ON THE NUMBER OF COMPONENTS OF A COMPLETE INTERSECTION OF REAL QUADRICS

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Abstract. Our main results concern complete intersections of three real quadrics. We prove that the maximal number $B^0_2(N)$ of connected components that a regular complete intersection of three real quadrics in $\mathbb{P}^N$ can have differs at most by one from the maximal number of ovals of the submaximal depth $[(N-1)/2]$ of a real plane projective curve of degree $d = N+1$. As a consequence, we obtain a lower bound $\frac{1}{4}N^2 + O(N)$ and an upper bound $\frac{3}{8}N^2 + O(N)$ for $B^0_2(N)$.

1. Introduction

1.1. Statement of the problem and principal results. The question on the maximal number of connected components that a real projective variety of a given (multi-)degree may have remains one of the most difficult and least understood problems in topology of real algebraic varieties. Besides the trivial case of varieties of dimension zero, essentially the only general situation where this problem is solved is that of curves: the answer is given by the famous Harnack inequality in the case of plane curves [15], and by a combination of the Castelnuovo–Halphen [7], [14] and Harnack–Klein [19] inequalities in the case of curves in projective spaces of higher dimension, see [16] and [24]. The immediate generalization of the Harnack inequality given by the Smith theory, the so called *Smith inequality* (see, *e.g.*, [9]), involves all Betti numbers of the real part and thus gives too rough a bound when applied to the problem of the number of connected components in a straightforward manner (see, *e.g.*, the discussion in Section 6.7).

In this paper, we address the problem of the maximal number of connected components in the case of varieties defined by equations of degree two, *i.e.*, complete intersections of quadrics. To be more precise, let us denote by

$$B^0_r(N), \quad 0 \leqslant r \leqslant N-1,$$

the maximal number of connected components that a regular complete intersection of $r+1$ real quadrics in $\mathbb{P}^N_{\mathbb{R}}$ can have. Certainly, as we study regular complete intersections of even degree, the actual number of connected components covers the whole range of values between 0 and $B^0_r(N)$.

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In the following three extremal cases, the answer is easy and well known:

- $B^0_0(N) = 1$ for all $N \geqslant 2$ (a single quadric),
- $B^0_1(N) = 2$ for all $N \geqslant 3$ (intersection of two quadrics), and
- $B^0_{N-1}(N) = 2^N$ for all $N \geqslant 1$ (intersection of dimension zero).

To our knowledge, very little was known in the next case $r = 2$ (intersection of three quadrics); even the fact that $B^0_2(N) \to \infty$ as $N \to \infty$ does not seem to have been observed before. Our principal result here is the following theorem, providing a lower bound $\frac{1}{4}N^2 + O(N)$ and an upper bound $\frac{3}{8}N^2 + O(N)$ for $B^0_2(N)$.

1.1.1. Theorem. *For all $N \geqslant 4$, one has*

$$\frac{1}{4}(N-1)(N+5) - 2 < B^0_2(N) \leqslant \frac{3}{2}k(k-1) + 2,$$

where $k = [\frac{1}{2}N] + 1$.

The proof of Theorem 1.1.1 found in Section 4.5 is based on a real version of Dixon's correspondence [10] between nets of quadrics (*i.e.*, linear systems generated by three independent quadrics) and plane curves equipped with a non-vanishing theta characteristic. Another tool is an Agrachev spectral sequence [1] computing the homology of a complete intersection of quadrics in terms of its spectral variety. The following intermediate result seems to be of an independent interest.

1.1.2. Definition. Define the *Hilbert number* $\operatorname{Hilb}(d)$ as the maximal number of ovals of the submaximal depth $[d/2]-1$ that a nonsingular real plane algebraic curve of degree d may have. (Recall that the depth of an oval of a curve of degree $d = N+1$ does not exceed $[(N+1)/2]$. A brief introduction to topology of nonsingular real plane algebraic curves is found in Section 4.2.)

1.1.3. Theorem. *For any integer $N \geqslant 4$, one has*

$$\operatorname{Hilb}(N+1) \leqslant B^0_2(N) \leqslant \operatorname{Hilb}(N+1) + 1.$$

This theorem is proved in Section 4.4. The few known values of $\operatorname{Hilb}(N+1)$ and $B^0_2(N)$ are given by the following table.

N	3	4	5	6	7
$\operatorname{Hilb}(N+1)$	4	6	9	13	17 or 18
$B^0_2(N)$	8	6	10	13 or 14	17, 18, or 19

It is worth mentioning that $B^0_2(4) = \operatorname{Hilb}(5)$, whereas $B^0_2(5) = \operatorname{Hilb}(6) + 1$, see Sections 6.4 and 6.5, respectively. (The case $N = 3$ is not covered by Theorem 1.1.3.) At present, we do not know the precise relation between the two sequences.

For completeness, we also discuss another extremal case, namely that of curves. Here, the maximal number of components is attained on the M-curves, and the statement should be a special case of the general Viro–Itenberg construction producing maximal complete intersections of any multidegree (see [17] for a simplified version of this construction). The result is the following theorem, which is proved in §5 by means of a Harnack like construction.

1.1.4. Theorem. *For all $N \geqslant 2$, one has $B^0_{N-2}(N) = 2^{N-2}(N-3) + 2$.*

1.2. Conventions. Unless indicated explicitly, the coefficients of all homology and cohomology groups are $\mathbb{Z}_2$. For a compact complex curve C, we freely identify $H^1(C; R) = H_1(C; R)$ (for any coefficient ring R) *via* the Poincaré duality. We do not distinguish between line bundles, invertible sheaves, and classes of linear equivalence of divisors, switching freely from one to another.

A *quod erat demonstrandum* symbol '□' after a statement means that no proof will follow: either the statement is obvious and the proof is straightforward, or the proof has already been explained, or a reference is given at the beginning of the section.

1.3. Content of the paper. The bulk of the paper, except §5 where Theorem 1.1.4 is proved, is devoted to the proof of Theorems 1.1.1 and 1.1.3. In §2, we collect the necessary material on the theta characteristics, including the real version of the theory. In §3, we introduce and study the spectral curve of a net and discuss Dixon's correspondence. The aim is to introduce the Spin- and index (semi-)orientations of the real part of the spectral curve, the former coming from the theta characteristic and the latter, directly from the topology of the net, and to show that the two semi-orientations coincide. In §4, we introduce the Agrachev spectral sequence computing the Betti numbers of the common zero locus of a net in terms of its index function. The sequence is used to prove Theorem 1.1.3, relating the number $B^0_2(N)$ in question and topology of real plane algebraic curves of of degree $N + 1$. Then, we cite a few known estimates on the number of the ovals of a curve (see Corollaries 4.5.2 and 4.5.4) and deduce Theorem 1.1.1. Finally, in §6 we discuss a few particular cases of nets and address several related questions.

1.4. Acknowledgements. This paper responds to the following question posed to us by D. Pasechnik and B. Shapiro: is the number of connected components of an intersection of $(r + 1)$ real quadrics in $\mathbb{P}^N_{\mathbb{R}}$ bounded by a constant $C(r)$ independent of N? We are grateful to them for the motivation of our involvement into this circle of problems. The paper was essentially completed during our stay at *Centre Interfacultaire Bernoulli*, *École polytechnique fédérale de Lausanne*. The current version was prepared during the stay of the second and third authors at the *Max-Planck-Institut für Mathematik*, Bonn. We extend our gratitude to these institutions for their hospitality and excellent working conditions.

2. Theta characteristics

For the reader's convenience, we cite a number of known results related to the (real) theta characteristics on algebraic curves. Appropriate references are given at the beginning of each section.

To avoid various 'boundary effects', *we only consider curves of genus at least* 2. *For real curves, we assume that the real part is nonempty.*

2.1. Complex curves (see [5], [20]). Recall that a *theta characteristic* on a nonsingular compact complex curve C is a line bundle θ on C such that θ^2 is isomorphic to the canonical bundle K_C. In topological terms, a theta characteristic is merely a Spin-structure on the topological surface C. One associates with a theta characteristic θ the integer $h(\theta) = \dim H^0(C; \theta)$ and its $\mathbb{Z}_2$-residue $\phi(\theta) = h(\theta) \bmod 2$.

Let $\mathfrak{S} \subset \operatorname{Pic}^{g-1} C$ be the set of theta characteristics on C. The map $\phi\colon \mathfrak{S} \to \mathbb{Z}_2$ has the following fundamental properties:

(1) ϕ is preserved under deformations;
(2) ϕ is a quadratic extension of the intersection index form;
(3) $\operatorname{Arf} \phi = 0$ (equivalently, ϕ vanishes at $2^{g-1}(2^g + 1)$ points).

To precise the meaning of items (2) and (3), notice that $\mathfrak{S}$ is an affine space over $H^1(C)$, so that (2) is equivalent to the identity

$$\phi(a + x + y) - \phi(x) - \phi(y) + \phi(a) = \langle x, y\rangle,$$

while $\operatorname{Arf} \phi$ is the usual Arf-invariant of ϕ after $\mathfrak{S}$ is identified with $H^1(C)$ by choosing for zero any element $\theta \in \mathfrak{S}$ with $\phi(\theta) = 0$.

A theta characteristic θ is called *even* if $\phi(\theta) = 0$; otherwise it is called *odd*. An even theta characteristic is called *non-zero* (or *non-vanishing*) if $h(\theta) = 0$.

Recall that there are canonical bijections between the set of Spin-structures on C, the set of quadratic extensions of the intersection index form on $H^1(C)$, and the set of theta characteristics on C. In particular, a theta characteristic $\theta \in \mathfrak{S}$ is uniquely determined by the quadratic function ϕ_θ on $H^1(C)$ given by $\phi_\theta(x) = \phi(\theta+x)-\phi(\theta)$. One has $\operatorname{Arf} \phi_\theta = \phi(\theta)$.

2.1.1. The moduli space of pairs (C, θ), where C is a curve of a given genus and θ is a theta characteristic on C, has two connected components, formed by even and odd theta characteristics. If C is restricted to nonsingular plane curves of a given degree d, the result is almost the same. Namely, if d is even, there are still two connected components, while if d is odd, there is an additional component formed by the pairs $(C, \frac{1}{2}(d-3)H)$, where H is the hyperplane section divisor. In topological terms, the extra component consists of the pairs $(C, \mathcal{R})$, where $\mathcal{R}$ is the Rokhlin function (see [25]); it is even if $d = \pm 1 \bmod 8$ and odd if $d = \pm 3 \bmod 8$. The other theta characteristics still form two connected components, distinguished by the parity.

2.2. Real curves (see [12], [13]). Now, let C be a real curve, *i.e.*, a complex curve equipped with an anti-holomorphic involution $c\colon C \to C$ (a *real structure*). Recall that we always assume that the genus $g = g(C) > 1$ and that the *real part* $C_{\mathbb{R}} = \operatorname{Fix} c$ is nonempty.

Consider the set

$$\mathfrak{S}_{\mathbb{R}} = \mathfrak{S} \cap \operatorname{Pic}^{g-1}_{\mathbb{R}} C$$

of *real* (*i.e.*, c-invariant) theta characteristics. Under the assumptions above, there are canonical bijections between $\mathfrak{S}_{\mathbb{R}}$, the set of c-invariant Spin-structures on C, and the set of c_*-invariant quadratic extensions of the intersection index form on the group $H^1(C)$.

The set $\mathfrak{S}_{\mathbb{R}}$ of real theta characteristics is a principal homogeneous space over the $\mathbb{Z}_2$-torus $(J_{\mathbb{R}})_2$ of torsion 2 elements in the real part $J_{\mathbb{R}}(C)$ of the Jacobian $J(C)$. In particular, $\operatorname{Card} \mathfrak{S}_{\mathbb{R}} = 2^{g+r}$, where $r = b_0(C_{\mathbb{R}}) - 1$. More precisely, $\mathfrak{S}_{\mathbb{R}}$ admits a free action of the $\mathbb{Z}_2$-torus $(J^0_{\mathbb{R}})_2$, where $J^0_{\mathbb{R}} \subset J_{\mathbb{R}}$ is the component of zero. This action has 2^r orbits, which are distinguished by the restrictions $\phi_\theta\colon (J^0_{\mathbb{R}})_2 \to \mathbb{Z}_2$, which are linear forms. Indeed, the form ϕ_θ depends only on the orbit of an element $\theta \in \mathfrak{S}_{\mathbb{R}}$, and the forms defined by elements θ_1, θ_2 in distinct orbits of the action differ.

Alternatively, one can distinguish the orbits above as follows. Realize an element $\theta \in \mathfrak{S}_{\mathbb{R}}$ by a real divisor D and, for each real component $C_i \subset C_{\mathbb{R}}, 1 \leqslant i \leqslant r+1$, count the residue $c_i(\theta) = \operatorname{Card}(C_i \cap D) \bmod 2$. The residues $(c_i(\theta)) \in \mathbb{Z}_2^{r+1}$ are subject to relation $\sum c_i(\theta) = g - 1 \mod 2$ and determine the orbit.

In most cases, within each of the above 2^r orbits, the numbers of even and odd theta characteristics coincide. The only exception to this rule is the orbit given by $c_1(\theta) = \ldots = c_r(\theta) = 1$ in the case when C is a dividing curve.

2.2.1. Lemma. *With one exception, any (real) even theta characteristic on a (real) nonsingular plane curve becomes non-zero after a small (real) perturbation of the curve in the plane. The exception is Rokhlin's theta characteristic* $\frac{1}{2}(d-3)H$ *on a curve of degree* $d = \pm 1 \bmod 8$, *see* 2.1.1.

Proof. As is well known, the vanishing of a theta characteristic is a condition analytic with respect to the coefficients of the curve. (Essentially, this statement follows from the fact that the Riemann Θ-divisor depends on the coefficients analytically.) Hence, in the space of pairs (C, θ), where C is a nonsingular plane curve of degree d and θ is a theta characteristic on C, the pairs (C, θ) with non-vanishing θ form a Zariski open set. Since there does exist a curve of degree d with a non-zero theta characteristic (*e.g.*, any nonsingular spectral curve, see 3.3.1), this set is nonempty and hence dense in the (only) component formed by the even theta characteristics other than $\frac{1}{2}(d-3)H$. □

3. Linear systems of quadrics

3.1. Preliminaries. Consider an injective linear map $x \mapsto q_x$ from $\mathbb{C}^{r+1}$ to the space $S^2\mathbb{C}^{N+1}$ of homogeneous quadratic polynomials on $\mathbb{C}^{N+1}$. It defines a linear system of quadrics in $\mathbb{P}^N$ of dimension r, *i.e.*, an r-subspace in the projective space $\mathcal{C}_2(\mathbb{P}^N)$ of quadrics. Conversely, any linear system of quadrics is defined by a unique, up to obvious equivalence, linear map as above.

Occasionally, we will fix coordinates $(u_0, \ldots, u_N)$ in $\mathbb{C}^{N+1}$ and represent q_x by a matrix Q_x, so that $q_x(u) = \langle Q_x u, u\rangle$. Clearly, the map $x \mapsto Q_x$ is also linear.

Define the common zero set

$$V = \bigl\{u \in \mathbb{P}^N \bigm| q_x(u) = 0 \text{ for all } x \in \mathbb{C}^{r+1}\bigr\} \subset \mathbb{P}^N$$

and the *Lagrange hypersurface*

$$L = \bigl\{(x, u) \in \mathbb{P}^r \times \mathbb{P}^N \bigm| q_x(u) = 0\bigr\} \subset \mathbb{P}^r \times \mathbb{P}^N.$$

(As usual, the vanishing condition $q_x(u) = 0$ does not depend on the choice of the representatives of x and u.) The following statement is straightforward.

3.1.1. Lemma. *An intersection of quadrics V is regular if and only if the associated Lagrange hypersurface L is nonsingular.* □

3.2. The spectral variety. Define the *spectral variety* C of a linear system of quadrics $x \mapsto q_x$ *via*

$$C = \bigl\{x \in \mathbb{P}^r \bigm| \det Q_x = 0\bigr\} \subset \mathbb{P}^r.$$

Clearly, this definition does not depend on the choice of the matrix representation $x \mapsto Q_x$: the spectral variety is formed by the elements of the linear system which are singular quadrics. More precisely (as a scheme), C is the intersection of the linear system with the discriminant hypersurface $\Delta \subset \mathcal{C}_2(\mathbb{P}^N)$.

In what follows, we assume that C is a proper subset of $\mathbb{P}^r$, *i.e.*, we exclude the possibility $C = \mathbb{P}^r$, as in this case all quadrics in the system have a common singular point and, hence, V is not a regular intersection. Under this assumption, $C \subset \mathbb{P}^r$ is a hypersurface of degree $N+1$, possibly not reduced. By the dimension argument, C is necessarily singular whenever $r \geqslant 3$. Furthermore, even in the case $r = 1$ or 2 (pencils or nets), the spectral variety of a regular intersection may still be singular.

3.2.1. Lemma. *Let x be an isolated point of the spectral variety C of a pencil. Then x is a simple point of C if and only if the quadric $\{q_x = 0\}$ has a single singular point, and this point is not a base point of the pencil.* □

3.2.2. Corollary. *If $x \in C$ is a smooth point, then* $\operatorname{corank} q_x = 1$, i.e., *the quadric $\{q_x(u) = 0\}$ has a single singular point.*

Proof. Restrict the system to a generic pencil through the point. □

3.2.3. Lemma. *If the spectral variety C of a linear system is nonsingular, then the complete intersection V is regular.*

Proof. It is easy to see that the common zero set V of a linear system is a regular complete intersection if and only if none of the members of the system has a singular point in V. Thus, it suffices to observe that a generic pencil through a point $x \in C$ is transversal to C; hence, x is a simple point of its discriminant variety and the only singular point of $\{q_x = 0\}$ is not in V, see Lemma 3.2.1. □

3.3. The Dixon construction (see [10], [11]). From now on, we confine ourselves to the case of nets, *i.e.*, $r = 2$. As explained in 3.2, each net gives rise to its spectral curve, which is a curve $C \subset \mathbb{P}^2$ of degree $d = N + 1$; if the net is generic, C is nonsingular.

3.3.1. Assume that the spectral curve C is nonsingular. Then, at each point $x \in C$, the kernel $\operatorname{Ker} Q_x \subset \mathbb{R}^{N+1}$ is a 1-subspace, see Lemma 3.2.2. The correspondence $x \mapsto \operatorname{Ker} Q_x$ defines a line bundle $\mathcal{K}$ on C or, after a twist, a line bundle $\mathcal{L} = \mathcal{K}(d-1)$. The latter has the following properties: $\mathcal{L}^2 = \mathcal{O}_C(d-1)$ (so that $\deg \mathcal{L} = \frac{1}{2}d(d-1)$) and $H^0(C, \mathcal{L}(-1)) = 0$. Thus, switching to $\theta = \mathcal{L}(-1)$, we obtain a non-vanishing theta characteristic on C; it is called the *spectral theta characteristic* of the net.

The following theorem is due to A. C. Dixon [10].

3.3.2. Theorem. *Given a non-vanishing theta characteristic θ on a nonsingular plane curve C of degree $N+1$, there exists a unique, up to projective transformation of $\mathbb{P}^N$, net of quadrics in $\mathbb{P}^N$ such that C is its spectral curve and θ is its spectral theta characteristic.* □

3.3.3. The original proof by Dixon contains an explicit construction of the net. We outline this construction below. Pick a basis $\phi_{11}, \phi_{12}, \ldots, \phi_{1d} \in H^0(C, \mathcal{L})$ and let

$$v_{11} = \phi_{11}^2,\ v_{12} = \phi_{11}\phi_{12},\ \ldots,\ v_{1d} = \phi_{11}\phi_{1d} \in H^0(C, \mathcal{L}^2).$$

Since the restriction map $H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(d-1)) \to H^0(C; \mathcal{O}_C(d-1)) = H^0(C; \mathcal{L}^2)$ is onto, we can regard v_{1i} as homogeneous polynomials of degree $d-1$ in the coordinates x_0, x_1, x_2 in $\mathbb{P}^2$. Let also $U(x_0, x_1, x_2) = 0$ be the equation of C. The curve $\{v_{12} = 0\}$ passes through all points of intersection of C and $\{v_{11} = 0\}$. Hence, there are homogeneous polynomials v_{22}, w_{1122} of degrees $d-1$, $d-2$, respectively, such that

$$v_{12}^2 = v_{11}v_{22} - Uw_{1122}.$$

In the same way, we get polynomials v_{rs}, w_{11rs}, $2 \leqslant s, r \leqslant d$, such that

$$v_{1r}v_{1s} = v_{11}v_{rs} - Uw_{11rs}.$$

Obviously, $v_{rs} = v_{sr}$ and $w_{11rs} = w_{11sr}$. It is shown in [10] that

(1) the algebraic complement A_{rs} in the $(d \times d)$ symmetric matrix $[v_{ij}]$ is of the form $U^{d-2}\beta_{rs}$, where β_{rs} are certain linear forms, and
(2) the determinant $\det[\beta_{ij}]$ is a constant non-zero multiple of U.

(It is the non-vanishing of the theta characteristic that is used to show that the latter determinant is non-zero.) Thus, C is the spectral curve of the net $Q_x = [\beta_{ij}]$.

It is immediate that the construction works over any field of characteristic zero. Hence, we obtain the following real version of the Dixon theorem.

3.3.4. Theorem. *Given a nonsingular real plane curve C of degree $N+1 \geqslant 4$ and with nonempty real part, and a non-vanishing real theta characteristic θ on C, there exists a unique, up to real projective transformation of $\mathbb{P}^N_{\mathbb{R}}$, real net of quadrics in $\mathbb{P}^N_{\mathbb{R}}$ such that C is its spectral curve and θ is its spectral theta characteristic.* □

3.4. The Spin**-orientation** (*cf.* [21], [22]). Let (C, c) be a real curve equipped with a real theta characteristic θ. As above, assume that $C_{\mathbb{R}} \neq \varnothing$. Then, the real structure of C lifts to a real structure (*i.e.*, a fiberwise anti-linear involution) $c\colon \theta \to \theta$, which is unique up to a phase $e^{i\phi}$, $\phi \in \mathbb{R}$. If an isomorphism $\theta^2 = K_C$ is fixed, one can choose a lift compatible with the canonical action of c on K_C; such a lift is unique up to multiplication by i.

Fix a lift $c\colon \theta \to \theta$ as above and pick a c-real meromorphic section ω of θ. Then, ω^2 is a real meromorphic 1-form with zeros and poles of even multiplicities. Therefore, it determines an orientation of $C_{\mathbb{R}}$. This orientation does not depend on the choice of ω, and it is reversed when switching from c to ic. Thus, it is, in fact, a semi-orientation of $C_{\mathbb{R}}$; it is called the Spin-*orientation* defined by θ.

The definition above can be made closer to original Dixon's construction outlined in 3.3.3. One can replace ω by a meromorphic section ω' of $\theta(1)$ and treat $(\omega')^2$ as a real meromorphic 1-form with values in $\mathcal{O}_C(2)$; the latter is trivial over $C_{\mathbb{R}}$.

The following, more topological, definition is equivalent to the previous one. Recall that a semi-orientation is essentially a rule comparing orientations of pairs of components. Let θ be a c-invariant Spin-structure on C, and let $\phi_\theta\colon H_1(C) \to \mathbb{Z}_2$ be the associated quadratic extension. Pick a point p_i on each real component C_i of $C_{\mathbb{R}}$. For each pair p_i, p_j, $i \neq j$, pick a simple smooth path connecting p_i and p_j in the complement $C \smallsetminus C_{\mathbb{R}}$ and transversal to $C_{\mathbb{R}}$ at the ends, and let γ_{ij} be the loop obtained by combining the path with its c-conjugate. Pick an orientation of C_i and transfer it to C_j by a vector field normal to the path above. The two orientations are considered coherent with respect to the Spin-orientation defined by θ if and only if $\phi_\theta([\gamma_{ij}]) = 0$.

3.4.1. Lemma. *Assuming that $C_{\mathbb{R}} \neq \varnothing$, any semi-orientation of $C_{\mathbb{R}}$ is the* Spin-*orientation for a suitable real even theta characteristic.* □

Proof. Observe that the c_*-invariant classes $\{[\gamma_{1i}], [C_i]\}$ with $i \geqslant 2$ (see above) form a standard symplectic basis in a certain nondegenerate c_*-invariant subgroup $S \subset H_1(C)$. On the complement $S^\perp$, one can pick any c_*-invariant quadratic extension with Arf-invariant 0. Then, the values $\phi_\theta([\gamma_{1i}])$ can be chosen arbitrarily (thus producing any given Spin-orientation), and the values $\phi_\theta([C_i])$, $i \geqslant 2$, can be adjusted (*e.g.*, made all 0) to make the resulting theta characteristic even. □

3.4.2. If $d = \pm 1 \bmod 8$ and θ is the exceptional theta characteristic $\frac{1}{2}(d-3)H$, see 2.1.1, the Spin-orientation defined by θ is given by the residue $\operatorname{res}(p^2\Omega/U)$, where $U = 0$ is the equation of C as above, $\Omega = x_0dx_1\wedge dx_2 - x_1dx_0\wedge dx_2 + x_2dx_0\wedge dx_1$ is a non-vanishing section of $K_{\mathbb{P}^2}(3) \cong \mathcal{O}_{\mathbb{P}^2}$, and p is any real homogeneous polynomial of degree $(d-3)/2$. In affine coordinates, this orientation is given by $p^2\, dx\wedge dy/dU$. Such a semi-orientation is called *alternating*: it is the only semi-orientation of $C_\mathbb{R}$ induced by alternating orientations of the components of $\mathbb{P}^2_\mathbb{R} \smallsetminus C_\mathbb{R}$.

3.5. The index function (*cf.* [2]). Fix a real dimension r linear system $x \mapsto q_x$ of quadrics in $\mathbb{P}^N$. Consider the sphere $S^r = (\mathbb{R}^{r+1} \smallsetminus 0)/\mathbb{R}_+$ and denote by $\tilde{C} \subset S^r$ the pull-back of the real part $C_\mathbb{R} \subset \mathbb{P}^r_\mathbb{R}$ of the spectral variety under the double covering $S^r \to \mathbb{P}^r_\mathbb{R}$. Define the *index function*

$$\operatorname{ind}\colon S^r \to \mathbb{Z}$$

by sending a point $x \in S^r$ to the negative index of inertia of the quadratic form q_x. The following statement is obvious. (For 3.5.1(3), one should use Lemma 3.2.2.)

3.5.1. Proposition. *The index function* ind *has the following properties*:

(1) ind *is lower semicontinuous*;
(2) ind *is locally constant on* $S^r \smallsetminus \tilde{C}$;
(3) ind *jumps by* ± 1 *when crossing* $\tilde{C}$ *transversally at its regular point*;
(4) *one has* $\operatorname{ind}(-x) = N + 1 - (\operatorname{ind} x + \operatorname{corank} q_x)$. □

Due to 3.5.1(3), ind defines a coorientation of $\tilde{C}$ at all its smooth points. This coorientation is reversed by the antipodal map $a\colon S^r \to S^r$, $x \mapsto -x$, see 3.5.1(4). If $r = 2$ and the real part $C_\mathbb{R}$ of the spectral curve is nonsingular, this coorientation defines a semi-orientation of $C_\mathbb{R}$ as follows: pick an orientation of S^2 and use it to convert the coorientation to an orientation $\tilde{o}$ of $\tilde{C}$; since the antipodal map a is orientation reversing, $\tilde{o}$ is preserved by a and, hence, descends to an orientation o of $C_\mathbb{R}$. The latter is defined up to the total reversing (due to the initial choice of an orientation of S^2); hence, it is in fact a semi-orientation. It is called the *index orientation* of $C_\mathbb{R}$.

Conversely, any semi-orientation of $C_\mathbb{R}$ can be defined as above by a function $\operatorname{ind}\colon S^r \to \mathbb{Z}$ satisfying 3.5.1(1)–(4); the latter is unique up to the antipodal map.

3.5.2. Theorem (*cf.* [28]). *Assume that the real part $C_\mathbb{R}$ of the spectral curve of a real net of quadrics is nonsingular. Then the index orientation of $C_\mathbb{R}$ coincides with its* Spin-*orientation defined by the spectral theta characteristic.*

Proof. The semi-orientation of $C_\mathbb{R}$ is given by the residue $\operatorname{res}(v_{11}\Omega/U)$, *cf.* 3.4.2, and it is sufficient to check that the index function is larger on the side of $U = 0$ where $v_{11}/U > 0$. Since the kernel $\sum x_iQ_i$, regarded as a section of the projectivization of the trivial bundle over C, is given by $v = (v_{11}, v_{12}, \ldots, v_{1d})$, it remains to notice that, over $C_\mathbb{R}$, one has $\sum x_i\langle Q_iv, v\rangle = v_{11}\det(\sum x_iQ_i) = v_{11}U$. □

3.5.3. Theorem. *Let C be a nonsingular real plane curve of degree $d = N+1$ and with nonempty real part, and let o be a semi-orientation of $C_{\mathbb{R}}$. Assume that either $d \neq \pm 1 \bmod 8$ or o is not the alternating semi-orientation, see* 3.4.2. *Then, after a small real perturbation of C, there exists a regular intersection of three real quadrics in $\mathbb{P}^N_{\mathbb{R}}$ which has C as its spectral curve and o as its spectral* Spin*-orientation.*

3.5.4. Remark. According to 3.4.2, the only case not covered by Theorem 3.5.3 is when $d = \pm 1 \bmod 8$ and the index function ind assumes only the two middle values $(d \pm 1)/2$.

Proof of 3.5.3. By Lemma 3.4.1, there exists a real even theta characteristic θ that has o as its Spin-orientation. Using Lemma 2.2.1, one can make θ non-vanishing by a small real perturbation of C, and it remains to apply Theorem 3.3.4. □

4. The topology of the zero locus of a net

4.1. The spectral sequence. Consider a real dimension r linear system $x \mapsto q_x$ of quadrics in $\mathbb{P}^N$, see Section 3.1 for the notation. Let $V_{\mathbb{R}} \subset \mathbb{P}^N_{\mathbb{R}}$, $C_{\mathbb{R}} \subset \mathbb{P}^r_{\mathbb{R}}$, and $L_{\mathbb{R}} \subset \mathbb{P}^r_{\mathbb{R}} \times \mathbb{P}^N_{\mathbb{R}}$ be the real parts of the common zero set, spectral variety, and Lagrange hypersurface, respectively.

Consider the sphere $S^r = (\mathbb{R}^{r+1} \smallsetminus 0)/\mathbb{R}_+$ and the lift $\tilde{C} \subset S^r$ of $C_{\mathbb{R}}$, *cf.* 3.5, and let $\tilde{L} = \{(x,u) \in S^r \times \mathbb{P}^N_{\mathbb{R}} \mid q_x(u) = 0\} \subset S^r \times \mathbb{P}^N_{\mathbb{R}}$ be the lift of $L_{\mathbb{R}}$ and

$$L_+ = \{(x,u) \in S^r \times \mathbb{P}^N_{\mathbb{R}} \mid q_x(u) > 0\} \subset S^r \times \mathbb{P}^N_{\mathbb{R}}$$

its *positive complement*. (Clearly, the conditions $q_x(u) = 0$ and $q_x(u) > 0$ do not depend on the choice of representatives of $x \in S^r$ and $u \in \mathbb{P}^N_{\mathbb{R}}$.)

4.1.1. Lemma. *The projection $S^r \times \mathbb{P}^N_{\mathbb{R}} \to \mathbb{P}^N_{\mathbb{R}}$ restricts to a homotopy equivalence $L_+ \to \mathbb{P}^N_{\mathbb{R}} \smallsetminus V_{\mathbb{R}}$.*

Proof. Denote by p the restriction of the projection to L_+. The pull-back $p^{-1}(u)$ of a point $u \in V_{\mathbb{R}}$ is empty; hence, p sends L_+ to $\mathbb{P}^N_{\mathbb{R}} \smallsetminus V_{\mathbb{R}}$. On the other hand, the restriction $L_+ \to \mathbb{P}^N_{\mathbb{R}} \smallsetminus V_{\mathbb{R}}$ is a locally trivial fibration, and for each point $u \in \mathbb{P}^N_{\mathbb{R}} \smallsetminus V_{\mathbb{R}}$, the fiber $p^{-1}(u)$ is the open hemisphere $\{x \in S^r \mid q_x(u) > 0\}$, hence contractible. □

4.1.2. Proposition. *One has $b^0(L_+) = b^1(L_+) = 1$ and, if $V_{\mathbb{R}}$ is nonsingular, also $b^2(L_+) = b^0(V_{\mathbb{R}}) + 1$.*

Proof. Due to Lemma 4.1.1, one has $H^*(L_+) = H^*(\mathbb{P}^N_{\mathbb{R}} \smallsetminus V_{\mathbb{R}})$, and the statement follows from the Poincaré-Lefschetz duality $H^i(\mathbb{P}^N_{\mathbb{R}} \smallsetminus V_{\mathbb{R}}) = H_{N-i}(\mathbb{P}^N_{\mathbb{R}}, V_{\mathbb{R}})$ and the exact sequence of the pair $(\mathbb{P}^N_{\mathbb{R}}, V_{\mathbb{R}})$. For the last statement one needs, in addition, to know that the inclusion homomorphism $H_{N-3}(V_{\mathbb{R}}) \to H_{N-3}(\mathbb{P}^N_{\mathbb{R}})$ is trivial, *i.e.*, that every 3-plane P intersects each component of $V_{\mathbb{R}}$ at an even number of points. By restricting the system to a 4-plane containing P, one reduces the problem to the case $N = 4$. In this case, $V \subset \mathbb{P}^N$ is the canonical embedding of a genus 5 curve, *cf.* Section 6.4, and the statement is obvious. □

From now on, we assume that the real part $C_{\mathbb{R}}$ of the spectral hypersurface is nonsingular. Consider the ascending filtration

$$\varnothing = \Omega_{-1} \subset \Omega_0 \subset \Omega_1 \subset \ldots \subset \Omega_{N+1} = S^r, \quad \Omega_i = \{x \in S^r \mid \operatorname{ind} x \leqslant i\}. \tag{4.1.3}$$

Due to Proposition 3.5.1(1), all Ω_i are closed subsets.

4.1.4. Theorem (*cf.* [1]). *There is a spectral sequence*

$$E_2^{pq} = H^p(\Omega_{N-q}) \Rightarrow H^{p+q}(L_+).$$

Proof. The sequence in question is the Leray spectral sequence of the projection $\pi\colon L_+ \to S^r$. Let $\mathcal{Z}_2$ be the constant sheaf on L_+ with the fiber $\mathbb{Z}_2$. Then the sequence is $E_2^{pq} = H^p(S^r, R^q\pi_*\mathcal{Z}_2) \Rightarrow H^{p+q}(L_+)$. Given a point $x \in S^r$, the stalk $(R^q\pi_*\mathcal{Z}_2)|_x$ equals $H^q(\pi^{-1}U_x)$, where $U_x \ni x$ is a small neighborhood of x regular with respect to a triangulation of S^r compatible with the filtration. If $x \notin \tilde{C}$ and $\operatorname{ind} x = i$, then

$$\pi^{-1}U_x \sim \pi^{-1}x = \{u \in \mathbb{P}^N_{\mathbb{R}} \mid q_x(u) > 0\} \sim \mathbb{P}^{N-i}_{\mathbb{R}}.$$

(The fiber $\pi^{-1}x$ is a D^i-bundle over $\mathbb{P}^{N-i}_{\mathbb{R}}$; if $i = N+1$, the fiber is empty.) However, if $x \in \tilde{C}$ and $\operatorname{ind} x = i$, then $\pi^{-1}U_x \sim \pi^{-1}x'$, where $x' \in (U_x \cap \Omega_i) \smallsetminus \tilde{C}$. Thus, $(R^q\pi_*\mathcal{Z}_2)|_x = \mathbb{Z}_2$ or 0 if x does (respectively, does not) belong to Ω_{N-q}, and the statement is immediate. □

4.1.5. Remark. It is worth mentioning that there are spectral sequences, similar to the one introduced in Theorem 4.1.4, which compute the cohomology of the double coverings of $\mathbb{P}^N_{\mathbb{R}} \smallsetminus V_{\mathbb{R}}$ and $V_{\mathbb{R}}$ sitting in S^N, see [2].

4.2. Elements of topology of real plane curves. Let $C \subset \mathbb{P}^2$ be a nonsingular real curve of degree d. Recall that the real part $C_{\mathbb{R}}$ splits into a number of *ovals* (*i.e.*, embedded circles contractible in $\mathbb{P}^2_{\mathbb{R}}$) and, if d is odd, one *one-sided component* (*i.e.*, an embedded circle isotopic to $\mathbb{P}^1_{\mathbb{R}}$.) The complement of each oval $\mathfrak{o}$ has two connected components, exactly one of them being contractible; this contractible component is called the *interior* of $\mathfrak{o}$.

On the set of ovals of $C_{\mathbb{R}}$, there is a natural partial order: an oval $\mathfrak{o}$ is said to *contain* another oval $\mathfrak{o}'$, $\mathfrak{o} \prec \mathfrak{o}'$, if $\mathfrak{o}'$ lies in the interior of $\mathfrak{o}$. An oval is called *empty* if it does not contain another oval. The *depth* $\operatorname{dp}\mathfrak{o}$ of an oval $\mathfrak{o}$ is the number of elements in the maximal descending chain starting at $\mathfrak{o}$. (Such a chain is unique.) Every oval $\mathfrak{o}$ of depth > 1 has a unique immediate predecessor; it is denoted by $\operatorname{pred}\mathfrak{o}$.

A *nest* of C is a linearly ordered chain of ovals of $C_{\mathbb{R}}$; the *depth* of a nest is the number of its elements. The following statement is a simple and well known consequence of the Bézout theorem.

4.2.1. Proposition. *Let C be a nonsingular real plane curve of degree d. Then*

(1) *C cannot have a nest of depth greater than $D_{\max} = D_{\max}(d) = [d/2]$;*
(2) *if C has a nest of depth $D_{\max}$* (a maximal nest)*, it has no other ovals*;
(3) *if C has a nest $\mathfrak{o}_1 \prec \ldots \prec \mathfrak{o}_k$ of depth $k = D_{\max} - 1$* (a submaximal nest) *but no maximal nest, then all ovals other than $\mathfrak{o}_1, \ldots, \mathfrak{o}_{k-1}$ are empty.* □

Let $\operatorname{pr}\colon S^2 \to \mathbb{P}^2_{\mathbb{R}}$ be the orientation double covering, and let $\tilde{C} = \operatorname{pr}^{-1} C_{\mathbb{R}}$. The pull-back of an oval $\mathfrak{o}$ of C consists of two disjoint circles $\mathfrak{o}'$, $\mathfrak{o}''$; such circles are called *ovals* of $\tilde{C}$. The antipodal map $x \mapsto -x$ of S^2 induces an involution on the set of ovals of $\tilde{C}$; we denote it by bar, $\mathfrak{o} \mapsto \bar{\mathfrak{o}}$. The pull-back of the one-sided component of $C_{\mathbb{R}}$ is connected; it is called the *equator*. The *tropical components* are the components of $S^2 \smallsetminus \tilde{C}$ whose image is the (only) component of $\mathbb{P}^2_{\mathbb{R}} \smallsetminus C_{\mathbb{R}}$ outer to all ovals. The *interior* $\operatorname{int}\mathfrak{o}$ of an oval $\mathfrak{o}$ of $\tilde{C}$ is the component of the complement of $\mathfrak{o}$ that projects to the interior of $\operatorname{pr}\mathfrak{o}$ in $\mathbb{P}^2_{\mathbb{R}}$. As in the case of $C_{\mathbb{R}}$, one can use the notion of interior to define the partial order, depth, nests, *etc.* The projection pr and the antipodal involution induce strictly increasing maps of the sets of ovals.

Now, consider an (abstract) index function $\operatorname{ind}\colon S^2 \to \mathbb{Z}$ satisfying 3.5.1(1)–(4) (where $N+1=d$ and $\operatorname{corank} q_x = \chi_{\tilde{C}}(x)$ is the characteristic function of $\tilde{C}$) and use it to define the filtration Ω_* as in (4.1.3). For an oval $\mathfrak{o}$ of $\tilde{C}$, define $i(\mathfrak{o})$ as the value of ind immediately inside $\mathfrak{o}$. (Note that, in general, $i(\mathfrak{o})$ is *not* the restriction of ind to $\mathfrak{o}$ as a subset of S^r.) Then, in view of Proposition 3.5.1, one has

$$\tfrac{1}{2}N \leqslant \operatorname{ind}|_T \leqslant \tfrac{1}{2}N+1 \quad \text{for each tropical component } T, \tag{4.2.2}$$

$$i(\mathfrak{o}) \leqslant N+1-(D_{\max}-\operatorname{dp}\mathfrak{o}) \quad \text{for each oval } \mathfrak{o}, \tag{4.2.3}$$

$$i(\mathfrak{o}) = N+1-(D_{\max}-\operatorname{dp}\mathfrak{o}) \bmod 2 \quad \text{if } N \text{ is odd.} \tag{4.2.4}$$

Here, as above, $D_{\max}=[(N+1)/2]$. Keeping in mind the applications, we state the restrictions in terms of $N=d-1$. (Certainly, congruence (4.2.4) simplifies to $i(\mathrm{dp})\mathfrak{o}+\operatorname{dp}\mathfrak{o}=D_{\max} \bmod 2$; however, we leave it in the form convenient for the further applications.)

4.2.5. Corollary. *If $N \geqslant 5$, then Ω_{N-2} contains the tropical components.* □

4.2.6. Lemma. *Assume that $b^0(\Omega_q)>1$ for some integer $q>\frac{1}{2}N$. Then $\tilde{C}$ has a nest $\mathfrak{o} \prec \mathfrak{o}'$ such that $i(\mathfrak{o})=q+1$ and $i(\mathfrak{o}')=q$.*

Proof. Due to (4.2.2), the assumption $q>\frac{1}{2}N$ implies that Ω_q contains the tropical components. Then, one can take for $\mathfrak{o}'$ the oval bounding from outside another component of Ω_q, and let $\mathfrak{o}=\operatorname{pred}\mathfrak{o}'$. □

4.2.7. Lemma. *Let $N \geqslant 7$, and assume that the curve $\tilde{C}$ has a nest $\mathfrak{o}_{k-1} \prec \mathfrak{o}_k$, $k=D_{\max}-1$, with $i(\mathfrak{o}_{k-1})=N-1$. Then $\Omega_{N-2} \supset S^2 \smallsetminus \operatorname{int}\operatorname{pred}\mathfrak{o}_{k-1}$.*

Proof. Due to (4.2.3), the nest $\mathfrak{o}_{k-1} \prec \mathfrak{o}_k$ in the statement can be completed to a submaximal nest $\mathfrak{o}_1 \prec \ldots \prec \mathfrak{o}_k$.

First, assume that either $\tilde{C}$ has no maximal nest or the innermost oval of $\tilde{C}$ is inside $\mathfrak{o}_k$. Then $\operatorname{dp}\mathfrak{o}_s=s$ and $\mathfrak{o}_s=\operatorname{pred}\mathfrak{o}_{s+1}$ for all s. Due to (4.2.3) and 3.5.1(3), one has $i(\mathfrak{o}_{k-s})=N-s$ for $s=1,\ldots,k-1$. Then, due to 3.5.1(4), $i(\bar{\mathfrak{o}}_{k-s})=s+1$ and hence $i(\bar{\mathfrak{o}}_k)\leqslant 3$. Proposition 4.2.1 implies that all ovals other than $\mathfrak{o}_{k-s}$, $\bar{\mathfrak{o}}_{k-s}$, $s=0,\ldots,k-1$, are empty. For such an oval $\mathfrak{o}$, one has $i(\mathfrak{o})\leqslant[\frac{1}{2}N]+2$ if $\operatorname{dp}\mathfrak{o}=1$, see (4.2.2), and $i(\mathfrak{o})\leqslant 4$, $s+2$, or $N+1-s$ if $\operatorname{pred}\mathfrak{o}=\bar{\mathfrak{o}}_k$, $\bar{\mathfrak{o}}_{k-s}$, or $\mathfrak{o}_{k-s}$, respectively, $s=1,\ldots,k-1$. From the assumption $N\geqslant 7$, it follows that, for any oval $\mathfrak{o}$, one has $i(\mathfrak{o})\leqslant N-2$ unless $\mathfrak{o}\not\succcurlyeq\mathfrak{o}_{k-2}$. Together with Corollary 4.2.5, this observation implies the statement.

The case when $\tilde{C}$ has another oval $\mathfrak{o}' \prec \mathfrak{o}_i$ for some $i\leqslant k-1$ is treated similarly. In this case, $N=2k$ is even, see (4.2.4), and, renumbering the ovals consecutively from k down to 0, one has $i(\mathfrak{o}_{k-s})=N-s$, $i(\bar{\mathfrak{o}}_{k-s})=s+1$ for $s=1,\ldots,k$. □

Note that the condition $N \geqslant 7$ in Lemma 4.2.7 is, in particular, necessary for $\mathfrak{o}_{k-1}$ to have a predecessor, *i.e.*, for the statement to make sense. The remaining two interesting cases $N=5$ and 6 are treated in the next lemma.

4.2.8. Lemma. *Let $N = 5$ or 6, and assume that the curve $\tilde{C}$ has a nest $\mathfrak{o}_1 \prec \mathfrak{o}_2$ with $i(\mathfrak{o}_1) = N - 1$. Then either*

1. *for each pair $\mathfrak{o}$, $\bar{\mathfrak{o}}$ of antipodal ovals of depth 1, the set Ω_{N-2} contains the interior of exactly one of them, or*
2. *$N = 5$ and $\tilde{C}$ has another nested oval $\mathfrak{o}_3 \succ \mathfrak{o}_2$ with $i(\mathfrak{o}_3) = 2$. (These data determine the index function uniquely.)*

Proof. The proof repeats literally that of Lemma 4.2.7, with a careful analysis of the inequalities that do not hold for small values of N. □

4.3. The estimates. In this section, we consider a net of quadrics ($r = 2$) and assume that the spectral curve $\tilde{C} \subset S^2$ is nonsingular. Furthermore, we can assume that the index function takes values between 1 and N, as otherwise the net would contain an empty quadric and one would have $V = \varnothing$. Thus, one has $\varnothing = \Omega_{-1} = \Omega_0$ and $\Omega_N = \Omega_{N+1} = S^2$.

Denote $i_{\max} = \max_{x \in S^2} \operatorname{ind} x$. Thus, we assume that $i_{\max} \leqslant N$.

In addition, we can assume that $N \geqslant 3$, as for $N \leqslant 2$ a regular intersection of three quadrics in $\mathbb{P}^N$ is empty.

The spectral sequence E_r^{pq} given by Theorem 4.1.4 is concentrated in the strip $0 \leqslant p \leqslant 2$, and all potentially nontrivial differentials are $d_2^{0,q}\colon E_2^{0,q} \to E_2^{2,q-1}$, $q \geqslant 1$. Furthermore, one has

$$E_2^{0,q} = E_2^{2,q} = \mathbb{Z}_2, \quad E_2^{1,q} = 0 \quad \text{for } q = 0, \ldots, N - i_{\max}, \tag{4.3.1}$$

$$E_2^{0,q} = E_2^{1,q} = E_2^{2,q} = 0 \quad \text{for } q \geqslant i_{\max}, \quad \text{and} \tag{4.3.2}$$

$$E_2^{2,q} = 0 \quad \text{for } q > N - i_{\max}. \tag{4.3.3}$$

In particular, it follows that $d_2^{0,q} = 0$ for $q > N + 1 - i_{\max}$.

4.3.4. Corollary. *If $i_{\max} \leqslant N - 2$, then $b^0(V_{\mathbb{R}}) \leqslant 1$.* □

The assertion of Corollary 4.3.4 was first observed by Agrachev [1].

4.3.5. Lemma. *With one exception, $d_2^{0,1} = 0$. The exception is a curve $\tilde{C}$ with a maximal nest $\mathfrak{o}_1 \prec \ldots \prec \mathfrak{o}_k$, $k = D_{\max}$, so that $i(\mathfrak{o}_k) = N - 1$ and $i(\mathfrak{o}_{k-1}) = N$. In this exceptional case, one has $b^0(V_{\mathbb{R}}) = 1$.*

Proof. Since $b^1(L_+) = 1$, see Proposition 4.1.2, and $E_2^{1,0} = 0$, the differential $d_2^{0,1}$ is nontrivial if and only if $b^0(\Omega_{N-1}) = \dim E_2^{0,1} > 1$. Since $N \geqslant 3$, the exceptional case is covered by Lemma 4.2.6 and Proposition 4.2.1. □

4.3.6. Corollary. *If $i_{\max} = N - 1 \geqslant 2$, then one has $b^0(\Omega_{N-2}) - 1 \leqslant b^0(V_{\mathbb{R}}) \leqslant b^0(\Omega_{N-2})$.* □

4.3.7. Lemma. *Assume that $i_{\max} = N - 1 \geqslant 4$ and that $b^0(\Omega_{N-2}) > 1$. Then $\beta \leqslant b^0(V_{\mathbb{R}}) \leqslant \beta + 1$, where β is the number of ovals of $C_{\mathbb{R}}$ of depth $D_{\max} - 1$.*

Proof. In view of Corollary 4.3.6, it suffices to show that $b^0(\Omega_{N-2}) = \beta + 1$. Due to Lemma 4.2.6, the curve $\tilde{C}$ has a nest $\mathfrak{o} \prec \mathfrak{o}'$ with $i(\mathfrak{o}) = N - 1$, and then the set Ω_{N-2} is described by Lemmas 4.2.7, 4.2.8 and the assumption $i_{\max} = N - 1$. In view of Proposition 4.2.1, each oval of $C_{\mathbb{R}}$ of depth $\operatorname{dp} \mathfrak{o} + 1$ is inside $\operatorname{pr} \mathfrak{o}$, thus contributing an extra unit to $b^0(\Omega_{N-2})$. □

4.3.8. Lemma. *Assume that* $i_{\max} = N \geqslant 5$. *Then* $b^0(V_{\mathbb{R}})$ *is equal to the number* β *of ovals of* $C_{\mathbb{R}}$ *of depth* $D_{\max} - 1$.

Proof. If $(\tilde{C}, \operatorname{ind})$ is the exceptional index function mentioned in Lemma 4.3.5, then $b^0(V_{\mathbb{R}}) = \beta = 1$, and the statement holds. Otherwise, both differentials $d_2^{0,1}$ and $d_2^{0,2}$ vanish, see (4.3.3), and, using Proposition 4.1.2 and (4.3.1), one concludes that $b^0(V_{\mathbb{R}}) = \dim E_2^{0,2} + \dim E_2^{1,1} = b^0(\Omega_{N-2}) + b^1(\Omega_{N-1})$.

Pick an oval $\mathfrak{o}'$ with $i(\mathfrak{o}') = N$ and let $\mathfrak{o} = \operatorname{pred} \mathfrak{o}'$. The topology of Ω_{N-2} is given by Lemmas 4.2.7, 4.2.8. If $\operatorname{dp} \mathfrak{o} = D_{\max} - 1$, then $b^0(V_{\mathbb{R}}) = \beta = 1$. Otherwise ($\operatorname{dp} \mathfrak{o} = D_{\max} - 2$), one has $b^0(\Omega_{N-2}) = \beta_- + 1$ and $b^1(\Omega_{N-1}) = \beta_+ - 1$, where $\beta_- \geqslant 0$ and $\beta_+ > 0$ are the numbers of ovals $\mathfrak{o}'' \succ \mathfrak{o}$ with $i(\mathfrak{o}'') = N - 2$ and N, respectively; due to Proposition 4.2.1, one has $\beta_- + \beta_+ = \beta$. ☐

4.4. Proof of Theorem 1.1.3. The case $N = 4$ is covered by Theorem 1.1.4. (Note that $B_2^0(4) = \operatorname{Hilb}(5)$, see Section 6.4.) Alternatively, one can treat this case manually, trying various index functions on a curve of degree 5.

Assume that $N \geqslant 5$. The upper bound on $B_2^0(N)$ follows from Corollary 4.3.4 and Lemmas 4.3.7 and 4.3.8. For the lower bound, pick a generic real curve C of degree $d = N + 1$ with $\operatorname{Hilb}(d)$ ovals of depth $[d/2] - 1$. Select an oval $\mathfrak{o}_i$ in each pair $(\mathfrak{o}_i, \bar{\mathfrak{o}}_i)$ of antipodal outermost ovals of $\tilde{C}$; if d is even, make sure that all selected ovals are in the boundary of the same tropical component. Take for ind the 'monotonous' function defined *via* $i(\mathfrak{o}) = N + 1 - (D_{\max} - \operatorname{dp} \mathfrak{o})$ if $\mathfrak{o} \succcurlyeq \mathfrak{o}_i$ and $i(\mathfrak{o}) = D_{\max} - \operatorname{dp} \mathfrak{o}$ if $\mathfrak{o} \succcurlyeq \bar{\mathfrak{o}}_i$ for some i, see Figure 1 (where the cases $N = 7$ and $N = 8$ are shown schematically). Due to Theorem 3.5.3 (see also Remark 3.5.4), the pair (C, ind) is realized by a net of quadrics, and for this net one has $b^0(V) = \operatorname{Hilb}(d)$, see Lemma 4.3.8. ☐

FIGURE 1. 'Monotonous' index functions ($N = 7$ and $N = 8$)

4.5. Proof of Theorem 1.1.1. In this section, we make an attempt to estimate the Hilbert number $\operatorname{Hilb}(d)$ introduced in Definition 1.1.2.

Let C be a nonsingular real plane algebraic curve of degree d. An oval of $C_{\mathbb{R}}$ is said to be *even* (*odd*) if its depth is odd (respectively, even). An oval is called *hyperbolic* if it has more than one immediate successor (in the partial order defined in Section 4.2). The following statement is known as *generalized Petrovsky inequality.*

4.5.1. Theorem (see [4]). *Let* C *be a nonsingular real plane curve of even degree* $d = 2k$. *Then*

$$p - n^- \leqslant \frac{3}{2}k(k-1) + 1, \quad n - p^- \leqslant \frac{3}{2}k(k-1).$$

where p, n *are the numbers of even/odd ovals of* $C_{\mathbb{R}}$ *and* p^-, n^- *are the numbers of even/odd hyperbolic ovals.* ☐

4.5.2. Corollary. *One has* $\mathrm{Hilb}(d) \leqslant \frac{3}{2}k(k-1)+1$, *where* $k = [(d+1)/2]$.

Proof. Let $d = 2k$ be even, and let C be a curve of degree d with $m > 1$ ovals of depth $k-1$. All submaximal ovals are situated inside a nest $\mathfrak{o}_1 \prec \ldots \prec \mathfrak{o}_{k-2}$ of depth $D_{\max}-2 = k-2$, see Proposition 4.2.1. Assume that $k = 2l$ is even. Then the submaximal ovals are even, and one has $p \geqslant m+l-1$, counting as well the even ovals $\mathfrak{o}_1, \mathfrak{o}_3, \ldots \mathfrak{o}_{2l-1}$ in the nest. On the other hand, $n^- \leqslant l-1$, as all odd ovals other than $\mathfrak{o}_2, \mathfrak{o}_4, \ldots, \mathfrak{o}_{2l}$ are empty, hence not hyperbolic, see Proposition 4.2.1 again. Hence, the statement follows from the first inequality in Theorem 4.5.1. The case of k odd is treated similarly, using the second inequality in Theorem 4.5.1.

Let $d = 2k-1$ be odd, and consider a real curve C of degree d with a nest $\mathfrak{o}_1 \prec \ldots \prec \mathfrak{o}_{k-3}$ of depth $D_{\max} - 2 = k-3$ and $m \geqslant 2$ ovals $\mathfrak{o}', \mathfrak{o}'', \ldots$ of depth $k-2$. Pick a pair of points p' and p'' inside $\mathfrak{o}'$ and $\mathfrak{o}''$, respectively, and consider the line $L = (p_1p_2)$. From the Bézout theorem, it follows that all points of intersection of L and C are one point on the one-sided component of $C_{\mathbb{R}}$ and a pair of points on each of the ovals $\mathfrak{o}_1, \ldots, \mathfrak{o}_{k-3}, \mathfrak{o}', \mathfrak{o}''$. Furthermore, the pair $\mathfrak{o}'$, $\mathfrak{o}''$ can be chosen so that all other innermost ovals of $C_{\mathbb{R}}$ lie to one side of L in the interior of $\mathfrak{o}_{k-1}$ (which is divided by L into two components). According to the Brusotti theorem [6], the union $C+L$ can be perturbed to form a nonsingular curve of degree $2k$ with m ovals of depth $k-1$, see Figure 2 (where the curve and its perturbation are shown schematically in grey and black, respectively). Hence, the statement follows from the case of even degree considered above. □

FIGURE 2. The perturbation of $C+L$ with a deep nest

4.5.3. Theorem [16]. *For each integer* $d \geqslant 4$, *there is a nonsingular curve of degree* d *in* $\mathbb{P}^2_{\mathbb{R}}$ *with*

- 4 *ovals of depth* 1 *if* $d = 4$,
- 6 *ovals of depth* 1 *if* $d = 5$,
- $k(k+1)-3$ *ovals of depth* $[d/2]-1$ *if* $d = 2k$ *is even,*
- $k(k+2)-3$ *ovals of depth* $[d/2]-1$ *if* $d = 2k+1$ *is odd.* □

For the reader's convenience, we give a brief outline of the original construction due to Hilbert that produces curves as in Theorem 4.5.3.

For even degrees, one can use an inductive procedure which produces a sequence of curves $C^{(2k)}$, $\deg C^{(2k)} = 2k$. Let $E = \{p_E = 0\}$ be an ellipse in $\mathbb{P}^2_{\mathbb{R}}$. The curve $C^{(2)}$ is defined by a polynomial $p^{(2)}$ of the form

$$p^{(2)} = p_E + \varepsilon^{(2)} l_1^{(2)} l_2^{(2)}$$

where $\varepsilon^{(2)} > 0$ is a real number, $|\varepsilon^{(2)}| \ll 1$, and $l_1^{(2)}$ and $l_2^{(2)}$ are real polynomials of degree 1 such that the pair of lines $\{l_1^{(2)} l_2^{(2)} = 0\}$ intersects E at four distinct real points. The intersection of the exterior of E and the interior of $C^{(2)}$ is formed by two disks $D_1^{(2)}$ and $D_2^{(2)}$. Inductively, we construct curves $C^{(2k)} = \{p^{(2k)} = 0\}$ with the following properties:

(1) $C^{(2k)}$ has an oval $\mathfrak{o}^{(2k)}$ of depth $k-1$ such that $\mathfrak{o}^{(2k)}$ intersects E at $4k$ distinct points, the orders of the intersection points on $\mathfrak{o}^{(2k)}$ and E coincide, and the intersection of the exterior of E and the exterior of $\mathfrak{o}^{(2k)}$ consists of a Möbius strip and $2k-1$ discs $D_1^{(2k)}, \ldots, D_{2k-1}^{(2k)}$ (shaded in Figure 3),

(2) one has
$$p^{(2k)} = p^{(2k-2)} p_E + \varepsilon^{(2k)} l_1^{(2k)} \ldots l_{2k}^{(2k)},$$
where $\varepsilon^{(2k)}$ is a real number, $|\varepsilon^{(2k)}| \ll 1$, and $l_1^{(2k)}, \ldots, l_{2k}^{(2k)}$ are certain polynomials of degree 1 such that the union of lines $\{l_1^{(2k)} \ldots l_{2k}^{(2k)} = 0\}$ intersects E at $4k$ distinct real points, all points belonging to $\partial D_1^{(2k-2)}$.

The sign of $\varepsilon^{(2k)}$ is chosen so that $\mathfrak{o}^{(2k-2)} \cup E$ produces $4k-4$ ovals of $C^{(2k)}$.

FIGURE 3. The oval $\mathfrak{o}^{(2k)}$ and the disks $D_i^{(2k)}$ (shaded)

The above properties imply that each curve $C^{(2k)}$ has the required number of ovals of depth $k-1$ (and next curve $C^{(2k+2)}$ still satisfies (1)).

FIGURE 4. Hilbert's construction in degree 4

The curves $C^{(2k+1)}$ of odd degree $2k+1$ are constructed similarly, starting from a curve $C^{(3)}$ defined by a polynomial of the form
$$p^{(3)} = l p_E + \varepsilon^{(3)} l_1^{(3)} l_2^{(3)} l_3^{(3)},$$

where $\varepsilon^{(3)} > 0$ is a sufficiently small real number, l is a polynomial of degree 1 defining a line disjoint from E, and $l_1^{(3)}$, $l_2^{(3)}$, and $l_3^{(3)}$ are polynomials of degree 1 such that the union of lines $\{l_1^{(3)} l_2^{(3)} l_3^{(3)} = 0\}$ intersects E at six distinct real points.

4.5.4. Corollary. *One has* $\operatorname{Hilb}(d) > \frac{1}{4}(d-2)(d+4) - 2$. $\square$

4.5.5. Remark. S. Orevkov informed us that he could construct real pseudo-holomorphic curves of degree d with

$$\frac{161}{512} d^2 + O(d)$$

ovals of depth $[\frac{d}{2}] - 1$, and that he expects that this estimate is not final. We do not know any similar example in the algebraic category.

Proof of Theorem 1.1.1. The statement of the theorem follows from Theorem 1.1.3 and the bounds on $\operatorname{Hilb}(d)$ given by Corollaries 4.5.2 and 4.5.4. $\square$

5. Intersections of quadrics of dimension one

In this section, we consider the case $N = r + 2$, *i.e.*, one-dimensional complete intersections of quadrics.

5.1. Proof of Theorem 1.1.4: the upper bound. Let V be a regular complete intersection of $N-1$ quadrics in $\mathbb{P}^N$. Iterating the adjunction formula, one finds that the genus $g(V)$ of the curve V satisfies the relation

$$2g(V) - 2 = 2^{N-1}\bigl(2(N-1) - (N+1)\bigr) = 2^{N-1}(N-3);$$

hence, $g(V) = 2^{N-2}(N-3) + 1$ and the Harnack inequality gives the upper bound $B^0_{N-2}(N) \le 2^{N-2}(N-3) + 2$. $\square$

5.2. Proof of Theorem 1.1.4: the construction. To prove the lower bound $B^0_{N-2}(N) \geqslant 2^{N-2}(N-3) + 2$, for each integer $N \geqslant 2$ we construct a homogeneous quadratic polynomial $q^{(N)} \in \mathbb{R}[x_0, \ldots, x_N]$ and a pair $(l_1^{(N)}, l_2^{(N)})$ of linear forms $l_i^{(N)} \in \mathbb{R}[x_0, \ldots, x_N]$, $i = 1$, 2, with the following properties:

(1) the common zero set $V^{(N)} = \{q^{(2)} = \ldots = q^{(N)} = 0\} \subset \mathbb{P}^N$ is a regular complete intersection;
(2) the real part $V_{\mathbb{R}}^{(N)}$ has $2^{N-2}(N-3) + 2$ connected components;
(3) there is a distinguished component $\mathfrak{o}^{(N)} \subset V_{\mathbb{R}}^{(N)}$, which has two disjoint closed arcs $A_1^{(N)}$, $A_2^{(N)}$ such that the interior of $A_i^{(N)}$, $i = 1$, 2, contains all 2^{N-1} points of intersection of the hyperplane $L_i^{(N)} = \{l_i^{(N)} = 0\}$ with $V^{(N)}$.

Property (2) gives the desired lower bound.

The construction is by induction. Let

$$l_1^{(2)} = x_2, \quad l_2^{(2)} = x_2 - x_1, \quad \text{and} \quad q^{(2)} = l_1^{(2)} l_2^{(2)} + (x_1 - x_0)(x_1 - 2x_0).$$

Assume that, for all integers $2 \leqslant k \leqslant N$, polynomials $q^{(k)}$, $l_1^{(k)}$, and $l_2^{(k)}$ satisfying conditions (1)–(3) above are constructed. Let $\tilde{l}_2^{(N)} = l_2^{(N)} - \delta^{(N)} l_1^{(N)}$, where $\delta^{(N)} > 0$

FIGURE 5. Construction of one-dimensional intersection of quadrics

is a real number so small that, for all $t \in [0, \delta^{(N)}]$, the line $\{l_2^{(N)} - t^{(N)} l_1^{(N)} = 0\}$ intersects $\mathfrak{o}^{(N)}$ at 2^{N-1} distinct real points which all belong to the arc $A_2^{(N)}$. Put

$$l_1^{(N+1)} = x_{N+1} \quad \text{and} \quad l_2^{(N+1)} = x_{N+1} - l_1^{(N)}.$$

The intersection of the cone $\{q^{(2)} = \ldots = q^{(N)} = 0\} \subset \mathbb{P}_{\mathbb{R}}^{N+1}$ (over $V_{\mathbb{R}}^{(N)}$) and the hyperplane $L_2^{(N+1)} = \{l_2^{(N+1)} = 0\}$ is a copy of $V_{\mathbb{R}}^{(N)}$, see Figure 5.

Put

$$q^{(N+1)} = l_1^{(N+1)} l_2^{(N+1)} + \varepsilon^{(N+1)} l_2^{(N)} \tilde{l}_2^{(N)},$$

where $\varepsilon^{(N+1)} > 0$ is a sufficiently small real number. One can observe that, on the hyperplane $\{l_1^{(N)} = 0\} \subset \mathbb{P}_{\mathbb{R}}^{N+1}$, the polynomial $q^{(N+1)}$ has no zeroes outside the subspace $\{x_{N+1} = l_2^{(N)} = 0\}$.

The new curve $V^{(N+1)}$ is a regular complete intersection, and its real part has $2^{N-1}(N-2)+2$ connected components. Indeed, each component $\mathfrak{o} \subset V_{\mathbb{R}}^{(N)}$ other than $\mathfrak{o}^{(N)}$ gives rise to two components of $V_{\mathbb{R}}^{(N+1)}$, whereas $\mathfrak{o}^{(N)}$ gives rise to 2^{N-1} components of $V_{\mathbb{R}}^{(N+1)}$, each component being the perturbation of the union $\mathfrak{a}_j \cup \mathfrak{a}'_j$, where $\mathfrak{a}_j \subset \mathfrak{o}^{(N)}$, $j = 0, \ldots, 2^{N-1} - 1$, is the arc bounded by two consecutive (in $\mathfrak{o}^{(N)}$) points of the intersection $L_1^{(N)} \cap \mathfrak{o}^{(N)}$ (and not containing other intersection points) and $\mathfrak{a}'_j$ is the copy of $\mathfrak{a}_j$ in $L_2^{(N+1)}$. All but one arcs $\mathfrak{a}_j$ belong to $A_1^{(N)}$ and produce 'small' components; the arc $\mathfrak{a}_0$ bounded by the two outermost (from the point of view of $A_1^{(N)}$) intersection points produces the 'long' component, which we take for $\mathfrak{o}^{(N+1)}$. Finally, observe that the new component $\mathfrak{o}^{(N+1)}$ has two arcs $A_i^{(N+1)}$, $i = 1, 2$, satisfying condition (3) above: they are the perturbations of the arc $A_2^{(N)} \subset L_1^{(N+1)}$ and its copy in $L_2^{(N+1)}$. $\square$

6. CONCLUDING REMARKS

In this section, we consider the few first special cases, $N = 2$, 3, 4, and 5, where, in fact, a complete deformation classification can be given. We also briefly discuss the other Betti numbers and the maximality of common zero sets of nets of quadrics; however, we merely outline the directions of the further investigation, leaving all details for a subsequent paper.

6.1. Empty intersections of quadrics. Consider a complete intersection V of $(r+1)$ real quadrics in $\mathbb{P}^N$, and assume that $V_{\mathbb{R}} = \varnothing$. Choosing generators $q_0, q_1, \ldots, q_r$ of the linear system, we obtain a map

$$S^N = (\mathbb{R}^{N+1} \smallsetminus 0)/\mathbb{R}_+ \to S^r = (\mathbb{R}^{r+1} \smallsetminus 0)/\mathbb{R}_+, \quad u \mapsto (q_0(u), \ldots, q_r(u))/\mathbb{R}_+.$$

Clearly, the homotopy class of this map, which can be regarded as an element of the group $\pi_N(S^r)$ modulo the antipodal involution, is a deformation invariant of the system. Besides, the map is even (the images of u and $-u$ coincide); hence, it also induces certain maps $\mathbb{P}^N_{\mathbb{R}} \to S^r$, $S^N \to \mathbb{P}^r_{\mathbb{R}}$, and $\mathbb{P}^N_{\mathbb{R}} \to \mathbb{P}^r_{\mathbb{R}}$, and their homotopy classes are also deformation invariant. Below, among other topics, we consider a few special cases where these classes do distinguish empty regular intersections.

In general, the deformation classifications of linear systems of quadrics, quadratic (rational) maps $S^N \to S^r$ (or $\mathbb{P}^N_{\mathbb{R}} \to \mathbb{P}^r_{\mathbb{R}}$), and spectral hypersurfaces (*e.g.*, spectral curves, even endowed with a theta characteristic) are different problems. We will illustrate this by examples.

6.2. Three conics. We start with the case $r = N = 2$, *i.e.*, a net of conics in $\mathbb{P}^2_{\mathbb{R}}$. The spectral curve is a cubic $C \subset \mathbb{P}^2$, and the regularity condition implies that the common zero set must be empty (even over $\mathbb{C}$). There are two deformation classes of complete intersections of three conics; they can be distinguished by the $\mathbb{Z}_2$-Kronecker invariant, *i.e.*, the mod2 degree of the associated map $\mathbb{P}^2_{\mathbb{R}} \to S^2$, see 6.1. If $\deg = 0 \bmod 2$, the index function takes all three values 0, 1, and 2; otherwise, the index function only takes the middle value 1. (Alternatively, the Kronecker invariant counts the parity of the number of real solutions of the system $q_a = q_b = 0$, $q_c > 0$ in $\mathbb{P}^2_{\mathbb{R}} = \mathbb{P}^N_{\mathbb{R}}$, where a, b, c represent any triple of non-collinear points in $\mathbb{P}^2_{\mathbb{R}} = \mathbb{P}^r_{\mathbb{R}}$).

The classification of generic nets of conics can be obtained using the results of [8]. Note that, considering generic quadratic maps $\mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$ rather than regular complete intersections, there are four deformation classes; they can be distinguished by the topology of $C_{\mathbb{R}}$ and the spectral theta characteristic.

6.3. Spectral curves of degree 4. Next special case is an intersection of three quadrics in $\mathbb{P}^3_{\mathbb{R}}$. Here, a regular intersection $V_{\mathbb{R}}$ may consist of 0, 2, 4, 6, or 8 real points, and the spectral curve is a quartic $C \subset \mathbb{P}^2_{\mathbb{R}}$. Assuming C nonsingular and computing the Euler characteristic (*e.g.*, using Theorem 4.1.4 or the general formula for the Euler characteristic found in [3]) one can see that, if $V \neq \varnothing$, the real part $C_{\mathbb{R}}$ consists of $\frac{1}{2}\operatorname{Card} V_{\mathbb{R}}$ empty ovals. In this case, $\operatorname{Card} V_{\mathbb{R}}$ determines the net up to deformation. If $V_{\mathbb{R}} = \varnothing$, then either $C_{\mathbb{R}} = \varnothing$ or $C_{\mathbb{R}}$ is a nest of depth two. Such nets form two deformation classes, the homotopy class of the associated quadratic map, see 6.1, being either 0 or $1 \in \pi_3(S^2)/\pm 1$.

6.4. Canonical curves of genus 5 in $\mathbb{P}^4_{\mathbb{R}}$. Regular complete intersections of three quadrics in $\mathbb{P}^4$ are canonical curves of genus 5. Thus, the set of projective classes of such (real) intersections is embedded into the moduli space of (real) curves of genus 5. As is known, see, *e.g.*, [27], the image of this embedding is the complement of the strata formed by the hyperelliptic curves, trigonal curves, and curves with a vanishing theta constant. Since each of the three strata has positive codimension, the known classification of real forms of curves of a given genus (applied to $g = 5$) implies that the maximal number of connected components that a regular complete intersection of three real quadrics in $\mathbb{P}^4_{\mathbb{R}}$ can have equals $6 = \operatorname{Hilb}(5)$.

6.5. $K3$-surfaces of degree 8 in $\mathbb{P}^5_{\mathbb{R}}$. A regular complete intersection of three quadrics in the projective space of dimension 5 is a $K3$-surface with a (primitive) polarization of degree 8. Thus, as in the previous case, the set of projective classes of intersections is embedded into the moduli space of $K3$-surfaces with a polarization of degree 8, the complement consisting of a few strata of positive codimension (for details, see [26]). In particular, any generic $K3$-surface with a polarization of degree 8 is indeed a complete intersection of three quadrics. The deformation classification of $K3$-surfaces can be obtained using the results of V. V. Nikuin [23]. The case of maximal real $K3$-surfaces is, as usually, particulary simple: there are three deformation classes, distinguished by the topology of the real part, which can be $S_{10} \sqcup S$, $S_6 \sqcup 5S$, or $S_2 \sqcup 9S$. In particular, the maximal number of connected components of a complete intersection of three real quadrics in $\mathbb{P}^5_{\mathbb{R}}$ equals $10 = \text{Hilb}(6) + 1$. (There is another shape with 10 connected components, the $K3$-surface with the real part $S_1 \sqcup 9S$. However, one can easily show that a $K3$-surface of degree 8 cannot have 10 spheres.)

6.6. Other Betti numbers. The techniques of this paper can be used to estimate the other Betti numbers as well. For $0 \leqslant i < \frac{1}{2}(N-3)$, it would give a bound of the form

$$B^i_2(N) - \text{Hilb}_{i+1}(N+1) = O(1),$$

where $B^i_2(N)$ is the maximal i-th Betti number of a regular complete intersection of three real quadrics in $\mathbb{P}^N_{\mathbb{R}}$ and $\text{Hilb}_{i+1}(N+1)$ is the maximal number of ovals of depth $\geqslant (D_{\max} - i - 1) = [\frac{1}{2}(N-1)] - i$ that a nonsingular real plane curve of degree $d = N+1$ may have. The possible discrepancy is due to a couple of unknown differentials in the spectral sequence and the inclusion homomorphism $H^{N-3-i}(\mathbb{P}^N_{\mathbb{R}}) \to H^{N-3-i}(V_{\mathbb{R}})$.

6.7. The examples are asymptotically maximal. Recall that, given a real algebraic variety X, the Smith inequality states that

$$\dim H_*(X_{\mathbb{R}}) \leqslant \dim H_*(X). \tag{6.7.1}$$

(As usual, all homology groups are with the $\mathbb{Z}_2$ coefficients.) If the equality holds, X is said to be *maximal*, or an *M-variety.* In particular, if X is a nonsingular plane curve of degree $N+1$, the Smith inequality (6.7.1) implies that the number of connected components of $X_{\mathbb{R}}$ does not exceed $g + 1 = \frac{1}{2}N(N-1) + 1$. The Hilbert curves used in Section 4 to construct nets with a big number of connected components are known to be maximal. Using the spectral sequence of Theorem 4.1.4, one can easily see that, under the choice of the index function made in the proof of Theorem 1.1.3, the dimension $\dim H_*(L_+)$ equals $2(g+1)+2$ if N is odd or $2(g+1)+1$ if N is even.

On the other hand, for the common zero set V (as for any projective variety) there is a certain constant l such that the inclusion homomorphism $H_i(V_{\mathbb{R}}) \to H_i(\mathbb{P}^N_{\mathbb{R}})$ is nontrivial for all $i \leqslant l$ and trivial for all $i > l$ (see [18]; in our case $1 \leqslant l < \frac{1}{2}N$). Hence, by the Poincaré–Lefschetz duality and Lemma 4.1.1, one has

$$\dim H_*(L_+) = \dim H_*(\mathbb{P}^N_{\mathbb{R}}, V_{\mathbb{R}}) = \dim H_*(V_{\mathbb{R}}) + N - 2l - 1.$$

Finally, one can easily find $\dim H_*(V)$: it equals $4(k^2-1)$ if $N = 2k$ is even and $4(k^2-k)$ if $N = 2k-1$ is odd. Now, combining the above computation, one observes that the intersections of quadrics constructed from the Hilbert curves using monotonous index functions are asymptotically maximal in the sense that

$$\dim H_*(V_{\mathbb{R}}) = \dim H_*(V) + O(N) = N^2 + O(N).$$

The latter identity shows that the upper bound for $B_2^0(N)$ provided by the Smith inequality is too rough: this bound is of the form $\frac{1}{2}N^2+O(N)$, whereas, as is shown in this paper, $B_2^0(N)$ does not exceed $\frac{3}{8}N^2+O(N)$. When the intersection is of even dimension, one can improve the leading coefficient in the bound by combining the Smith inequality and the generalized Comessatti inequality; however, the resulting estimate is still too rough.

6.8. The examples are not maximal. Another interesting consequence of the computation of the previous section is the fact that, starting from $N = 6$, the complete intersections of quadrics maximizing the number of components are never truly maximal in the sense of the Smith inequality (6.7.1): one has

$$N + O(1) \leqslant \dim H_*(V) - \dim H_*(V_{\mathbb{R}}) \leqslant 2N + O(1).$$

Using the spectral sequence of Theorem 4.1.4, one can easily show that a maximal complete intersection V of three real quadrics in $\mathbb{P}^N_{\mathbb{R}}$ must have index function taking values between $\frac{1}{2}(N-1)$ and $\frac{1}{2}(N+3)$ (*cf.* (4.3.1)–(4.3.3)); the real part $V_{\mathbb{R}}$ has large Betti numbers in two or three middle dimensions, (most) other Betti numbers being equal to 3.

Apparently, it is the Harnack M-curves that are suitable for obtaining nets with maximal common zero locus. However, at present we do not know much about the differentials in the spectral sequence or the constant l introduced in the previous section. It may happen that these data are controlled by the additional flexibility (the values on the components of $C_{\mathbb{R}}$) in the choice of the real Spin-structure on the spectral curve.

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