type fibers only?

5.6.6. Problem. Are the surfaces in Theorem 5.6.4 Galois conjugate (within a fixed series and for a fixed value of s)?

5.6.7. Problem. Are the braid monodromies of maximal trigonal curves defined by pseudo-trees Hurwitz equivalent (see *e.g.* [48])?

It is shown in [29] that, with very few exceptions, the fundamental groups of maximal trigonal curves defined by pseudo-trees are abelian, hence they do not distinguish the braid monodromies.

For a simple trigonal curve $C \subset \Sigma_k$, the Hurwitz equivalence class of its braid monodromy determines and is determined by the fiberwise diffeomorphism class of the pair $(\Sigma_k, C \cup E)$, *cf.* [12] and [49]. In general, the problem of distinguishing the Hurwitz equivalence classes seems very difficult, but in this particular case, where the monodromy takes values in the relatively simple group $\mathbb{B}_3$, one may hope to get a reasonable solution. For example, the transcendental lattice T_X defined above can be reconstructed directly from the braid monodromy, *cf.* [2] and [29], and it is obviously invariant under Hurwitz equivalence. Thus, one can generalize Problem 5.6.7 as follows.

5.6.8. Problem. Given a braid monodromy $\beta_1, \ldots, \beta_r \in \mathbb{B}_3$ with all β_i conjugate to σ_1, is its Hurwitz equivalence class determined by the product $\beta_1 \ldots \beta_r \in \mathbb{B}_3$ and the transcendental lattice T? By the product $\beta_1 \ldots \beta_r$ only?

In Orevkov's terminology, the last question can be restated as follows: is the natural map from the $\mathbb{B}_3$ valued braid monodromy monoid to $\mathbb{B}_3$ injective?

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