Journal für die reine und angewandte Mathematik, **551** (2002)

REAL RATIONAL SURFACES ARE QUASI-SIMPLE

Alex Degtyarev and Viatcheslav Kharlamov

Abstract. We show that real rational (over $\mathbb{C}$) surfaces are quasi-simple, i.e., that such a surface is determined up to deformation in the class of real surfaces by the topological type of its real structure.

1. Introduction

The list of all minimal models of complex rational surfaces is exhausted by the projective plane $\mathbb{P}^2$ and the geometrically ruled surfaces $\Sigma_a \to \mathbb{P}^1$, $a \in \mathbb{Z}_+$. As is well known, this implies that diffeomorphic rational surfaces (or, equivalently, rational surfaces with isomorphic cohomology rings) are deformation equivalent. In the present paper we treat the corresponding deformation problem for surfaces defined over $\mathbb{R}$ and rational over $\mathbb{C}$ (which we call *real rational surfaces*, see the definition in Section 2). We show that two real rational surfaces are deformation equivalent over $\mathbb{R}$ (i.e., in the class of real surfaces) if and only if their real structures are diffeomorphic. The prove is based on the classification of $\mathbb{R}$-minimal real rational surfaces going back to A. Comessatti [Co1].

Note that the same phenomenon, which we call quasi-simplicity of the class of surfaces, is observed for real Abelian surfaces (essentially due to A. Comessatti [Co2]), for real hyperelliptic surfaces (F. Catanese and P. Frediani [CF]), for real $K3$-surfaces (essentially due to V. Nikulin [N]), and for real Enriques surfaces (A. Degtyarev and V. Kharlamov; the quasi-simplicity statement was announced in [DK1], and the complete list of deformation classes of real Enriques surfaces was obtained in collaboration with I. Itenberg in [DIK]). It is natural to expect that such a simple behaviour would no longer take place for more complicated surfaces, like those of general type. Note, however, that the so called *Bogomolov-Miayoka-Yau surfaces*, which are of general type, are also quasi-simple, see [KK]. It is also worth mentioning that the first non trivial quasi-simplicity result, concerning real cubic surfaces in $\mathbb{P}^3$, was discovered by F. Klein and L. Schläfli (see, for example, our survey [DK3]). To our surprise, apart from this result, we could not find any trace of deformation classification of real rational surfaces in the literature. (The topology of the real part of a real rational surface is discussed in [S].)

It is worth mentioning that in all cases above, including the case of rational surfaces, the deformation class is determined, in fact, by the **topological** type of the real structure. In particular, as a consequence, the topological type of the real structure determines it up to diffeomorphism.

1991 *Mathematics Subject Classification.* 14J26, 14J10, and 14P25.
Key words and phrases. Real algebraic surface, rational surface, deformation, conic bundles.

The quasi-simplicity of real rational surfaces was obtained during our work on the deformation classification of real Enriques surfaces. One particular case, concerning the $\mathbb{R}$-minimal rational surfaces, is mentioned in [DK2]. Another one, concerning the Del Pezzo surfaces, is contained in [DIK]; this case is used explicitly in the classification of real Enriques surfaces. Here we treat the remaining cases and complete the proof.

Remark. In spite of its quasi-simplicity (or due to some uncertainty principle?) the class of real rational surfaces seems to be one of the most difficult for study in what concerns other geometric properties. For example, it is unknown whether the number of isomorphism classes of real structures on a given rational surface is finite, although the corresponding finiteness result is known to hold for all surfaces of Kodaira dimension $\geqslant 1$ and for minimal Abelian, hyperelliptic, $K3$-, and Enriques surfaces, see [DIK].

Acknowledgements. We are grateful to J. Kollár for his helpful remarks and for his interest in the quasi-simplicity phenomenon that gave us an additional stimulus for publishing this proof. This paper was written during the first author's stay at *Université Louis Pasteur*, Strasbourg, supported by the European Doctoral College and by a joint *CNRS–TÜBİTAK* exchange program.

2. Main Result

2.1. Basic definitions. From now on by a *complex surface* we mean a compact complex analytic variety of complex dimension 2. A *real structure* on a complex surface X is an anti-holomorphic involution $X \to X$. A surface supplied with a real structure is called a *real surface*; the real structure is usually denoted by conj. The fixed point set $\operatorname{Fix}\operatorname{conj} = X_{\mathbb{R}}$ is called the *real part* of X. Topologically, $X_{\mathbb{R}}$ is a closed 2-manifolds. To describe its homeomorphism type, we will list its connected components in an expression of the form $nS \sqcup n_1 S_1 \sqcup \dots \sqcup m_1 V_1 \sqcup \dots$, where $S = S_0 = S^2$ stands for the 2-sphere, $S_p = \#_p(S^1 \times S^1)$, for the orientable surface of genus g, and $V_q = \#_q \mathbb{P}^2_{\mathbb{R}}$, for the nonorientable surface of genus q.

Remark. Of course, if X is an algebraic surface defined over $\mathbb{R}$, then conj is the Galois involution and $X_{\mathbb{R}}$ is the set of $\mathbb{R}$-rational closed points of X. However, although all surfaces treated in this paper are in fact algebraic, in general we prefer the more geometric complex analytic approach.

By a *real rational surface* we mean a real surface birationally equivalent to $\mathbb{P}^2$ **over** $\mathbb{C}$. Note that this terminology differs from the commonly accepted one: 'most' real rational surfaces in our sense are not in fact rational over $\mathbb{R}$; Theorem 2.4.1 below gives a number of examples.

The meaning of many common notions and constructions depends on whether they are considered over $\mathbb{C}$ or over $\mathbb{R}$. (The rationality of a real surface is a good example.) The conj-equivariant category is usually designated with the word 'real'; for example, a *real curve* on a real surface is a conj-invariant analytic curve. However, sometimes this approach is unacceptable; in these cases we will use the prefix $\mathbb{R}$. Thus, we will speak about $\mathbb{R}$-*minimal* surfaces (instead of 'really minimal real surfaces'): a real surface X is called $\mathbb{R}$-minimal if any conj-invariant degree 1 holomorphic map $X \to Y$ to another real surface Y is a biholomorphism. Clearly, X is $\mathbb{R}$-minimal if and only if it is free of real (-1)-curves and of pairs of disjoint (-1)-curves transposed by conj.

The following is the key notion in the topological study of analytic varieties. The definition of deformation equivalence below applies to both complex and real categories; in the former case the prefix 'real' should be omitted.

2.1.1. Definition. A *deformation* of complex surfaces is a proper analytic submersion $p\colon Z \to D^2$, where Z is a 3-dimensional analytic variety and $D^2 \subset \mathbb{C}$ a disk. If Z is real and p is equivariant, the deformation is called real. Two (real) surfaces X' and X'' are called *deformation equivalent* if they can be connected by a chain $X' = X_0, \dots, X_k = X''$ so that X_i and X_{i-1} are isomorphic to (real) fibers of a (real) deformation.

Unless stated otherwise, the deformation equivalence is understood in the same category as the surfaces considered, i.e., real for real surfaces and complex otherwise.

2.1.2. Definition. A real surface S is called *quasi-simple* if it is deformation equivalent to any other real surface S' such that (1) S' is deformation equivalent to S as a complex surface, and (2) the real structure of S' is diffeomorphic to that of S.

Now we are ready to state our main result.

2.2. Main theorem. *Real rational surfaces are quasi-simple.*

In Section 6 below we reduce Theorem 2.2 to the following statement, which is also of an independent interest.

2.3. Minimal surfaces. *Any two $\mathbb{R}$-minimal real rational surfaces with diffeomorphic real structures are deformation equivalent in the class of $\mathbb{R}$-minimal real rational surfaces.*

Remark. In view of the modern advances in topology of 4-manifolds one should carefully distinguish between the smooth and topological categories, and the former seems to us more appropriate in definitions like 2.1.2. Note however, that in all known examples (see Introduction) the quasi-simplicity holds in a stronger sense: homeomorphic real structures (within a fixed complex deformation class) are deformation equivalent. The same remark applies to rational surfaces: everywhere in this paper one can replace the word 'diffeomorphic' with 'homeomorphic' (and even 'homotopy equivalent', as all the invariants can be expressed in terms of homology, see Theorem 6.4.1).

Remark. Alternatively, Theorem 2.2 can be strengthened in a different direction. Indeed, due to R. Friedman and Z. Qin [FQ], a surface diffeomorphic to a rational one is rational. Thus, one can state that, *given a real rational surface X, any real surface X' diffeomorphic to X (as a real surface) is deformation equivalent to X.*

Remark. In 2.1.2 we define the quasi-simplicity as a property of a real surface or, equivalently, of a real deformation class. One could also define as a property of a complex deformation class (i.e., as the requirement that all the real surfaces that are $\mathbb{C}$-deformation equivalent to a given one should be quasi-simple). In all examples known so far the quasi-simplicity does hold for whole complex deformation classes.

2.4. Plan of the proof. The proof of Theorem 2.3 is based on the following classification of $\mathbb{R}$-minimal real rational surfaces, due to Comessatti [Co1] (see also [S], the survey [MT], and the lecture notes [Ko]):

2.4.1. Theorem. *Each $\mathbb{R}$-minimal real rational surface X is one of the following*:

(1) *projective plane $\mathbb{P}^2$ with its standard real structure*: $X_\mathbb{R} = V_1$;
(2) *quadric $\mathbb{P}^1 \times \mathbb{P}^1$ with one of its four nonequivalent real structures*: $X_\mathbb{R} = S_1$, $X_\mathbb{R} = S$, *and two structures with* $X_\mathbb{R} = \varnothing$ (*see* 2.5.1 *below*);
(3) *geometrically ruled surface Σ_a, $a \geqslant 2$* (*see* 2.5.2 *below*), *with $X_\mathbb{R} = V_2$ and the standard real structure, if a is odd, and with $X_\mathbb{R} = S_1$ or $\varnothing$ and one of the two respective nonequivalent structures, if a is even*;
(4) *relatively minimal real conic bundles over $\mathbb{P}^1$* (*see Section* 3) *with $2m \geqslant 4$ singular fibers*: $X_\mathbb{R} = mS$;
(5) *unnodal Del Pezzo surfaces of degree $d = 1$ or 2*: $X_\mathbb{R} = V_1 \sqcup 4S$, *if $d = 1$, and $X_\mathbb{R} = 3S$ or $4S$, if $d = 2$.*

Conversely, any rational surface from the above list is $\mathbb{R}$-minimal. □

Remark. An $\mathbb{R}$-minimal unnodal Del Pezzo surface of degree 2 with $X_\mathbb{R} = 3S$ can also be represented as a conic bundle over $\mathbb{P}^1$ with six reducible fibers. Note that 2.4.1(5) does **not** list all real structures on (unnodal) Del Pezzo surfaces of degree 1 and 2; most of them are not $\mathbb{R}$-minimal (see [DIK, §17.3]).

To prove Theorem 2.3, we first show that the statement holds for each family listed in 2.4.1 separately. The conic bundles 2.4.1(4) are treated in Section 3; the other classes have already been considered elsewhere, see references in Section 4. Then it remains to show that whenever the corresponding complex deformation classes overlap, the real surfaces are deformation equivalent; this is done in 4.2.

2.5. More notation. Here we cite a few known results and introduce some notation used throughout the paper.

2.5.1. Recall that the quadric $Y = \mathbb{P}^1 \times \mathbb{P}^1$ has two nonisomorphic real structures with $Y_\mathbb{R} = \varnothing$. They are both product structures: $c_1 \times c_1$ and $c_0 \times c_1$, where c_0 is the ordinary complex conjugation on $\mathbb{P}^1$ (with $\mathbb{P}^1_\mathbb{R} = S^1$) and c_1 is the quaternionic real structure (with empty real point set). The quotient manifold $Y/(c_1 \times c_1)$ is not Spin; the quotient $Y/(c_0 \times c_1)$ is Spin. The real structure $c_1 \times c_1$ is inherited from $\mathbb{P}^3$ (with its standard real structure); we will call it *standard.*

The two real structures with $Y_\mathbb{R} = \varnothing$ become equivalent after a single blow-up at a pair of conjugate points (see, e.g., [DIK, Lemma in 17.3.1]).

2.5.2. We denote by Σ_a, $a \geqslant 0$, the geometrically ruled rational surface (i.e., relatively minimal conic bundle over $\mathbb{P}^1$, see Section 3) that has a section of square $(-a)$. (Unless $a = 0$, such a section is unique; it is called the *exceptional section.*) All these surfaces except Σ_1 are minimal. (Clearly, $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and Σ_1 is the plane $\mathbb{P}^2$ blown up at one point.) Recall that, up to isomorphism, Σ_a, $a > 0$, has one real structure if a is odd (and then the real part is V_2), and it has two real structures if a is even (the real part being either S_1 or $\varnothing$).

We will denote by $|e_0|$ and $|e_\infty|$ the classes of the exceptional section and of generic section, respectively, so that $e_0^2 = -a$, $e_\infty^2 = a$, and $e_0 \cdot e_\infty = 0$. (If $a = 0$, then $e_0 = e_\infty$.) The class of the fiber will be denoted by $|l|$; one has $l^2 = 0$ and $l \cdot e_0 = l \cdot e_\infty = 1$. As is well know, the semigroup of classes of effective divisors on Σ_a is generated by $|e_0|$ and $|l|$ and one has $|e_\infty| = |e_0 + al|$.

2.5.3. Given a topological space X, we denote by $b_i(X) = \operatorname{rk} H_i(X)$ and $\beta_i(X) = \dim H_i(X; \mathbb{Z}_2)$ its $\mathbb{Z}$- and $\mathbb{Z}_2$-Betti numbers, respectively. The corresponding total

Betti numbers are denoted by $b_*(X) = \sum_{i\geqslant 0} b_i(X)$ and $\beta_*(X) = \sum_{i\geqslant 0} \beta_i(X)$. Recall the following Smith inequality: *for any real variety X one has*

$$\beta_*(X_{\mathbb{R}}) \leqslant \beta_*(X) \quad \text{and} \quad \beta_*(X_{\mathbb{R}}) = \beta_*(X) \bmod 2.$$

Thus, one has $\beta_*(X_{\mathbb{R}}) = \beta_*(X) - 2d$ for some nonnegative integer d; in this case X is called an *$(M-d)$-variety.*

If X is a curve, the Smith inequality reduces to the so called Harnack inequality: *the number of connected components of the real part of a nonsingular irreducible real curve X of genus g does not exceed $g+1$.* Thus, M-curves of genus g are those with $(g+1)$ components in the real part.

2.5.4. A *cyclic order* on a finite set $\mathcal{S}$ is an embedding $\mathcal{S} \to S^1$ considered up to auto-homeomorphism of the circle S^1. In other words, we consider cyclic orders up to reflections. Alternatively, a cyclic order can be defined as a symmetric binary relation 'to be neighbors' such that each element of $\mathcal{S}$ has exactly two neighbors.

3. Conic Bundles

Recall that a *conic bundle* is a regular map $p\colon X \to C$, where X is a compact algebraic surface and C is a curve, such that a generic fiber of p is an irreducible rational curve. If both X and C are real and p is equivariant, the conic bundle is called real.

As is well known, over $\mathbb{C}$ any conic bundle blows down to the ruling of a geometrically ruled surface. Over $\mathbb{R}$, a real conic bundle is relatively minimal if and only if all its singular fibers are real and each singular fiber consists of a pair of conjugate (-1)-curves. As it follows from 2.4.1, a real conic bundle is relatively minimal over $\mathbb{R}$ if and only if it is $\mathbb{R}$-minimal.

3.1. Birational classification. An *elementary transformation* of a (complex) conic bundle $X \to C$ at a point P in a nondegenerate fiber F of X is the blow-up of P followed by the blow-down of F. If $X \to C$ is real, a *real elementary transformation* of X is either an elementary transformation at a real point of X or a pair of elementary transformations at two conjugate points $P_1, P_2 \in X$ that belong to two distinct conjugate fibers of $X \to C$.

Next two statements are found in [S] and [MT].

3.1.1. Theorem. *Any* (*real*) *fiberwise birational map between two relatively minimal* (*real*) *conic bundles decomposes into a product of* (*real*) *elementary transformations.* □

3.1.2. Theorem. *Any conic bundle $X \to \mathbb{P}^1$ is fiberwise birationally equivalent to a conic bundle $Y \to \mathbb{P}^1$ which is a branched double covering of a geometrically ruled surface. Furthermore, Y can be chosen so that in appropriate affine coordinates (x, y, z) it is given by the equation $x^2 + y^2 = p(z)$, where p is a polynomial, and the projection $Y \to \mathbb{P}^1$ is the restriction to Y of the map $(x, y, z) \mapsto z$.*

If $X \to \mathbb{P}^1$ is real and $X_{\mathbb{R}} \neq \varnothing$, the above statement holds in the equivariant setting, i.e., the bundle $Y \to \mathbb{P}^1$, the coordinates and the equation $x^2 + y^2 = p(z)$, and the birational equivalence can be chosen real. □

The following is straightforward:

3.1.3. Lemma. *Let a real conic bundle Y be the double covering of a geometrically ruled surface Σ_a branched over a nonsingular curve B. Assume that Y has $2m \geqslant 4$ singular fibers. Then one has $|a| \leqslant m$ and $a = m \bmod 2$ and the branch curve belongs to the linear system $|2e_\infty + (m-a)l|$. Furthermore, Y is $\mathbb{R}$-minimal if and only if B is an M-curve.* □

3.2. Double ruled surfaces. *Two $\mathbb{R}$-minimal real rational conic bundles Y_1, Y_2 which are double branched covering of geometrically ruled surfaces are deformation equivalent if and only if they have the same number of singular fibers.*

Proof. We concentrate on the 'if' part of the statement; the 'only if' part is obvious.

If Y_1 and Y_2 cover the same geometrically ruled surface Σ_a, then the statement follows from the classification of real curves in the linear system $|2e_\infty + (m-a)l|$ (see, e.g., [DK3] or [DIK, A2.2.8]): any two such M-curves are rigidly isotopic.

Thus, in view of 3.1.3, it remains to deform an M-curve in $|2e_\infty + (m-a)l|$ on Σ_a to an M-curve in $|2e_\infty + (m-a-2)l|$ on Σ_{a+2}, $a \leqslant m-2$. This can be done as follows (cf. [W]). Consider a generic M-curve $C \in |2e_\infty + (m-a-1)l|$ on the surface Σ_{a+1}. Pick a real point P_0 of intersection of C and the exceptional section E_0 of Σ_{a+1}. (Such a point exists since $m-a-1 > 0$ is odd.) The desired deformation is parametrized by the real points P_t of the germ of C at P_0. It consists of the surfaces obtained from Σ_{a+1} by the elementary transformations at P_t (and the corresponding transforms of C). If $t = 0$, the resulting surface is Σ_{a+2}, otherwise it is Σ_a. □

3.3. The general case. *Two $\mathbb{R}$-minimal real rational conic bundles Y_1, Y_2 are deformation equivalent if and only if they have the same number of singular fibers.*

The statement is an immediate consequence of 3.2 and the following lemma.

3.3.1. Lemma. *Any $\mathbb{R}$-minimal real rational conic bundle is deformation equivalent to a conic bundle obtained by a double covering of a geometrically ruled surface.*

Proof. In view of 3.1.2 and 3.1.1, any real rational conic bundle is obtained from a double covering by a sequence of real elementary transformations. Up to deformation (moving the blow-up centers), one can assume that the transformations are independent, and it suffices to prove that a real elementary transformation applied to a double covering results in a surface deformation equivalent to a double covering.

Two complex conjugate blow-up centers can be merged to a single double real center (recall that the real part of the surface is nonempty), and then taken apart to two real centers. Thus, all the blow-up centers can be assumed real. In view of 3.1.3, each component of the real part of the surface intersects the branch curve of the covering; moving the blow-up centers to the branch curve, one obtains a sequence of real elementary transformations respecting the deck translation of the covering. Hence, the resulting surface possesses an involution, i.e., it is a double covering. □

4. Other Minimal Surfaces

In this section we complete the proof of 2.3. In fact, we prove the following slightly stronger statement.

4.1. Theorem. *With two exceptions, the real part* $X_{\mathbb{R}}$ *of an* $\mathbb{R}$*-minimal real rational surface* X *determines* X *up to deformation in the class of* $\mathbb{R}$*-minimal real rational surfaces. The exceptions are*:

(1) *surfaces with* $X_{\mathbb{R}} = \varnothing$, $X = \mathbb{P}^1 \times \mathbb{P}^1$, *with the two possible real structures, see* 2.5.1, *which are not deformation equivalent*;
(2) *surfaces with* $X_{\mathbb{R}} = 4S$, *see* 2.4.1(4) *and* (5).

In each of the exceptional cases the two families differ by the topology of (X, conj). *In case* (1), *they differ by* $w_2(X/\operatorname{conj})$, *see* 2.5.1. *In case* (2), *the conic bundles in* 2.4.1(4) *are* $(M-2)$*-surfaces, whereas the Del Pezzo surfaces of degree* 2 *in* 2.4.1(5) *are* $(M-1)$*-surfaces.*

Remark. As shown in the proof, the other surfaces with empty real part, $X = \Sigma_a$, $a \geqslant 2$, are deformation equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$ with the standard real structure without real points, see 2.5.1.

4.2. Proof of 4.1. First, note that each family listed in 2.4.1 consists of deformation equivalent surfaces. Indeed, the classification of real structures on $\mathbb{P}^2$ and geometrically ruled surfaces (2.4.1(1)–(3)) is well known and reduces mainly to college linear algebra, the classification of $\mathbb{R}$-minimal real rational conic bundles (2.4.1(4)) is given in 3.3, and the deformation classification of real Del Pezzo surfaces (2.4.1(5)) can be found, e.g., in [DIK, §17.3]. Note also that **all** surfaces listed in 2.4.1 are $\mathbb{R}$-minimal; hence, their classification up to real deformation coincides with the classification up to deformation in the class of $\mathbb{R}$-minimal real surfaces. Thus, to complete the proof it remains to show that, with the exception of the two cases mentioned in the statement, the distinct families of surfaces with homeomorphic real parts are, in fact, deformation equivalent. Below we consider these families case by case.

Cases 1, 2: $X = \Sigma_a$, $X_{\mathbb{R}} = S_1$ *if* a *is even or* $X_{\mathbb{R}} = V_2$ *if* a *is odd.* A real deformation $\Sigma_a \mapsto \Sigma_{a+2}$ can be constructed as in the proof of 3.2, with the branch curve ignored, i.e., both the surfaces occur in a family of real elementary transformations of the surface Σ_{a+1} (with nonempty real part).

Case 3: $X = \Sigma_a$ *with* a *even,* $X_{\mathbb{R}} = \varnothing$. (We do not exclude the case when X is an empty quadric in $\mathbb{P}^3$ with its standard real structure.) A deformation $\Sigma_a \mapsto \Sigma_{a+4}$ can be constructed as above, as a family of elementary transformations of Σ_{a+2} with a pair of complex conjugate blow-up centers. For a deformation $\Sigma_0 \mapsto \Sigma_2$ we use a slightly modified, equivariant, version of the standard construction. Namely, take two copies of $\mathbb{C}^1 \times \mathbb{P}^1 \times B$, $B = \{t \in \mathbb{C} \mid |t| < 1\}$, with coordinates $u, (y_0 : y_1), t$ and $v, (z_0 : z_1), t$, and glue them together along the subsets $u \neq 0$ and $v \neq 0$ via the relations $uv = 1$ and $y_1 z_0 = v^2 y_0 z_1 + tv(y_0 z_0 + y_1 z_1)$. Clearly, the result is a deformation of rational ruled surfaces, t being the parameter and the $\mathbb{P}^1$ factors being the fibers of the rulings. It is equivariant under the real structure $(u, y_0, y_1, t) \mapsto (-\bar{v}, -\bar{y}_1, \bar{y}_0, -\bar{t})$ (as so are the gluing relations). In particular, the real part of the parameter space consists of the imaginary values of t, and the real parts of the fibers of the deformation are empty. The fiber over $t = 0$ is, in fact, the standard representation of Σ_2 as a toric variety. If $t \neq 0$, one can see that the curves $(uy_1 - ty_0)/y_1 = \text{const}$ form a family of disjoint (and, hence, of self-intersection 0) sections of the ruling. Hence, the surface is Σ_0. (To verify that the curves are also disjoint in the other chart, one can use the identity $(uy_1 - ty_0)/y_1 = (1+t^2)z_1/(vz_1 + tz_0)$, which follows from the gluing relations.) □

5. Canonical Minimal Models

In order to prove Theorem 2.2, we consider a real rational surface X and show that, up to deformation, X has a certain canonical minimal model whose topology and, hence, deformation type is determined by the topology of the real structure of X. Then, given two such surfaces X_1, X_2 with diffeomorphic real structures, one can assume that they are both obtained by a sequence of blow-ups from the same $\mathbb{R}$-minimal surface Y, and it would remain to show that the sets of blow-up centers in Y are also determined by the topology of X_i up to equivariant isotopy in Y. This is done in Section 6.

5.1. Essential invariant classes. Given a real rational surface (X, conj), denote by $L_+(X)$ the eigenlattice

$$L_+(X) = \operatorname{Ker}\bigl[(1 - \operatorname{conj}_*)\colon H_2(X) \to H_2(X)\bigr]$$

supplied with the induced intersection form.

5.1.1. Lemma. *The lattice $L_+(X)$ is negative definite and even. If another real rational surface X' is obtained from X by a sequence of r blow-ups at real points and q blow-ups at pairs of complex conjugate points, then $L_+(X') = L_+(X) \oplus q\langle -2\rangle$.*

Proof. By the Hodge index theorem, the signature of the intersection form on $H_2(S)$ is of the form $(1, k)$, i.e., the positive inertia index is 1. By the usual averaging arguments, one can find a conj-invariant Kähler metric on X. The Poincaré dual $\gamma \in H_2(X; \mathbb{R})$ of the fundamental class of such a metric belongs to the (-1)-eigenspace $\operatorname{Ker}(1 + \operatorname{conj}_*)$. Hence, the latter contains a vector with positive square, and its orthogonal complement $L_+(X) \otimes \mathbb{R}$ is negative definite.

The fact that $L_+(X)$ is even is a well known common property of real surfaces. It follows, for example, from the fact that the Stiefel-Whitney class $w_2(X)$ (which coincides with the mod 2-reduction of the first Chern class $c_1(X)$) can be realized by an algebraic cycle which is defined over $\mathbb{R}$ and thus belongs to the (-1)-eigenlattice of conj_*.

The other assertions of the lemma are straightforward. □

As is well known, a definite integral lattice admits a unique (up to reordering) decomposition into irreducible orthogonal summands. Denote by $E_+(X)$ the *essential part* of $L_+(X)$ obtained by removing from $L_+(X)$ the summands isomorphic to $\langle -2\rangle$. Due to Lemma 5.1.1 this lattice is well defined and depends only on the topology of (X, conj). The following is also an immediate consequence of Lemma 5.1.1.

5.1.2. Corollary. *The lattice $E_+(X)$ is a birational invariant of a real rational surface X, i.e., any real birational transformation $X_1 \dashrightarrow X_2$ of real rational surfaces induces an isomorphism $E_+(X_1) = E_+(X_2)$.* □

5.2. Canonical minimal models. *A real rational surface X is $\mathbb{R}$-deformation equivalent to a surface obtained by a sequence of real blow-ups from an $\mathbb{R}$-minimal real rational surface Y described below.*

(1) *If $b_0(X_\mathbb{R}) = 0$ and $b_*(X) > 4$, then $Y = \mathbb{P}^1 \times \mathbb{P}^1$ with the standard real structure with $Y_\mathbb{R} = \varnothing$ (see 2.5.1).*
(2) *If $b_0(X_\mathbb{R}) = 0$ and $b_*(X) = 4$, then $Y = \mathbb{P}^1 \times \mathbb{P}^1$ with one of the two real structures with $Y_\mathbb{R} = \varnothing$ (see 2.5.1).*

(3) *If* $b_0(X_{\mathbb{R}}) = 1$ *and* $X_{\mathbb{R}}$ *is orientable, then* $Y = \mathbb{P}^1 \times \mathbb{P}^1$ *with* $Y_{\mathbb{R}} = S$ *or* S_1.
(4) *If* $b_0(X_{\mathbb{R}}) = 1$ *and* $X_{\mathbb{R}}$ *is nonorientable, then* $Y = \mathbb{P}^2$.
(5) *If* $b_0(X_{\mathbb{R}}) = 2$ *or* 3, *then* Y *is an* $\mathbb{R}$*-minimal conic bundle.*
(6) *If* $b_0(X_{\mathbb{R}}) = 4$, *then* Y *is a conic bundle or a Del Pezzo surface of degree* 2. *The two cases are distinguished by* $E_+(X) = D_8$ *or* E_7, *respectively.*
(7) *If* $b_0(X_{\mathbb{R}}) = 5$, *then* Y *is a conic bundle or a Del Pezzo surface of degree* 1. *The two cases are distinguished by* $E_+(X) = D_{10}$ *or* E_8, *respectively.*
(8) *If* $b_0(X_{\mathbb{R}}) \geqslant 6$, *then* Y *is an* $\mathbb{R}$*-minimal conic bundle.*

Thus, the deformation class of X *admits a canonical* $\mathbb{R}$*-minimal model whose deformation type is determined by the topology of* (X, conj).

Remark. In fact, in 5.2(1) one can choose and fix any of the two real structures on Y with $Y_{\mathbb{R}} = \varnothing$, as they become equivalent after a single blow-up at a pair of conjugate points.

Proof. Since the number of connected components of the real part of a surface is obviously preserved by a blow-up, the possible minimal models of X can be found using Theorem 2.4.1. These models can further be deformed using Theorem 2.3. This completes the proof in all cases except (1) and (4). (In cases (6) and (7) the lattice $E_+(X)$ is found by a direct calculation. In view of Corollary 5.1.2 it suffices to calculate the lattice for the minimal models. In fact, for the two Del Pezzo surfaces in question one has $E_+ = K^\perp$, where K is the canonical class, and the latter lattice is well known, see, e.g., [M] or [Dem].)

In cases (1) and (2) Theorems 2.4.1 and 2.3 yield $Y = \mathbb{P}^1 \times \mathbb{P}^1$ with one of the two real structures with empty real part. The two structures become isomorphic after a blow-up at a pair of conjugate points (see, e.g., [DIK, Lemma in 17.3.1]); hence, if X is not $\mathbb{R}$-minimal, case (1), one can choose any structure.

In case (4) Theorems 2.4.1 and 2.3 yield either $Y = \mathbb{P}^2$, or $Y = \Sigma_{2k+1}$, or $Y = \mathbb{P}^1 \times \mathbb{P}^1$ with $Y_{\mathbb{R}} \neq \varnothing$. The ruled surfaces Σ_{2k+1} are deformation equivalent to Σ_1 (cf. 4.2, Cases 1, 2), which blows down to $\mathbb{P}^2$. If $Y = \mathbb{P}^1 \times \mathbb{P}^1$, then the passage to X contains at least one blow-up at a real point (recall that $X_{\mathbb{R}}$ is nonorientable); after such a blow-up the surface also blows down to $\mathbb{P}^2$. □

6. Proof of the Main Theorem

6.1. Cyclic order of components. Let $Y \to \mathbb{P}^1$ be an $\mathbb{R}$-minimal real rational conic bundle, $Y_{\mathbb{R}} = mS$. Then the projections of the connected components of $Y_{\mathbb{R}}$ are disjoint segments in $\mathbb{P}^1_{\mathbb{R}} = S^1$. This defines a canonical cyclic order on the set of components of $Y_{\mathbb{R}}$. From Proposition 6.1.1 below it follows, in particular, that this cyclic order does not depend on the conic bundle structure of Y.

6.1.1. Proposition. *Let* X *be a real rational surface obtained by a sequence of real blow-ups from an* $\mathbb{R}$*-minimal real rational conic bundle* $Y \to \mathbb{P}^1$. *Then the cyclic order of the components of* $Y_{\mathbb{R}}$ *is determined by the classes realized by the components of* $X_{\mathbb{R}}$ *in* $H_2(X/\mathrm{conj}\,; \mathbb{Z}_2)$. *In particular, the set of connected components of* $X_{\mathbb{R}}$ *also has a natural cyclic order.*

Proof. Recall that for any real surface X the projection $X \to X/\mathrm{conj}$ induces an isomorphism $L_+(X) \otimes \mathbb{Q} = H_2(X/\mathrm{conj}\,; \mathbb{Q})$. Hence, from Lemma 5.1.1 it follows that $H_2(Y/\mathrm{conj})$ is a negative definite lattice and an easy calculation for the minimal models shows that $H_2(Y/\mathrm{conj}) = 2m\langle -1 \rangle$. (Alternatively, this statement can be derived from Donaldson's theorem [Do] or from the fact that Y/conj decomposes into connected sum of several copies of $\overline{\mathbb{P}^2}$, see [Fi].) Thus, $H_2(Y/\mathrm{conj})$ has a canonical (up to reordering and multiplication by (-1)) basis $e_1, \ldots, e_{2m}$ such that $e_i^2 = -1$ and $e_i e_j = 0$ for $i \neq j$. This basis can easily be visualized. Let $F_1, \ldots, F_m$ be the components of $Y_\mathbb{R}$ (numbered according to their cyclic order) and $f_i \in H_2(Y/\mathrm{conj})$ their classes. Then one can take for e_{2i+1} the class realized by the image in Y/conj of a vanishing cycle in Y corresponding to joining the components F_i and F_{i+1}. (To unify the notation, we use the 'cyclic' numbering of the components and classes: $F_{i+km} = F_i$, $e_{i+2km} = e_i$, etc.) Let, further, $g_1, \ldots, g_{2m} \in H_2(Y/\mathrm{conj})$ be the classes realized by the images of the singular fibers of Y, numbered so that F_i is adjacent to the fibers corresponding to g_{2i} and g_{2i+1}. One has $g_i^2 = -2$ and $g_i g_j = 0$ for $i \neq j$. It follows that these elements generate the lattice over $\mathbb{Q}$, and calculating the intersection numbers, one obtains $e_{2i+1} = \frac{1}{2}(g_{2i+2} - g_{2i+1})$. Hence, the other m generators have the form $e_{2i+2} = \frac{1}{2}(g_{2i+2} + g_{2i+1})$ and, under appropriate choice of the signs, one has $f_i = e_{2i+2} - e_{2i+1} - e_{2i} - e_{2i-1}$.

Since $H_2(Y/\mathrm{conj})$ is torsion free (as $Y_\mathbb{R} \neq \varnothing$), the group $H_2(Y/\mathrm{conj}; \mathbb{Z}_2) = H_2(Y/\mathrm{conj}) \otimes \mathbb{Z}_2$ also has a canonical basis $\bar{e}_i = e_i \bmod 2$, $i = 1, \ldots, 2m$ and, denoting by $\bar{f}_i = f_i \bmod 2$ the class realized by F_i, one has $\bar{f}_i = \bar{e}_{2i+2} - \bar{e}_{2i+1} - \bar{e}_{2i} - \bar{e}_{2i-1}$. Hence, the cyclic order is determined as follows: *two components F_i, F_j are neighbors if and only if there is a standard basis element $\bar{e}$ such that $\bar{e}\bar{f}_i = \bar{e}\bar{f}_j = 1$.*

It remains to note that the above rule extends literally to any surface X obtained from Y by a sequence of blow-ups. Indeed, since $H_2(X/\mathrm{conj}) = H_2(Y/\mathrm{conj}) \oplus q\langle -1\rangle$ (the extra summands corresponding to the pairs of complex conjugate exceptional divisors), this lattice and hence the group $H_2(X/\mathrm{conj}; \mathbb{Z}_2) = H_2(X/\mathrm{conj}) \otimes \mathbb{Z}_2$ has a canonical basis, which can be used to determine the cyclic order. □

6.1.2. Proposition. *Any cyclic permutation and/or reversing of the order of the components of the real part of an $\mathbb{R}$-minimal real rational conic bundle can be realized by a deformation and automorphism of the surface.*

Proof. The statement can easily be proved by constructing explicit symmetric surfaces. A cyclic permutation sending each component to one of its neighbors can be realized in the family Y_t of surfaces given by the affine equation $x^2 + y^2 = (\omega z)^m + (\omega z)^{-m}$, where $\omega = \exp(2\pi i t/m)$, $t \in [0,1]$ is the parameter of the deformation, and the real structure is given by $(x, y, z) \mapsto (\bar{x}, \bar{y}, \bar{z}^{-1})$. The order of the components is reversed by the automorphism $(x, y, z) \mapsto (x, y, z^{-1})$ of Y_0. □

Let now Y be a real degree 1 Del Pezzo surface with $Y_\mathbb{R} = V_1 \sqcup 4S$. Then Y can canonically be represented as the covering of a quadratic cone branched over the vertex and a real M-curve $B \in |3e_\infty|$ (see, e.g., [Dem]; since the map is defined by the anti-bicanonical system, it is necessarily real), the spherical components of $Y_\mathbb{R}$ corresponding to the ovals of the branch curve. Hence, the ruling of the cone determines a canonical cyclic order on the set of the spherical components of $Y_\mathbb{R}$. Next two statements are proved similar to 6.1.1 and 6.1.2.

6.1.3. Proposition. *Let X be a real rational surface obtained by a sequence of real blow-ups from a real degree 1 Del Pezzo surface Y with $Y_\mathbb{R} = V_1 \sqcup 4S$. Then the cyclic order of the spherical components of $Y_\mathbb{R}$ and the V_1-component of $Y_\mathbb{R}$ are determined by the classes realized by the components of $X_\mathbb{R}$ in $H_2(X; \mathbb{Z}_2)$ and*

$H_2(X/\mathrm{conj}\,;\mathbb{Z}_2)$. *In particular, $X_{\mathbb{R}}$ has a distinguished component that blows down to V_1, and the set of the other components of $X_{\mathbb{R}}$ has a natural cyclic order.* □

Remark. The rule for determining the cyclic order of the components is the same as in the case of conic bundles. The component of $X_{\mathbb{R}}$ that blows down to V_1 can be distinguished as the 'essentially nonorientable' one: it is the only component whose class in

$$\mathrm{Ker}\big[(1-\mathrm{conj}_*)\colon H_2(X;\mathbb{Z}_2)\to H_2(X;\mathbb{Z}_2)\big]$$

does not belong to the image of $L_+(X)$.

6.1.4. Proposition. *Any cyclic permutation and/or reversing of the order of the spherical components of the real part of a real degree 1 Del Pezzo surface Y with $Y_{\mathbb{R}}=V_1\sqcup 4S$ can be realized by a deformation and automorphism of the surface.* □

6.2. Permutations of components. Let us show that the connected components of the real part of all other $\mathbb{R}$-minimal real rational surfaces can be permuted by a deformation.

6.2.1. Proposition. *Let Y be either an $\mathbb{R}$-minimal real rational conic bundle with $Y_{\mathbb{R}}=2S$ or $3S$ or a real degree 2 Del Pezzo surface with $Y_{\mathbb{R}}=4S$. Then any permutation of the components of $Y_{\mathbb{R}}$ can be realized by a deformation.*

Proof. A surface as in the statement with $m=2$, 3, or 4 connected components can be constructed from a plane M-quartic B with $(4-m)$ isolated double points: one blows up the double points of B, blows down the line through them (in the case $m=2$), and constructs the double covering of the resulting surface branched over the transform of B. The statement follows now from the well known fact that the ovals of an M-quartic can be permuted by a rigid isotopy (which, in turn, can easily be proved by constructing symmetric quartics and using the connectedness of the group $PGL(3,\mathbb{R})$). □

6.3. Proof of Theorem 2.2. Let X_1, X_2 be two real rational surfaces with diffeomorphic real structures. In view of 5.2, X_1 and X_2 are deformation equivalent to some surfaces X_1', X_2' that blow down to the same $\mathbb{R}$-minimal model Y. Let $R_i\subset Y_{\mathbb{R}}$ and $I_i\subset Y\smallsetminus Y_{\mathbb{R}}$ be the corresponding sets of blow-up centers, $i=1,2$. (Up to deformation one can assume that all blow-ups are independent.) Clearly, the numbers $\#R_1=\#R_2$ and $\#I_1=\#I_2$ are determined by the topology of the involutions, and since $Y\smallsetminus Y_{\mathbb{R}}$ is connected, one can assume that $I_1=I_2$.

Let $F_1,\ldots,F_k$ be the components of $Y_{\mathbb{R}}$. From 6.1.1–6.1.4 and 6.2.1 it follows that the components of one of the two copies of Y can be reordered by a deformation and automorphism of the surface so that for each $j=1,\ldots,k$ one has $\#(R_1\cap F_j)=\#(R_2\cap F_j)$. Then one can also isotope R_1 to R_2, which results in a deformation of X_1' to X_2'. □

6.4. Invariants. In fact, we have proved that the deformation type of a real rational surface is determined by certain homological information. More precisely, extracting the invariants used in the proofs (see 5.2 and 6.1), one arrives at the following statement.

6.4.1. Theorem. *Two real rational surfaces X', X'' are deformation equivalent if and only if there is a homeomorphism $X_{\mathbb{R}}'\to X_{\mathbb{R}}''$ preserving the additional structures described below and one has $E_+(X')=E_+(X'')$ (see 5.1.2; this condition holds automatically unless $b_0(X_{\mathbb{R}}')=b_0(X_{\mathbb{R}}'')=4$ or 5).*

The additional structures to be preserved are:

(1) *a distinguished 'essentially nonorientable' component of* $X_{\mathbb{R}}$*, defined in the case* $E_+(X) = E_8$*: this is the only component whose class in* $H_2(X;\mathbb{Z}_2)$ *does not belong to the image of* $L_+(X)$*;*

(2) *a cyclic order on the set of the other components, defined whenever* $b_0(X) \geqslant 4$ *as follows: two components* F_1*,* F_2 *are neighbors if and only if there is a canonical basis element* $\bar{e} \in H_2(X/\mathrm{conj};\mathbb{Z}_2) = H_2(X/\mathrm{conj}) \otimes \mathbb{Z}_2$ *(see the proof of* 6.1.1*) such that* $\bar{e}[F_1] = \bar{e}[F_2] = 1$*.* □

References

[CF] F. Catanese, P. Frediani, *Real hyperelliptic surfaces and the orbifold fundamental group*, `math.AG/0012003`, 2000.

[Co1] A. Comessatti, *Fondamenti per la geometria sopra le superficie razionali dal punto di vista reale*, Math. Ann. **43** (1912), 1–72.

[Co2] A. Comessatti, *Sulle varietà abeliane reali,* I *e* II, Ann. Mat. Pura Appl. **2 and 4** (1924 and 1926), 67–106 and 27–71.

[DIK] A. Degtyarev, I. Itenberg, V. Kharlamov, *Real Enriques surfaces*, Lecture Notes in Math., vol. 1746, Springer–Verlag, 2000.

[DK1] A. Degtyarev, V. Kharlamov, *On the moduli space of real Enriques surfaces*, C.R. Acad. Sci. **324** (1997), no. 3, 317–322.

[DK2] A. Degtyarev, V. Kharlamov, *Around real Enriques surfaces*, Revista Matemática Univ. Complutense Madrid **10** (1997), 93–109.

[DK3] A. Degtyarev, V. Kharlamov, *Topological properties of real algebraic varieties: de côtè de chez Rokhlin.*, Russ. Math. Surveys. **55** (2000), no. 4, 735–814.

[Dem] M. Demazure, *Surfaces de Del Pezzo,* II–V, Lecture Notes in Math., vol. 777, Springer–Verlag, 1980, pp. 23–69.

[Do] S. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), no. 2, 279–315.

[Fi] S. Finashin, *Decomposability of quotients by complex conjugation for rational and Enriques surfaces*, Topology Appl. **79** (1997), no. 2, 121–128.

[FQ] R. Friedman, Z. Qin, *On complex surfaces diffeomorphic to rational surfaces*, Invent. Math. **120** (1995), 81–117.

[KK] V. Kharlamov, Vik. Kulikov, *On real structures of rigid surfaces*, `math.AG/0101098`, 2001.

[Ko] J. Kollár, *Real Algebraic Surfaces*, `math.AG/9712003`, 2000.

[M] Yu. I. Manin, *Cubic forms. Algebra, geometry, arithmetic*, Translated from the Russian by M. Hazewinkel. Second edition. North-Holland Mathematical Library, 4., North-Holland Publishing Co., Amsterdam-New York, 1986., pp. x+326.

[MT] Yu. I. Manin, M. A. Tsfasman, *Rational varieties: algebra, geometry and arithmetics*, Russian Math. Surveys **41** (1986), no. 2, 51–116.

[N] V. V. Nikulin, *Integer quadratic forms and some of their geometrical applications*, Math. USSR–Izv. **43** (1979), 103–167.

[S] R. Silhol, *Real algebraic surfaces*, Lecture Notes in Math., vol. 1392, Springer–Verlag, 1989.

[W] J. Y. Welschinger, *Courbes flexibles réelles sur les surfaces réglées de base* $\mathbb{P}^1$, preprint, 2000.

Bilkent University, Turkey
E-mail address: `degt@fen.bilkent.edu.tr`

Institut de Recherche Mathématique Avancée,
Université Louis Pasteur et CNRS, France
E-mail address: `kharlam@math.u-strasbg.fr`