TOWARDS THE GENERALIZED
SHAPIRO AND SHAPIRO CONJECTURE

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To my teacher Oleg Viro at his 60th birthday

ABSTRACT. We find a new, asymptotically better, bound $g \leqslant \frac{1}{4}d^2 + O(d)$ on the genus of a curve that may violate the generalized total reality conjecture. The bound covers all known cases except $g = 0$ (the original conjecture).

1. INTRODUCTION

The original (rational) total reality conjecture suggested by B. and M. Shapiro in 1993 states that, if all flattening points of a regular curve $\mathbb{P}^1 \to \mathbb{P}^n$ belong to the real line $\mathbb{P}^1_{\mathbb{R}} \subset \mathbb{P}^1$, then the curve can be made real by an appropriate projective transformation of $\mathbb{P}^n$. (The *flattening points* are the points in the source $\mathbb{P}^1$ where the first n derivatives of the map are linearly dependent. In the case $n = 1$, a curve is a meromorphic function and the flattening points are its critical points.) There is a number of interesting and not always straightforward restatements of this conjecture, in terms of the Wronsky map, Schubert calculus, dynamical systems, *etc.* Although supported by extensive numerical evidence, the conjecture proved extremely difficult to settle. It was not before 2002 that the first result appeared, due to A. Eremenko and A. Gabrielov [4], settling the case $n = 1$, *i.e.*, meromorphic functions on $\mathbb{P}^1$. Later, a number of sporadic results were announced, and the conjecture was proved in full generality in 2005 by E. Mukhin, V. Tarasov, and A. Varchenko, see [6]. The proof, revealing a deep connection between Schubert calculus and theory of integrable system, is based on the Bethe ansatz method in the Gaudin model.

In the meanwhile, a number of generalizations of the conjecture were suggested. In this paper, we deal with one of them, see [3] and Problem 1.1 below, replacing the source $\mathbb{P}^1$ with an arbitrary compact complex curve (but, however, restricting n to 1, *i.e.*, to the case of meromorphic functions). Due to the lack of evidence, the authors chose to state the assertion as a problem rather than a conjecture.

Recall that a *real variety* is a complex algebraic (analytic) variety X supplied with a *real structure*, *i.e.*, an anti-holomorphic involution $c: X \to X$. Given two real varieties (X, c) and (Y, c'), a regular map $f: X \to Y$ is called *real* if it commutes with the real structures: $f \circ c = c' \circ f$.

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1.1. Problem (see [3]). *Let (C, c) be a real curve and let $f\colon C \to \mathbb{P}^1$ be a regular map such that*:

(1) *all critical points and critical values of f are pairwise distinct;*
(2) *all critical points of f are real.*

Is it true that f is real with respect to an appropriate real structure in $\mathbb{P}^1$?

The condition that the critical points of f are distinct includes, in particular, the requirement that each critical point is simple, *i.e.*, has ramification index 2.

A pair of integers $g \geqslant 0$, $d \geqslant 1$ is said to have the *total reality property* if the answer to Problem 1.1 is in the affirmative for any curve C of genus g and map f of degree d. At present, the total reality property is known for the following pairs (g, d):

– $(0, d)$ for any $d \geqslant 1$ (the original conjecture, see [4]);
– (g, d) for any $d \geqslant 1$ and $g > G_1(d) := \frac{1}{3}(d^2 - 4d + 3)$, see [3];
– (g, d) for any $g \geqslant 0$ and $d \leqslant 4$, see [3] and [1].

The principle result of the present paper is the following theorem.

1.2. Theorem. *Any pair (g, d) with $d \geqslant 1$ and g satisfying the inequality*

$$g > G_0(d) := \begin{cases} k^2 - 2k, & \text{if } d = 2k \text{ is even,} \\ k^2 - \dfrac{10}{3}k + \dfrac{7}{3}, & \text{if } d = 2k - 1 \text{ is odd} \end{cases}$$

has the total reality property.

1.3. Remark. Note that one has $G_0(d) - G_1(d) \leqslant -\frac{1}{3}(k-1)^2 \leqslant 0$, where $k = [\frac{1}{2}(d+1)]$. Theorem 1.2 covers the values $d = 2, 3$ and leaves only $g = 0$ for $d = 4$, reducing the generalized conjecture to the classical one. The new bound is also asymptotically better: $G_0(d) = \frac{1}{4}d^2 + O(d) < G_1(d) = \frac{1}{3}d^2 + O(d)$.

1.4. Contents of the paper. In §2, we outline the reduction of Problem 1.1 to the question of existence of certain real curves on the ellipsoid and restate Theorem 1.2 in the new terms, see Theorem 2.6. In §3, we briefly recall V. V. Nikulin's theory of discriminant forms and lattice extensions. In §4, we introduce a version of the Alexander module of a plane curve suited to the study of the resolution lattice in the homology of the double covering of the plane ramified at the curve. Finally, in §5, we prove Theorem 2.6 and, hence, Theorem 1.2.

2. The reduction

We briefly recall the reduction of Problem 1.1 to the problem of existence of a certain real curve on the ellipsoid. Details are found in [3].

2.1. Denote by conj: $z \mapsto \bar{z}$ the standard real structure on $\mathbb{P}^1 = \mathbb{C} \cup \infty$. The *ellipsoid* **E** is the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ with the real structure $(z, w) \mapsto (\operatorname{conj} w, \operatorname{conj} z)$. (It is indeed the real structure whose real part is homeomorphic to the 2-sphere.)

Let (C, c) be a real curve and let $f \colon C \to \mathbb{P}^1$ be a holomorphic map. Consider the *conjugate map* $\bar{f} = \mathrm{conj} \circ f \circ c \colon C \to \mathbb{P}^1$ and let

$$\Phi = (f, \bar{f}) \colon C \to \mathbf{E}.$$

It is straightforward that Φ is holomorphic and real (with respect to the above real structure on $\mathbf{E}$). Hence, the image $\Phi(C)$ is a real algebraic curve in $\mathbf{E}$. (We exclude the possibility that $\Phi(C)$ is a point as we assume $f \neq \mathrm{const}$, *cf.* Condition 1.1(1).) In particular, the image $\Phi(C)$ has bi-degree (d', d') for some $d' \geqslant 1$.

2.2. Lemma (see [3]). *A holomorphic map $f \colon C \to \mathbb{P}^1$ is real with respect to some real structure on $\mathbb{P}^1$ if and only if there is a Möbius transformation $\varphi \colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $\bar{f} = \varphi \circ f$.* $\square$

2.3. Corollary (see [3]). *A holomorphic map $f \colon C \to \mathbb{P}^1$ is real with respect to some real structure on $\mathbb{P}^1$ if and only if the image $\Phi(C) \subset \mathbf{E}$, see above, is a curve of bi-degree $(1, 1)$.* $\square$

2.4. Let $p \colon \mathbf{E} \to \mathbb{P}^1$ be the projection to the first factor. In general, the map Φ as above splits into a ramified covering α and a generically one-to-one map β,

$$\Phi \colon C \xrightarrow{\alpha} C' \xrightarrow{\beta} \mathbf{E},$$

so that $d = \deg f = d' \deg \alpha$, where $d' = \deg(p \circ \beta)$ or, alternatively, (d', d') is the bi-degree of the image $\Phi(C) = \beta(C')$. Then, f itself splits into α and $p \circ \beta$. Hence, the critical values of f are those of $p \circ \beta$ and the images under $p \circ \beta$ of the ramification points of α. Thus, if f satisfies Condition 1.1(1), the splitting cannot be proper, *i.e.*, either $d = \deg \alpha$ and $d' = 1$ or $\deg \alpha = 1$ and $d = d'$. In the former case, f is real with respect to some real structure on $\mathbb{P}^1$, see Corollary 2.3. In the latter case, assuming that the critical points of f are real, Condition 1.1(2), the image $B = \Phi(C)$ is a curve of genus g with $2g + 2d - 2$ real ordinary cusps (type $\mathbf{A}_2$ singular points, the images of the critical points of f) and all other singularities with smooth branches.

Conversely, let $B \subset \mathbf{E}$ be a real curve of bi-degree (d, d), $d > 1$, and genus g with $2g + 2d - 2$ real ordinary cusps and all other singularities with smooth branches, and let $\rho \colon \tilde{B} \to B$ be the normalization of B. Then $f = p \circ \rho \colon \tilde{B} \to \mathbb{P}^1$ is a map that satisfies Conditions 1.1(1) and (2) but is not real with respect to any real structure on $\mathbb{P}^1$; hence, the pair (g, d) does not have the total reality property.

As a consequence, we obtain the following statement.

2.5. Theorem (see [3]). *A pair (g, d) has the total reality property if and only if there does not exist a real curve $B \subset \mathbf{E}$ of degree d and genus g with $2g + 2d - 2$ real ordinary cusps and all other singularities with smooth branches.* $\square$

Thus, Theorem 1.2 is equivalent to the following statement, which is actually proved in the paper.

2.6. Theorem. *Let $\mathbf{E}$ be the ellipsoid, and let $B \subset \mathbf{E}$ be a real curve of bi-degree (d, d) and genus g with $c = 2d + 2g - 2$ real ordinary cusps and other singularities with smooth branches. Then $g \leqslant G_0(d)$, see Theorem 1.2.*

2.7. Remark. It is worth mentioning that the bound $g > G_1(d)$ mentioned in the introduction is purely complex: it is derived from the adjunction formula for the virtual genus of a curve $B \subset \mathbf{E}$ as in Theorem 2.5. On the contrary, the proof of the conjecture for the case $(g, d) = (1, 4)$ found in [1] makes an essential use of the real structure, as an elliptic curve with eight ordinary cusps in $\mathbb{P}^1 \times \mathbb{P}^1$ does exist! Our proof of Theorem 2.6 also uses the assumption that all cusps are real.

2.8. In general, a curve B as in Theorem 2.6 may have rather complicated singularities. However, as the proof below is essentially topological, we follow S. Yu. Orevkov [9] and perturb B to a real *pseudo-holomorphic* curve with ordinary nodes (type $\mathbf{A}_1$) and ordinary cusps (type $\mathbf{A}_2$) only. By the genus formula, the number of nodes of such a curve is

$$(2.9) \qquad n = (d-1)^2 - g - c = d^2 - 4d - 1 - 3g.$$

3. Discriminant forms

In this section, we cite the techniques and a few results of Nikulin [8]. Most proofs are found in [8]; they are omitted.

3.1. A *lattice* is a finitely generated free abelian group L equipped with a symmetric bilinear form $b \colon L \otimes L \to \mathbb{Z}$. We abbreviate $b(x, y) = x \cdot y$ and $b(x, x) = x^2$. As the transition matrix between two integral bases has determinant ± 1, the determinant $\det L \in \mathbb{Z}$ (*i.e.*, the determinant of the Gram matrix of b in any basis of L) is well defined. A lattice L is called *nondegenerate* if $\det L \neq 0$; it is called *unimodular* if $\det L = \pm 1$ and *p-unimodular* if $\det L$ is prime to p (where p is a prime).

To fix the notation, we use $\sigma_+(L)$, $\sigma_-(L)$, and $\sigma(L) = \sigma_+(L) - \sigma_-(L)$ for, respectively, the positive and negative inertia indices and the signature of a lattice L.

3.2. Given a lattice L, the bilinear form extends to $L \otimes \mathbb{Q}$. If L is nondegenerate, the dual group $L^* = \operatorname{Hom}(L, \mathbb{Z})$ can be regarded as the subgroup

$$\left\{ x \in L \otimes \mathbb{Q} \mid x \cdot y \in \mathbb{Z} \text{ for all } x \in L \right\}.$$

In particular, $L \subset L^*$ and the quotient L^*/L is a finite group; it is called the *discriminant group* of L and is denoted by $\operatorname{discr} L$ or $\mathcal{L}$. The group $\mathcal{L}$ inherits from $L \otimes \mathbb{Q}$ a symmetric bilinear form $\mathcal{L} \otimes \mathcal{L} \to \mathbb{Q}/\mathbb{Z}$, called the *discriminant form*; when speaking about the discriminant groups, their (anti-)isomorphisms, *etc.*, we always assume that the discriminant form is taken into account. The following properties are straightforward:

(1) the discriminant form is nondegenerate, *i.e.*, the associated homomorphism $\mathcal{L} \to \operatorname{Hom}(\mathcal{L}, \mathbb{Q}/Z)$ is an isomorphism;
(2) one has $\#\mathcal{L} = |\det L|$;
(3) in particular, $\mathcal{L} = 0$ if and only if L is unimodular.

Following Nikulin, we denote by $\ell(\mathcal{L})$ the minimal number of generators of a finite abelian group $\mathcal{L}$. For a prime p, we denote by $\mathcal{L}_p$ the p-primary part of $\mathcal{L}$ and let $\ell_p(\mathcal{L}) = \ell(\mathcal{L}_p)$. Clearly, for a lattice L one has

(4) $\operatorname{rk} L \geqslant \ell(\mathcal{L}) \geqslant \ell_p(\mathcal{L})$ (for any prime p);
(5) L is p-unimodular if and only if $\mathcal{L}_p = 0$.

3.3. An *extension* of a lattice S is another lattice M containing L. All lattices below are assumed nondegenerate.

Let $M \supset S$ be a finite index extension of a lattice S. Since M is also a lattice, one has monomorphisms $S \hookrightarrow M \hookrightarrow M^* \hookrightarrow S^*$. Hence, the quotient $\mathcal{K} = M/S$ can be regarded as a subgroup of the discriminant $\mathcal{S} = \text{discr } S$; it is called the *kernel* of the extension $M \supset S$. The kernel is an isotropic subgroup, *i.e.*, $\mathcal{K}^\perp \subset \mathcal{K}$, and one has $\mathcal{M} = \mathcal{K}^\perp/\mathcal{K}$. In particular, in view of 3.2(1), for any prime p one has

$$\ell_p(\mathcal{M}) \geqslant \ell_p(\mathcal{L}) - 2\ell_p(\mathcal{K}).$$

Now, assume that $M \supset S$ is a *primitive* extension, *i.e.*, the quotient M/S is torsion free. Then the construction above applies to the finite index extension $M \supset S \oplus N$, where $N = S^\perp$, giving rise to the kernel $\mathcal{K} \subset \mathcal{S} \oplus \mathcal{N}$. Since both S and N are primitive in M, one has $\mathcal{K} \cap \mathcal{S} = \mathcal{K} \cap \mathcal{N} = 0$; hence, $\mathcal{K}$ is the graph of an anti-isometry κ between certain subgroups $\mathcal{S}' \subset \mathcal{S}$ and $\mathcal{N}' \subset \mathcal{N}$. If M is unimodular, then $\mathcal{S}' = \mathcal{S}$ and $\mathcal{N}' = \mathcal{N}$, *i.e.*, κ is an anti-isometry $\mathcal{S} \to \mathcal{N}$. Similarly, if M is p-unimodular for a certain prime p, then $\mathcal{S}'_p = \mathcal{S}_p$ and $\mathcal{N}'_p = \mathcal{N}_p$, *i.e.*, κ is an anti-isometry $\mathcal{S}_p \to \mathcal{N}_p$. In particular, $\ell(\mathcal{S}) = \ell(\mathcal{N})$ (respectively, $\ell_p(\mathcal{S}) = \ell_p(\mathcal{N})$). Combining these observations with 3.2(4), we arrive at the following statement.

3.4. Lemma. *Let p be a prime, and let $L \supset S$ be a p-unimodular extension of a nondegenerate lattice S. Denote by $\tilde{S}$ the primitive hull of S in L, and let $\mathcal{K}$ be the kernel of the finite index extension $\tilde{S} \supset S$. Then $\text{rk } S^\perp \geqslant \ell_p(\mathcal{S}) - 2\ell_p(\mathcal{K})$.* $\square$

4. The Alexander module

Here, we discuss (a version of) the Alexander module of a plane curve and its relation to the resolution lattice in the homology of the double covering of the plane ramified at the curve.

4.1. Let π be a group, and let $\kappa\colon \pi \twoheadrightarrow \mathbb{Z}_2$ be an epimorphism. Denote $K = \text{Ker } \kappa$ and define the *Alexander module* of π (more precisely, of κ) as the $\mathbb{Z}[\mathbb{Z}_2]$-module $A_\pi = K/[K,K]$, the generator t of $\mathbb{Z}_2$ acting *via* $x \mapsto [\bar{t}^{-1}\bar{x}\bar{t}] \in A_\pi$, where $\bar{t} \in \pi$ and $\bar{x} \in K$ are some representatives of t and x, respectively. (We simplify the usual definition and consider only the case needed in the sequel. A more general version and further details can be found in A. Libgober [7].)

Let $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible curve of even bi-degree $(d,d) = (2k, 2k)$, and let $\pi = \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus B)$. Recall that $\pi/[\pi,\pi] = \mathbb{Z}_{2k}$; hence, there is a unique epimorphism $\kappa\colon \pi \twoheadrightarrow \mathbb{Z}_2$. The resulting Alexander module $A_B = A_\pi$ will be called the *Alexander module* of B. The *reduced Alexander module* $\tilde{A}_B$ is the kernel of the canonical homomorphism $A_B \to \mathbb{Z}_k \subset \pi/[\pi,\pi]$. There is a natural exact sequence

$$(4.2) \qquad\qquad 0 \to \tilde{A}_B \to A_B \to \mathbb{Z}_k \to 0$$

of $\mathbb{Z}[\mathbb{Z}_2]$-modules (where the $\mathbb{Z}_2$-action on $\mathbb{Z}_d$ is trivial). The following statement is essentially contained in O. Zariski [10].

4.3. Lemma. *The exact sequence (4.2) splits*: one has $A_B = \tilde{A}_B \oplus \text{Ker}(1-t)$, *where t is the generator of $\mathbb{Z}_2$. Furthermore, $\tilde{A}_B$ is a finite group free of 2-torsion, and the action of t on $\tilde{A}_B$ is via the multiplication by (-1).*

Proof. Since A_B is a finitely generated abelian group, to prove that it is finite and free of 2-torsion it suffices to show that $\text{Hom}_\mathbb{Z}(\tilde{A}_B, \mathbb{Z}_2) = 0$. Assume the contrary.

Then the $\mathbb{Z}_2$-action in the 2-group $\mathrm{Hom}_{\mathbb{Z}}(\tilde{A}_B, \mathbb{Z}_2)$ has a fixed non-zero element, *i.e.*, there is an equivariant epimorphism $\tilde{A}_B \twoheadrightarrow \mathbb{Z}_2$. Hence, π factors to a group G that is an extension $0 \to \mathbb{Z}_2 \to G \to \mathbb{Z}_{2k} \to 0$. The group G is necessarily abelian and it is strictly larger than $\mathbb{Z}_{2k} = \pi/[\pi, \pi]$. This is a contradiction.

Since $\tilde{A}_B$ is finite and free of 2-torsion, one can divide by 2 and there is a splitting $\tilde{A}_B = \tilde{A}^+ \oplus \tilde{A}^-$, where $\tilde{A}^{\pm} = \mathrm{Ker}[(1 \pm t) \colon \tilde{A}_B \to \tilde{A}_B]$. Then, π factors to a group G that is a central extension $0 \to \tilde{A}^+ \to G \to \mathbb{Z}_{2k} \to 0$, and as above one concludes that $\tilde{A}^+ = 0$, *i.e.*, t acts on $\tilde{A}_B$ *via* (-1).

Pick a representative $a' \in A_B$ of a generator of $\mathbb{Z}_k = A_B/\tilde{A}_B$. Then, obviously, $(1 - t)a' \in \tilde{A}_B$, and replacing a' with $a' + \frac{1}{2}(1 - t)a'$, one obtains a t-invariant representative $a \in \mathrm{Ker}(1 - t)$. The multiple $ka \in \tilde{A}_B$ is both invariant and skew-invariant; since $\tilde{A}_B$ is free of 2-torsion, $ka = 0$ and the sequence splits. $\square$

4.4. Let $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible curve of even bi-degree $(d, d) = (2k, 2k)$ and with simple singularities only. Consider the double covering $X \to \mathbb{P}^1 \times \mathbb{P}^1$ and denote by $\tilde{X}$ the minimal resolution of singularities of X. Let $\tilde{B} \subset \tilde{X}$ be the proper pull-back of B, and let $E \subset \tilde{X}$ be the exceptional divisor contracted by the blow-down $\tilde{X} \to X$.

Recall that the minimal resolution of a simple surface singularity is diffeomorphic to its perturbation, see, *e.g.*, [2]. Hence, $\tilde{X}$ is diffeomorphic to the double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified at a nonsingular curve. In particular, $\pi_1(\tilde{X}) = 0$ and one has

$$(4.5) \qquad b_2(X) = \chi(X) - 2 = 8k^2 - 8k + 6, \quad \sigma(X) = -4k^2.$$

4.6. Denote $L = H_2(\tilde{X})$. We regard L as a lattice *via* the intersection index pairing on $\tilde{X}$. (Since $\tilde{X}$ is simply connected, L is a free abelian group. It is a unimodular lattice due to the Poincaré duality.) Let $\Sigma \subset L$ be the sublattice spanned by the components of E, and let $\tilde{\Sigma} \subset L$ be the primitive hull of Σ. Recall that Σ is a negative definite lattice. Let, further, $h_1, h_2 \subset L$ be the classes of the pull-backs of a pair of generic generatrices of $\mathbb{P}^1 \times \mathbb{P}^1$, so that $h_1^2 = h_2^2 = 0$, $h_1 \cdot h_2 = 2$.

4.7. Lemma. *If a curve B as above is irreducible, then there are natural isomorphisms $\tilde{A}_B = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Ext}_{\mathbb{Z}}(\mathcal{K}, \mathbb{Z})$, where $\mathcal{K}$ is the kernel of the extension $\tilde{\Sigma} \supset \Sigma$.*

Proof. One has $A_B = H_1(\tilde{X} \smallsetminus (\tilde{B} + E))$ as a group, the $\mathbb{Z}_2$-action being induced by the deck translation of the covering. Hence, by the Poincaré–Lefschetz duality, A_B is the cokernel of the inclusion homomorphism $i^* \colon H^2(\tilde{X}) \to H^2(\tilde{B} + E)$.

On the other hand, there is an orthogonal (with respect to the intersection index form in $\tilde{X}$) decomposition $H_2(\tilde{B} + E) = \Sigma \oplus \langle b \rangle$, where $b = k(h_1 + h_2)$ is the class realized by the divisorial pull-back of B in $\tilde{X}$. The cokernel of the restriction $i^* \colon H^2(X) \to \langle b \rangle^*$ is a cyclic group $\mathbb{Z}_k$ fixed by the deck translation. Hence, in view of Lemma 4.3,

$$\tilde{A}_B = \mathrm{Coker}[i^* \colon H^2(\tilde{X}) \to H^2(E)] = \mathrm{Coker}[L^* \to \Sigma^*] = \mathrm{discr}\, \Sigma/\mathcal{K}^{\perp}.$$

(We use the splitting $L^* \twoheadrightarrow \tilde{\Sigma}^* \to \Sigma^*$, the first map being an epimorphism as $L/\tilde{\Sigma}$ is torsion free.) Since the discriminant form is nondegenerate, see 3.2(1), one has $\mathrm{discr}\, \Sigma/\mathcal{K}^{\perp} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}, \mathbb{Q}/\mathbb{Z})$. Since $\mathcal{K}$ is a finite group, applying the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}, \cdot)$ to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, one obtains an isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Ext}_{\mathbb{Z}}(\mathcal{K}, \mathbb{Z})$. $\square$

4.8. Corollary. *In the notation of Lemma 4.7, if B is irreducible and the group $\pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus B)$ is abelian, then $\mathcal{K} = 0$.* $\square$

4.9. Corollary. *In the notation of Lemma 4.7, if B is an irreducible curve of bi-degree (d, d), $d = 2k \geqslant 2$, then $\mathcal{K}$ is free of 2-torsion and $\ell(\mathcal{K}) \leqslant d - 2$.*

Proof. Due to Lemma 4.7, one can replace $\mathcal{K}$ with $\tilde{A}_B$. Then, the statement on the 2-torsion is given by Lemma 4.3, and it suffices to estimate the numbers $\ell_p(\tilde{A}_B) = \ell(\tilde{A}_B \otimes \mathbb{Z}_p)$ for odd primes p.

Due to the Zariski–van Kampen theorem [5] applied to one of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$, there is an epimorphism $\pi_1(L \smallsetminus B) = F_{d-1} \twoheadrightarrow \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus B)$, where L is a generic generatrix of $\mathbb{P}^1 \times \mathbb{P}^1$ and F_{d-1} is the free group on $d - 1$ generators. Hence, A_B is a quotient of the Alexander module

$$A_{F_{d-1}} = \mathbb{Z}[\mathbb{Z}_2]/(t - 1) \oplus \bigoplus_{d-2} \mathbb{Z}[\mathbb{Z}_2].$$

For an odd prime p, there is a splitting $A_{F_{d-1}} \otimes \mathbb{Z}_p = A_p^+ \oplus A_p^-$ (over the field $\mathbb{Z}_p$) into the eigenspaces of the action of $\mathbb{Z}_2$, and, due to Lemma 4.3, the group $\tilde{A}_B \otimes \mathbb{Z}_p$ is a quotient of $A_p^- = \bigoplus_{d-2} \mathbb{Z}_p$. $\square$

4.10. Remark. All statements in this section hold for pseudo-holomorphic curves as well, *cf.* 2.8. For Corollary 4.9, it suffices to assume that B is a small perturbation of an algebraic curve of bi-degree (d, d). Then, one still has an epimorphism $F_{d-1} \twoheadrightarrow \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus B)$, and the proof applies literally.

5. Proof of Theorem 1.2

As explained in §2, it suffices to prove Theorem 2.6. We consider the cases of d even and d odd separately.

5.1. Let $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible curve of even bi-degree (d, d), $d = 2k$. Assume that all singularities of B are simple and let $\tilde{X}$ be the minimal resolution of singularities of the double covering $X \to \mathbb{P}^1 \times \mathbb{P}^1$ ramified at B, *cf.* 4.4. As in 4.6, consider the unimodular lattice $L = H_2(\tilde{X})$.

Let $c \colon \tilde{X} \to \tilde{X}$ be a real structure on $\tilde{X}$, and denote by $L^\pm$ the (± 1)-eigenlattices of the induced involution c_* of L. The following statements are well known:

(1) $L^\pm$ are the orthogonal complements of each other;
(2) $L^\pm$ are p-unimodular for any odd prime p;
(3) one has $\sigma_+(L^+) = \sigma_+(L_-) - 1$.

Since also $\sigma_+(L^+) + \sigma_+(L^-) = \sigma_+(L) = 2k^2 - 4k + 3$, see (4.5), one arrives at $\sigma_+(L^+) = \sigma_+(L^-) - 1 = (k - 1)^2$ and, further, at

$$(5.2) \qquad \operatorname{rk} L^- = (7k^2 - 6k + 5) - \sigma_-(L^+).$$

5.3. Remark. The common proof of Property 5.1(3) uses the Hodge structure. However, there is another (also very well known) proof that also applies to almost complex manifolds. Let $\tilde{X}_{\mathbb{R}} = \operatorname{Fix} c$ be the real part of $\tilde{X}$. Then, the normal bundle of $\tilde{X}_{\mathbb{R}}$ in $\tilde{X}$ is i times its tangent bundle; hence, the normal Euler number $\tilde{X}_{\mathbb{R}} \circ \tilde{X}_{\mathbb{R}}$ equals (-1) times the index of any tangent vector field on $\tilde{X}_{\mathbb{R}}$, *i.e.*, $-\chi(\tilde{X}_{\mathbb{R}})$. Now, one has $\sigma(L^+) - \sigma(L^-) = \tilde{X}_{\mathbb{R}} \circ \tilde{X}_{\mathbb{R}} = -\chi(\tilde{X}_{\mathbb{R}})$ (by the Hirzebruch G-signature theorem) and $\operatorname{rk} L^+ - \operatorname{rk} L^- = \chi(X_{\mathbb{R}}) - 2$ (by the Lefschetz fixed point theorem). Adding the two equations, one obtains 5.1(3).

5.4. The case of $d = 2k$ even. Perturbing, if necessary, B in the class of real pseudo-holomorphic curves, see 2.8, one can assume that all singularities of B are c real ordinary cusps and n ordinary nodes, where

$$(5.5) \qquad c = 2d + 2g - 2 \quad \text{and} \quad n = d^2 - 4d - 1 - 3g,$$

see Theorem 2.5 and (2.9). Let $n = r + 2s$, where r and s are the numbers of, respectively, real nodes and pairs of conjugate nodes.

5.6. Consider the double covering $\tilde{X}$, see 4.4, lift the real structure on $\mathbf{E}$ to a real structure c on $\tilde{X}$, and let $L^{\pm} \subset L$ be the corresponding eigenlattices, see 5.1. In the notation of 4.6, let $\Sigma^{\pm} = \Sigma \cap L^{\pm}$. Then

- each real cusp of B contributes a sublattice $\mathbf{A}_2$ to Σ^-;
- each real node of B contributes a sublattice $\mathbf{A}_1 = [-2]$ to Σ^-;
- each pair of conjugate nodes contributes $[-4]$ to Σ^- and $[-4]$ to Σ^+.

In addition, the classes h_1, h_2 of two generic generatrices of $\mathbf{E}$ span a hyperbolic plane orthogonal to Σ, see 4.6. It contributes

- a sublattice $[4] \subset L^-$ spanned by $h_1 + h_2$, and
- a sublattice $[-4] \subset L^+$ spanned by $h_1 - h_2$.

(Recall that any real structure reverses the canonical complex orientation of pseudo-holomorphic curves.)

5.7. All sublattices of L^+ described above are negative definite; hence, their total rank $s + 1$ contributes to $\sigma_-(L^+)$. The total rank $2c + r + s + 1$ of the sublattices of L^- contributes to the rank of $S^- = \Sigma^- \oplus [4] \subset L^-$. Due to (5.2), one has

$$(5.8) \qquad 2c + n + 2 + \operatorname{rk} S^{\perp} \leqslant 7k^2 - 6k + 5,$$

where $S^{\perp}$ is the orthogonal complement of S^- in L^-. All summands of S^- other than $\mathbf{A}_2$ are 3-unimodular, whereas $\operatorname{discr} \mathbf{A}_2$ is the group $\mathbb{Z}_3$ spanned by an element of square $\frac{1}{3} \bmod \mathbb{Z}$. Let $\tilde{S}^- \supset S^-$ and $\tilde{\Sigma} \supset \Sigma$ be the primitive hulls, and denote by $\mathcal{K}^-$ and $\mathcal{K}$ the kernels of the corresponding finite index extensions, see 3.3. Clearly, $\ell_3(\mathcal{K}^-) \leqslant \ell_3(\mathcal{K})$ and, due to Corollary 4.9 (see also Remark 4.10), one has $\ell_3(\mathcal{K}) \leqslant d - 2$. Then, using Lemma 3.4, one obtains $\operatorname{rk} S^{\perp} \geqslant c - 2(d - 2)$, and combining the last inequality with (5.8), one arrives at

$$3c + n - 2(d - 2) \leqslant 7k^2 - 6k + 3.$$

It remains to substitute the expressions for c and n given by (5.5) and solve for g to get

$$g \leqslant k^2 - 2k + \frac{2}{3}.$$

Since g is an integer, the last inequality implies $g \leqslant G_0(2k)$ as in Theorem 2.6.

5.9. The case of $d = 2k - 1$ odd. As above, one can assume that B has c real ordinary cusps and $n = r + 2s$ ordinary nodes, see (5.5). Furthermore, one can assume that $c > 0$, as otherwise $g = 0$ and $d = 1$. Then, B has a real cusp and, hence, a real smooth point P. Let L_1, L_2 be the two generatrices of $\mathbf{E}$ passing through P. Choose P generic, so that each L_i, $i = 1, 2$, intersects B transversally at d points, and consider the real curve $B' = B + L_1 + L_2$ of even bi-degree $(2k, 2k)$,

applying to it the same double covering arguments as above. In addition to the nodes and cusps of B, the new curve B' has $(d-1)$ pairs of conjugate nodes and a real triple (type $\mathbf{D}_4$) point at P (with one real and two complex conjugate branches). Hence, in addition to the classes listed in 5.6, there are

– $(d-1)$ copies of $[-4]$ in each Σ^+, Σ^- (from the new conjugate nodes),
– a sublattice $[-4] \subset \Sigma^+$ (from the type $\mathbf{D}_4$ point), and
– a sublattice $\mathbf{A}_3 \subset \Sigma^-$ (from the type $\mathbf{D}_4$ point).

Thus, inequality (5.8) turns into

$$2c + n + 2(d-1) + 4 + 2 + \operatorname{rk} S^\perp \leqslant 7k^2 - 6k + 5.$$

We will show that $\operatorname{rk} S^\perp \geqslant c$. Then, substituting the expressions for c and n, see (5.5), and solving the resulting inequality in g, one would obtain $g \leqslant G_0(2k-1)$, as required.

5.10. In view of Lemma 3.4, in order to prove that $\operatorname{rk} S^\perp \geqslant c$, it suffices to show that $\ell_3(\mathcal{K}) = 0$ (*cf.* similar arguments in 5.7).

Perturb B' to a pseudo-holomorphic curve B'', keeping the cusps of B' and resolving the other singularities. (It would suffice to resolve the singular points resulting from the intersection $B \cap L_1$.) Then, applying the Zariski–van Kampen theorem [5] to the ruling containing L_1, it is easy to show that the fundamental group $\pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus B'')$ is cyclic. Indeed, let U be a small tubular neighborhood of L_1 in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $L'' \subset U$ be a generatrix transversal to B''. Obviously, the epimorphism $\pi_1(L_1'' \smallsetminus B'') \twoheadrightarrow \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus B'')$ given by the Zariski–van Kampen theorem factors through $\pi_1(U \smallsetminus B'')$, and the latter group is cyclic.

On the other hand, the new double covering $\tilde{X}'' \to \mathbb{P}^1 \times \mathbb{P}^1$ ramified at B'' is diffeomorphic to $\tilde{X}$, and the diffeomorphism can be chosen identical over the union of a collection of Milnor balls about the cusps of B'. Thus, since $\operatorname{discr} \mathbf{A}_1$ and $\operatorname{discr} \mathbf{D}_4$ are 2-torsion groups, the perturbation does not change $\mathcal{K} \otimes \mathbb{Z}_3$, and Corollary 4.8 (see also Remark 4.10) imply that $\mathcal{K} \otimes \mathbb{Z}_3 = 0$. $\square$

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