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ALEXANDER POLYNOMIAL OF A CURVE OF DEGREE SIX

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Abstract. The complete description of the Alexander polynomial of the complement of an irreducible sextic in $\mathbb{C}\mathrm{p}^2$ is given. Some general results about Alexander polynomials of algebraic curves are also obtained.

Introduction

In this paper we study the Alexander polynomial $\widetilde{\Delta}_B(t)$ of the complement of a complex projective plane algebraic curve $B \subset \mathbb{C}\mathrm{p}^2$. The original problem, which was suggested to me by O. Viro, was to find the dependence of the fundamental group of $\mathbb{C}\mathrm{p}^2 \smallsetminus B$ on the set of singularities of B. Alexander polynomial, being an invariant of the fundamental group, is supposed to reflect this dependence. (This approach, first suggested, perhaps, by O. Zariski [Z], was recently developed by A. Libgober [L1]–[L4]; see also A. Dimca [Di] and F. Loser and M. Vaquie [Lo]) In the paper, we do obtain an expression for $\widetilde{\Delta}_B(t)$ in terms of the singular points of B *and* their mutual position in $\mathbb{C}\mathrm{p}^2$ (see Corollary 5.2; a similar result can also be found in Esnault [E]). In fact, multiplicities of the roots of $\widetilde{\Delta}_B(t)$ depend on the dimensions of certain linear systems with prescribed base points at the singularities of B. (This generalizes Zariski's result [Z] on the Alexander polynomial of a cuspidal sextic.) Then, we obtain a lower estimate $\widetilde{\Delta}_B^{\min}(t)$ for $\widetilde{\Delta}_B(t)$, which depends *only* on the set of singularities of B. Thus, the problem eventually reduces to that of finding the curves for which $\widetilde{\Delta}_B(t) \neq \widetilde{\Delta}_B^{\min}(t)$. (We call such curves *abundant*.) We solve this problem for irreducible sextics. The main result of the paper is the following:

Key words and phrases. Alexander polynomial, algebraic curve.

Main theorem. *Let B be an irreducible sextic. Then the following statements are equivalent:*

(1) *B is abundant;*
(2) *B is the branch curve of a projection to $\mathbb{C}\mathrm{p}^2$ of a cubic surface $W \subset \mathbb{C}\mathrm{p}^3$ with only non-degenerated double points as singularities;*
(3) *B is given by $h^2 + c^3 = 0$, where h and c are some homogeneous polynomials of degree 3 and 2 respectively, and the singular set of B is of the form $aA_1 + \sum b_d A_{3d-1} + cE_6$, where $\sum db_d + 2c = 6$.*

The Alexander polynomial of an abundant irreducible sextic is $t^2 - t + 1$.

This theorem is proved in Section 6.

Remark. The simplest class of abundant sextics, namely, those with six cusps, was first found by O. Zariski [Z].

Remark. The problem of isotopy classification of sextics still remains open (see Section 7 for known results). Degree 6 is the first case when two irreducible curves with the same singular set may not be isotopic — they are distinguished by their Alexander polynomials. (Classification of the curves of degree 5 can be found in [D3], where it is proved that an irreducible quintic is determined up to isotopy by its set of singular points.) It would be very interesting to know if there are other invariants.

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1. Alexander polynomial

1.1. Cyclic coverings. Let X be a non-singular algebraic surface (over $\mathbb{C}$), B an effective divisor on X, and n a positive integer.

1.1.1. Definition. An *n-fold cyclic covering of X branched over B* is a proper analytic map $p'\colon \widetilde{X}' \to X$ such that:

(1) the restriction $p'\colon \widetilde{X}' \smallsetminus {p'}^{-1}B \to X \smallsetminus B$ is an n-fold cyclic covering;
(2) every point x of the support of B has a neighborhood U such that the restriction $p'\big|_U\colon {p'}^{-1}U \to U$ is isomorphic to the restriction of the canonical projection $U \times \mathbb{C}^1 \to U$ to the subvariety $\big\{(x,t) \in U \times \mathbb{C}^1 \bigm| t^n = s(x)\big\}$, where $s\colon U \to \mathbb{C}^1$ is a section whose zero-set is $B \cap U$.

1.1.2. Proposition (see, e.g., [H2] or [D1]). *Given X, B, and n as above, there is a natural one-to-one correspondence between the isomorphism classes of n-fold coverings of X branched over B and classes $E \in H^1(X; \mathcal{O}_X^*)$ such that $nE = [B]$.*

1.1.3. Definition. The above E will be called the *class* of the covering p'.

Remark. This correspondence can be described as follows: Denote by L_E and L_B the linear bundles over X corresponding to E and $[B]$ respectively, and fix an isomorphism $L_E^{\otimes n} \cong L_B$ and a section $s\colon X \to L_B$ with the zero-set B. Then p' is the restriction of the bundle projection $L_E \to X$ to the set $\widetilde{X}' \in L_E$ of the n-th roots of s (i.e., locally $\widetilde{X}'$ is given by $\bigl\{(x,t) \bigm| t^n = s(x)\bigr\}$).

1.2. Alexander polynomial. Let X be a topological space. Fix an epimorphism $\rho\colon H_1(X;\mathbb{Z}) \to \mathbb{Z}_n$ (where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the cyclic group of order n, $1 \leqslant n \leqslant \infty$), and consider the corresponding cyclic covering $p'\colon \widetilde{X}' \to X$. Then $H^1(\widetilde{X}';\mathbb{C})$ has a natural structure of a $\mathbb{C}[t]$-module, t acting via the homomorphism $\operatorname{tr}^*\colon H^1(\widetilde{X}';\mathbb{C}) \to H^1(\widetilde{X}';\mathbb{C})$ induced by the deck translation tr of the covering.

1.2.1. Definition. If $H^1(\widetilde{X}';\mathbb{C})$ is a finite dimensional $\mathbb{C}$-vector space, the characteristic polynomial of tr^* is called the *Alexander polynomial* of X (or, more precisely, pair (X,ρ) or covering p') and is denoted $\Delta_X(t)$. Thus, $\Delta_X(t) = \det(\operatorname{tr}^* - t\operatorname{id})$.

Other, equivalent, definitions of $\Delta_X(t)$ and a review of its basic properties can be found in Libgober [L1]. In particular, the following facts are proved there:

1.2.2. Proposition. *Suppose that X has homotopy type of a finite cell complex, and n is finite. Then:*

1. *$\Delta_X(t)$ is well defined and is a cyclotomic polynomial;*
2. *as a $\mathbb{C}[t]$-module, $H^1(\widetilde{X}';\mathbb{C})$ is the direct sum of m_i copies of the cyclic module $\mathbb{C}[t]/(t-\lambda_i)$ for each m_i-ply root λ_i of $\Delta_X(t)$;*
3. *p' induces an isomorphism $H^1(\widetilde{X}';\mathbb{C})/(t-1) = H^1(X;\mathbb{C})$.*

1.3. Alexander polynomial of an algebraic curve. Consider an algebraic curve B on an algebraic surface X. Suppose that its class $[B]$ is divisible by an integer n, and fix some $E \in H^1(X;\mathcal{O}_X^*)$ such that $nE = [B]$. Then, according to Proposition 1.1.2, there is a unique (up to isomorphism) covering $p'\colon \widetilde{X}' \to X$ branched over B whose class is E. The restriction $p'\colon \widetilde{X}' \smallsetminus {p'}^{-1}B \to X \smallsetminus B$ is an n-fold cyclic covering of $X \smallsetminus B$, and, according to Proposition 1.2.2, its Alexander polynomial is well defined.

1.3.1. Definition. The Alexander polynomial of the above restriction $p'\colon \widetilde{X}' \smallsetminus {p'}^{-1}B \to X \smallsetminus B$ is called the *Alexander polynomial* of B (or, more precisely, pair (B,E)) and is denoted by $\Delta_B(t)$. The $\mathbb{C}[t]$-module $H^1(\widetilde{X}' \smallsetminus {p'}^{-1}B;\mathbb{C})$ is called the *Alexander module* of B.

Let $\rho\colon \widetilde{X} \to \widetilde{X}'$ be a resolution of singularities of $\widetilde{X}'$ equivariant in respect to the deck translation, and let p be the composition $p'\circ\rho\colon \widetilde{X} \to X$. The following result is due to Libgober [L1]:

1.3.2. Proposition. *The inclusion homomorphism $H^1(\widetilde{X};\mathbb{C}) \to H^1(\widetilde{X} \smallsetminus p^{-1}B;\mathbb{C})$ is a monomorphism. If B is reduced, then p^* induces an isomorphism*

$$\operatorname{Coker}\bigl[H^1(X;\mathbb{C})\to H^1(X\smallsetminus B;\mathbb{C})\bigr] \xrightarrow[\cong]{p^*} \operatorname{Coker}\bigl[H^1(\widetilde{X};\mathbb{C})\to H^1(\widetilde{X}\smallsetminus p^{-1}B;\mathbb{C})\bigr].$$

Thus, $H^1(\widetilde{X};\mathbb{C})$ differs from the Alexander module of B by an easily controllable space with trivial tr^*-action. E.g., if B is an irreducible curve in $\mathbb{C}\mathrm{p}^2$ (this is the case that we are mainly interested in), there is a direct sum decomposition

$$H^1(\widetilde{X} \smallsetminus p^{-1}B;\mathbb{C}) = H^1(\widetilde{X};\mathbb{C}) \oplus \mathbb{C},$$

the latter summand being the invariant part of the Alexander module. For this reason, we will mainly deal with $H^1(\widetilde{X};\mathbb{C})$. The characteristic polynomial of the automorphism tr^* of $H^1(\widetilde{X};\mathbb{C})$ will be called the *reduced Alexander polynomial* of B and will be denoted by $\widetilde{\Delta}_B(t)$.

2. Resolution of singularities of $\widetilde{X}'$

In this section I give a brief review of Hirzebruch's [H1] results about the minimal resolution of the singularity of $\widetilde{X}' = \bigl\{(x,y,z)\in\mathbb{C}^3 \bigm| z^n = x^p y^q\bigr\}$. For the sake of simplicity we assume that g.c.d.$(n,p,q)=1$.

Let $p = p'p_0$, $q = q'q_0$, and $n = p'q'n_0$, where $p' = $ g.c.d.(n,p) and $q' = $ g.c.d.(n,q). Since p_0 and n_0 are relatively prime, there exists some $\overline{q}_0 \in \mathbb{N}$, $\overline{q}_0 < n_0$, such that $p_0\cdot\overline{q}_0 \equiv q_0 \pmod{n_0}$. Define two sequences $\lambda = \{\lambda_i\}_{i\geqslant 0}$ and $b = \{b_i\}_{i\geqslant 1}$ of non-negative integers as follows:

$$\lambda_0 = n_0, \qquad \lambda_1 = n_0 - \overline{q}_0,$$
$$\lambda_{i-1} = b_i\lambda_i - \lambda_{i+1}, \quad \lambda_{i+1} < \lambda_i, \;\; b_i \geqslant 2 \;\text{ for }\; i \geqslant 1.$$

The process terminates at some t so that $\lambda_t = 1$, $\lambda_{t+1} = 0$. Define also a sequence $\mu = \{\mu_i\}_{0\leqslant i\leqslant t+1}$ by putting $\mu_0 = 0$, $\mu_1 = 1$, and $\mu_{i+1} = b_i\mu_i - \mu_{i-1}$ for $i \geqslant 1$. It is shown in [H1] that $\mu_{t+1} = n_0$.

Let $\rho\colon \widetilde{X} \to \widetilde{X}'$ be the minimal resolution of the singularity of $\widetilde{X}'$ at $(0,0,0)$, and let $p'\colon \widetilde{X}' \to \mathbb{C}^2$ be the restriction to $\widetilde{X}'$ of the natural projection $\mathbb{C}^3 \to \mathbb{C}^2$, $(x,y,z) \mapsto (x,y)$. The following facts are proved in [H1] (see also [Li]):

2.1. $\rho^{-1}(0,0,0)$ consists of t non-singular rational curves $E_1, \ldots, E_t$. If we let E_0 and E_{t+1} be the reduced proper images of $\widetilde{X}' \cap \{y=0\}$ and $\widetilde{X}' \cap \{x=0\}$ respectively, then:

$$\begin{aligned} E_i^2 &= -b_i \quad \text{for } 1 \leqslant i \leqslant t,\\ E_i \circ E_{i+1} &= 1 \quad \text{for } 0 \leqslant i \leqslant t, \text{ and}\\ E_i \circ E_j &= 0 \quad \text{for } |i-j| > 1. \end{aligned}$$

2.2. There is a covering of $\widetilde{X}$ by affine charts $\mathcal{U}_i$ with coordinates (u_i, v_i), $0 \leqslant i \leqslant t$, such that:

(1) the transition functions are $u_{i+1} = v_i^{-1}$, $v_{i+1} = u_i v_i^{b_{i+1}}$;
(2) E_i is given by $\{v_i = 0\}$ (on $\mathcal{U}_i$) or $\{u_{i-1} = 0\}$ (on $\mathcal{U}_{i-1}$);
(3) the restriction of $p' \circ \rho$ to $\mathcal{U}_i$ is given by

$$x = u_i^{q'\lambda_{i+1}} v_i^{q'\lambda_i}, \qquad y = u_i^{p'\mu_{i+1}} v_i^{p'\mu_i}.$$

2.3. One has $\omega_{\widetilde{X}}(\widetilde{B}_{\mathrm{red}}) = (p' \circ \rho)^* \omega_{\mathbb{C}^2}(B_{\mathrm{red}})$, where $B_{\mathrm{red}} = \bigl\{(x,y) \in \mathbb{C}^2 \bigm| xy = 0\bigr\}$, and $\widetilde{B}_{\mathrm{red}}$ is the reduced full inverse image of B_{red}.

From 2.2(1) it follows that any meromorphic monomial function on $\widetilde{X}$ is locally given by $u_i^{\nu_{i+1}} v_i^{\nu_i}$, where $\nu = \{\nu_i\}_{0 \leqslant i \leqslant t+1}$, is a sequence of integers such that $\nu_{i-1} + \nu_{i+1} = b_i \nu_i$. We will call such sequences *admissible.* An admissible sequence will be called *non-negative* (*positive*) if $\nu_i \geqslant 0$ (resp., $\nu_i > 0$) for all i. (Obviously, non-negative sequences correspond to analytical functions.) We will need the following arithmetical properties of admissible sequences: (We suppose that the integers $b_i \geqslant 2$, $1 \leqslant i \leqslant t$, are fixed, and λ and μ are as above)

2.4. Proposition.

(1) *every admissible sequence has a unique representation* $\nu = r\lambda + s\mu$ *for some* $r, s \in \mathbb{Q}$;
(2) *if* ν *is non-negative and* $\nu_i = 0$ *for some* $i \neq 0, t+1$, *then* $\nu \equiv 0$;
(3) ν *is non-negative* (*positive*) *if and only if* $r, s \geqslant 0$ (*resp.*, $r, s > 0$).

Proof. Obviously, a linear combination of admissible sequences is admissible (provided that it is integral), and an admissible sequence is uniquely determined by its first two terms (ν_0, ν_1). Since $(\lambda_0, \lambda_1) = (n_0, n_0 - \overline{q}_0)$ and

$(\mu_0,\mu_1)=(0,1)$ are linearly independent, every such pair (ν_0,ν_1) is their linear combination. This proves Statement (1).

Statement (2) immediately follows from the relation $\nu_{i-1}+\nu_{i+1}=b_i\nu_i$: if $\nu_i=0$, then either $\nu_{i-1}=\nu_{i+1}=0$, or one of these numbers is negative. The latter case is impossible, the former one gives (by induction) $\nu\equiv 0$.

Since $\mu_0=\lambda_{t+1}=0$, one has $\nu_0=r\lambda_0$, $\nu_{t+1}=s\mu_{t+1}$. Hence, $\nu_0,\nu_{t+1}\geqslant 0$ if and only if $r,s\geqslant 0$, and $\nu_0,\nu_{t+1}>0$ (which, according to Statement (2), is equivalent to positivity of ν) if and only if $r,s>0$. □

2.5. Corollary. *If ν is a non-negative admissible sequence and $\nu_0\geqslant\lambda_0$ (or $\nu_{t+1}\geqslant\mu_{t+1}$), then $\nu-\lambda$ (resp., $\nu-\mu$) is still non-negative.*

3. The Hodge structure on $H^1(\widetilde X;\mathbb C)$.

Throughout this section we suppose fixed an algebraic curve B on an algebraic surface X, an n-fold cyclic covering $p'\colon \widetilde X'\to X$ branched over B, and a resolution $\rho\colon \widetilde X\to\widetilde X'$ of singularities of $\widetilde X$ equivariant in respect to the deck translation tr. We denote by $E\in H^1(X;\mathcal O_X^*)$ the class of p'.

Let $\xi_r=\exp(2\pi ir/n)$, $r=0,\dots,n-1$, be the n-th roots of unity. Since tr is an order n analytic automorphism of $\widetilde X$, the $\mathbb C$-linear space $H^1(\widetilde X;\mathbb C)$ splits into direct sum $\bigoplus H^1_r(\widetilde X;\mathbb C)$ of the eigenspaces, and this splitting is compatible with the Hodge decomposition $H^1(\widetilde X;\mathbb C)=H^{1,0}(\widetilde X)\oplus H^{0,1}(\widetilde X)$, where $H^{p,q}(\widetilde X)=H^p(\widetilde X;\Omega^q(\widetilde X))$. Let us denote by $h^{p,q}_r=h^{p,q}_r(\widetilde X)$ the dimension of the eigenspace $H^{p,q}_r(\widetilde X)$ of $H^{p,q}(\widetilde X)$ corresponding to the eigenvalue ξ_r. Then, obviously,

$$\widetilde\Delta_B(t)=\prod_{r=0}^{n-1}(t-\xi_r)^{h^{1,0}_r+h^{0,1}_r}.$$

Now from the fact that $H^{p,q}(\widetilde X)=\overline{H^{q,p}(\widetilde X)}$ (where the bar means complex conjugation) it follows that $h^{p,q}_r=h^{q,p}_{n-r}$. Hence, finally,

$$\widetilde\Delta_B(t)=\prod_{r=0}^{n-1}(t-\xi_r)^{h^{1,0}_r+h^{1,0}_{n-r}}. \tag{3.1}$$

Let us introduce the following notation: If D is a divisor on X with *rational* coefficients, and $\sum r_\alpha D_\alpha$ is its decomposition into the sum of pairwise distinct components, then we put $\lfloor D\rfloor=\sum\lfloor r_\alpha\rfloor D_\alpha$, $\lceil D\rceil=\sum\lceil r_\alpha\rceil D_\alpha$, and $D_{\mathrm{red}}=\sum D_\alpha$, where $\lfloor x\rfloor$ and $\lceil x\rceil$ are the lower and upper integral approximations of a rational x respectively.

Suppose now that B is a divisor with normal crossings (not necessarily reduced). In Esnault [E] (see also [D1]) the following fact is proved:

3.2. Proposition. *If B is a divisor with normal crossings, then the sheaf $\mathcal{O}_{\widetilde{X}}$ is p_*-acyclic (i.e., $R^m p_*\mathcal{O}_{\widetilde{X}} = 0$ for $m > 0$), and*

$$p_*\mathcal{O}_{\widetilde{X}} = \bigoplus_{r=0}^{n-1} \mathcal{O}_X\Big(\Big\lfloor \frac{r}{n}B\Big\rfloor - rE\Big),$$

the summands of the latter decomposition being the eigensheaves of tr^* *on* $p_*\mathcal{O}_{\widetilde{X}}$.

We will mainly use the following similar result about the dual sheaves $\omega_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$:

3.3. Proposition. *If B is a divisor with normal crossings, then the sheaf $\omega_{\widetilde{X}}$ is p_*-acyclic (i.e., $R^m p_*\omega_{\widetilde{X}} = 0$ for $m > 0$), and*

$$p_*\omega_{\widetilde{X}} = \bigoplus_{r=0}^{n-1} \omega_X\Big(rE - \Big\lfloor \frac{r}{n}B\Big\rfloor\Big),$$

the summands of the latter decomposition being the eigensheaves of tr^* *on* $p_*\omega_{\widetilde{X}}$ *(corresponding, respectively, to the eigenvalues ξ_{-r}).*

Proof. The first statement is an immediate consequence of the following result by J. Kóllar [K]:

3.4. Proposition (see [K]). *Let $p\colon X \to Y$ be a surjective map of a smooth projective variety X to a projective variety Y. Then $R^i p_*\omega_X = 0$ for all $i > \dim Y - \dim X$.*

According to 2.3, there is an exact sequence

$$0 \to \omega_{\widetilde{X}} \to p^*\omega_X(B_{\mathrm{red}}) \to p^*\omega_X(B_{\mathrm{red}})\big|_{\widetilde{B}_{\mathrm{red}}} \to 0,$$

where $\widetilde{B} = p^*B$. Since $\omega_X(B_{\mathrm{red}})$ is an invertible sheaf, the projection formula implies that $\mathcal{S} = p_*\omega_{\widetilde{X}} \otimes \big[\omega_X(B_{\mathrm{red}})\big]^{-1}$ is the subsheaf of $p_*\mathcal{O}_{\widetilde{X}}$ whose sections vanish on $\widetilde{B}_{\mathrm{red}}$.

Pick a point $\mathrm{pt} \in X$, and find the stock $\mathcal{S}\big|_{\mathrm{pt}} \in p_*\mathcal{O}_{\widetilde{X}}\big|_{\mathrm{pt}}$. Consider the following cases:

Case 1: $\mathrm{pt} \notin B$. In this case obviously $\mathcal{S}\big|_{\mathrm{pt}} = p_*\mathcal{O}_{\widetilde{X}}\big|_{\mathrm{pt}}$.

Case 2: pt *is a smooth point of* B_{red}. Let pt lie on a p-fold component of B given by $\{x = 0\}$, where (x, y) are some local coordinates on X about pt. We can assume that g.c.d.$(n, p) = 1$. (Otherwise both the left and right hand sides of 3.3 split into direct sums of $d =$ g.c.d.(n, p) isomorphic subsheavs corresponding to the d connected components of $\widetilde{X}$.) Then there are some local coordinates (u,v) in $\widetilde{X}$ such that p is given by $x = u^n$, $y = v$, and $\widetilde{B}_{\text{red}}$ is given by $\{u = 0\}$. In these coordinates $p_*\mathcal{O}_{\widetilde{X}}\big|_{\text{pt}}$ is generated (over $\mathcal{O}_X\big|_{\text{pt}}$) by the germs of $1, u, \ldots, u^{n-1}$, and $\mathcal{S}\big|_{\text{pt}}$ is generated by the germs of $u^n, u, \ldots, u^{n-1}$.

Case 3: pt *is a singular point of* B_{red}. In this case $\widetilde{X}'$ is given locally by $\{(x,y,z) \in \mathbb{C}^3 \mid z^n = x^p y^q\}$, and, similar to Case 2, we can assume that g.c.d.$(n,p,q) = 1$. Besides, we can consider the minimal resolution of $\widetilde{X}'$ and, hence, apply the results of Section 2. (We also use definitions and notation of Section 2.) According to Proposition 3.2, $p_*\mathcal{O}_{\widetilde{X}}\big|_{\text{pt}}$ is freely generated over $\mathcal{O}_X\big|_{\text{pt}}$ by the germs of n sections corresponding to some non-negative admissible sequences $\nu^{(r)} = \{\nu_i^{(r)}\}_{0\leqslant i\leqslant t+1}$, $r = 0, 1, \ldots, n$. These sequences must be minimal in the following sense: neither $\nu^{(r)} - q'\lambda$ nor $\nu^{(r)} - p'\mu$ is non-negative. (Otherwise one could multiply the corresponding section by $(x\circ p)^{-1}$ or $(y\circ p)^{-1}$ respectively, and the result would still be an analytical section.) Now, in order to pass to $\mathcal{S}\big|_{\text{pt}}$, one should replace those of $\nu^{(r)}$ for which $\nu_0^{(r)} = 0$ with $\nu^{(r)} + \lambda$, and those for which $\nu_{t+1}^{(r)} = 0$, with $\nu^{(r)} + \mu$. Let us prove that $\nu_0^{(r)} = 0$ if and only if rp is divisible by n. (Similarly one can prove that $\nu_{t+1}^{(r)} = 0$ if and only if rq is divisible by n.)

The deck translation tr acts on $\mathcal{U} = \{(u_0, v_0)\}$ via $u_0 \mapsto u_0\xi_\alpha$, $v_0 \mapsto v_0\xi_b$ for some $\alpha, \beta \in \mathbb{Z}_n$. Since the coordinates (x,y) in X are tr-invariant, 2.2(3) with $i = 0$ implies

$$\begin{cases} \alpha\cdot q'(n_0 - \overline{q}_0) + \beta\cdot q'n_0 \equiv 0, \\ \alpha\cdot p' \equiv 0. \end{cases} \qquad \pmod{p'q'n_0}$$

Besides, α and β generate $\mathbb{Z}_n$. From this it follows that $\alpha \equiv 0 \pmod{q'n_0}$ and β is invertible in $\mathbb{Z}_{q'n_0}$. Consider now a sequence $\nu^{(r)}$. Since tr multiplies the corresponding section by ξ_r, one has $\alpha\nu_1^{(r)} + \beta\nu_0^{(r)} \equiv r \pmod n$. Hence, if $\nu_0^{(r)} = 0$, then $r \equiv 0 \pmod{q'n_0}$ and $rp = rp'p_0$ is divisible by $n = p'q'n_0$. Conversely, if rp is divisible by n, then $r \equiv 0 \pmod{q'n_0}$ and $\beta\nu_0^{(r)} \equiv 0 \pmod{q'n_0}$. Since β is invertible in $\mathbb{Z}_{q'n_0}$, $\nu_0^{(r)}$ is divisible by $q'n_0$, and, since $\nu^{(r)}$ is minimal, $\nu_0^{(r)} = 0$ (cf. Corollary 2.5).

Thus, we have proved that the r-th summand of $\mathcal{S}$ is the subsheaf of the r-th summand of $p_*\mathcal{O}_{\widetilde{X}}$ whose sections vanish on those of the components B_α of B for which rp_α is divisible by n. (Here p_α is the multiplicity of B_α in B.) Hence,

$$\mathcal{S} = \bigoplus_{r=0}^{n-1} \mathcal{O}_X\left(\left\lceil \frac{r}{n}B\right\rceil - B_{\text{red}} - rE\right),$$

and multiplication by $\omega_X(B_{\mathrm{red}})$ gives 3.2. □

3.5. Corollary. *If B is a divisor with normal crossings, one has*

$$h_r^{1,0} = \dim_{\mathbb{C}} H^1\Bigl(X; \omega_X\Bigl(rE - \Bigl\lfloor \frac{n}{r} B \Bigr\rfloor\Bigr)\Bigr).$$

4. Spectrum of a singularity

Let us recollect some of Varchenko's results about orders of holomorphic forms in a neighborhood of an isolated singularity. Proofs of all the results stated in this section can be found in Varchenko [V]; see also [AVG] for a detailed review of the subject.

Let $f\colon (\mathbb{C}^2, O) \to (\mathbb{C}, 0)$ be a germ at an isolated singular point $O = (0, 0)$. Then, given a holomorphic (germ of a) differential form ω at O, there is defined its *order* $\operatorname{ord}\omega$. (The definition of $\operatorname{ord}\omega$ is rather sophisticated, so I prefer to omit it here. In short, $\operatorname{ord}\omega$ is the exponent of the leading term of some power series expansion of ω at O. Roughly speaking, $\operatorname{ord}\omega$ measures the rate of vanishing of ω at O compared to that of f.)

4.1. Proposition. $\operatorname{ord}\omega \in \mathbb{Q} \cap (-1, \infty)$ *for any holomorphic form ω. There is an integer N (depending on f) such that $N \cdot \operatorname{ord}\omega$ is an integer.*

4.2. Definition. The *spectrum* $\operatorname{Spec}_O f$ of the singularity f at O is the set of (not necessarily distinct) rational numbers defined as follows:

(1) $\operatorname{Spec} f$ is symmetric about 0;
(2) the multiplicity of a rational $q \leqslant 0$ in $\operatorname{Spec} f$ equals

$$\dim_{\mathbb{C}} \Bigl[\Omega^2(\mathbb{C}^2)\big|_O \,/\, \bigl\{\omega \in \Omega^2(\mathbb{C}^2)\big|_O \;\big|\; \operatorname{ord}\omega > q\bigr\}\Bigr].$$

(q belongs to $\operatorname{Spec} f$ iff its multiplicity is positive.)

Remark. This definition differs from that by Varchenko. The two definitions are equivalent for plane singularities. In [V] it is proved that $\operatorname{Spec} f$ is a well defined finite set.

Consider an embedded resolution $\sigma\colon Y \to \mathbb{C}^2$ of the singularity of f at O. Let the zero-divisor of $f \circ \sigma$ be $\sum r_i E_i +$ (proper inverse image of $\{f = 0\}$), where E_i are the exceptional curves of σ. Define the *weight* $g(\omega)$ of a holomorphic form ω (in respect to σ) to be $\max_i\{-(m_i + 1)/r_i\}$, where m_i is the multiplicity of the zero of $\sigma^*\omega$ on E_i.

4.3. Proposition. *For any holomorphic form ω one has* $\operatorname{ord}\omega \geqslant -g(\omega) - 1$. *If $g(\omega) \geqslant -1$, then* $\operatorname{ord}\omega = -g(\omega) - 1$

Suppose now that in some coordinates (x, y) in $\mathbb{C}^2$ the germ f is represented by a Taylor expansion with $\mathbb{C}$-non-degenerate principal part, and denote by D the Newton polygon of this series.

4.4. Proposition. *Let $r \leqslant 0$ be a rational, and $\omega = \varphi\, dx \wedge dy$ a holomorphic form at O. Then $\operatorname{ord} \omega > r$ if and only if the Newton polygon of φ lies in $\operatorname{Int}[(r+1)D - (1,1)]$, where the latter polygon is obtained from D by the pointwise multiplication by $(r+1)$ and shift by $(-1,-1)$.*

5. Curves with isolated singularities

Let now $B \in X$ be a reduced curve with only isolated singularities. As above, we suppose fixed an n-fold branched covering $p'\colon \widetilde{X}' \to X$ with the class $E \in H^1(X; \mathcal{O}_X^*)$. Consider the following commutative diagram:

$$\begin{array}{ccc} & & \widetilde{Y} \\ & & \big\downarrow{\scriptstyle \rho'} \\ \widetilde{X}' & \xleftarrow{\ \widetilde{\sigma}\ } & \widetilde{Y}' = \sigma^*\widetilde{X}' \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle q'=\sigma^*p'} \\ X & \xleftarrow{\ \sigma\ } & Y, \end{array}$$

where $\sigma\colon Y \to X$ is an embedded resolution of singularities of B, and $\rho'\colon \widetilde{Y} \to \widetilde{Y}'$ is a deck translation equivariant resolution of singularities of $\widetilde{Y}'$. Obviously, the covering $q'\colon \widetilde{Y}' \to Y$ is branched over $\overline{B} = \sigma^* B$, and its class is $\overline{E} = \sigma^* E$. The composition $\rho = \widetilde{\sigma} \circ \rho'$ is a tr-invariant resolution of singularities of $\widetilde{X}'$. Denote $p = p' \circ \rho$.

Our goal in this section is to find the sheaf $p_*\omega_{\widetilde{Y}} = \sigma_*\bigl((q' \circ \rho')_*\omega_{\widetilde{Y}}\bigr)$.

5.1. Theorem. *Let B be a curve with only isolated singularities. Then*

(1) *$\omega_{\widetilde{Y}}$ is p_*-acyclic;*
(2) *the eigensheaf $\mathcal{S}_r$ of $p_*\omega_{\widetilde{Y}}$ corresponding to the eigenvalue ξ_{-r}, $r = 0, 1, \ldots, n-1$ is the subsheaf of $\omega_X(rE)$ defined as follows: a germ φ at a point O belongs to $\mathcal{S}_r(-rE)\big|_O$ if and only if $\operatorname{ord}\varphi > r/n - 1$.*

Proof. The first statement follows immediately from Proposition 3.4. To prove the second one, we should just find the direct images under σ_* of $\omega_Y(r\overline{E} - \lfloor (r/n)\overline{B} \rfloor)$ (see Proposition 3.4). Due to the projection formula, it suffices to find $\sigma_*\omega_Y(-\lfloor (r/n)\overline{B} \rfloor)$. Let φ be a germ of a differential form at some $O \in X$. From the definitions it immediately follows that $\varphi \circ \sigma$ is a section of $\omega_Y(-\lfloor (r/n)\overline{B} \rfloor)$ if and only if $g(\varphi) < -r/n$. If this is the case, then, according to Proposition 4.3, $\operatorname{ord}\varphi > r/n - 1$. Conversely, if $\operatorname{ord}\varphi > r/n - 1$, then either $g(\varphi) < -1$, or $g(\varphi) = -\operatorname{ord}(\varphi) - 1 < -r/n$. In both the cases $g(\varphi) < -r/n$ and, hence, $\varphi \circ \sigma$ is a section of $\omega_Y(-\lfloor (r/n)\overline{B} \rfloor)$. $\square$

Remark. In general, $\mathcal{O}_{\tilde{Y}}$ is not p_*-acyclic.

If E is an ample class, the above result can be translated to the classical language of dimensions of linear systems. Let $\mathcal{L}_r$ be the linear system of global sections of the r-th eigensheaf of $p_*\omega_{\tilde{Y}}$. Denote by $\operatorname{vdim}\mathcal{L}_r$ its virtual dimension (i.e., the Euler characteristic of the corresponding sheaf). Then Riemann-Roch formula yields:

$$\operatorname{vdim}\mathcal{L}_r = \chi(X;\mathcal{O}_X) + \frac{1}{2}(r^2E^2 - rK_X \circ E) - \sum_{O\in\operatorname{Sing} B} d_{r/n}(O) - 1,$$

where

$$d_{r/n}(O) = \#\Big\{q \in \operatorname{Spec}_O B \;\Big|\; q \leqslant \frac{r}{n} - 1\Big\}.$$

5.2. Corollary. *If E is ample and $r < n$, one has $h^{1,0}_{n-r} = \dim\mathcal{L}_r - \operatorname{vdim}\mathcal{L}_r$.*

Proof. The quotient $\bigoplus \omega_X(rE)/p_*\omega_{\tilde{Y}}$ is concentrated at finitely many points. Hence, $H^2(X;p_*\omega_{\tilde{Y}}) = H^2(X;\bigoplus\omega_X(rE))$. Since E is ample, Sérre duality gives $H^2(X;\omega_X(rE)) = H^0(X;\mathcal{O}_X(-rE)) = 0$, and the statement follows. □

Remark. According to Theorem 5.1, $\mathcal{L}_r$ consists of curves linear equivalent to rE with certain prescribed base points at the singular points of B. In general, the local condition to a curve $C \in \mathcal{L}_r$ is expressed in terms of a resolution of singularities of B: one should pick some σ as above and see if the weight $g(C)$ satisfies $g(C) < -r/n$. However, in many cases Proposition 4.4 gives an easier description in terms of the Newton polygons.

5.3. Definition. Denote by $\tilde{\Delta}^{\min}_B(t)$ the minimal integral polynomial divisible by

$$\prod_{r=1}^{n-1}(t-\xi_r)^{m_r+m_{n-r}},$$

where

$$m_{n-r} = \min\Big\{0, \sum_{O\in\operatorname{Sing} B} d_{r/n}(O) - \chi(X;\mathcal{O}_X) - \frac{1}{2}(r^2E^2 - rK_X\circ E)\Big\}.$$

Since, obviously, $\dim\mathcal{L}_r \geqslant -1$ and $\tilde{\Delta}_B(t)$ is an integral polynomial, we obtain:

5.4. Corollary. *If E is ample, $\tilde{\Delta}_B(t)$ is divisible by $\tilde{\Delta}^{min}_B(t)$.*

5.5. Definition. An ample reduced curve $B \in X$ is called *abundant* if $\widetilde{\Delta}_B(t) \neq \widetilde{\Delta}_B^{min}(t)$

Remark. Corollary 5.2 gives the value of $\widetilde{\Delta}_B(t)$ in terms of the types of the singular points of B and their position in X. Corollary 5.4 gives a lower estimate, which depends only on the types of the singularities of B. Thus, in a way, abundant curves are those with abnormal behavior of the Alexander polynomial: it is greater then it has to be (for the given set of singularities).

6. Proof of the main theorem

Throughout this section we suppose that $B \subset \mathbb{C}\mathrm{p}^2$ is an irreducible curve of degree 6, and $E = [\mathbb{C}\mathrm{p}^1] \in H^1(\mathbb{C}\mathrm{p}^2; \mathcal{O}_{\mathbb{C}\mathrm{p}^2})$ is the class of a projective line. The Alexander polynomial $\widetilde{\Delta}_B(t)$ of B is the Alexander polynomial of the (desingularized) 6-fold branched covering corresponding to E.

For the sake of simplicity we will suppose that all the singular points of B are simple (i.e., A_p, D_q, E_6, E_7, or E_8). The curves of degree 6 which have at least one non-simple singular point are classified (see [D2]), and for such curves the statement of the theorem can be verified by a direct calculation using Corollary 5.2. (Note, though, that the consideration below requires just a slight modification to apply to such curves.)

Let O_i, $i = 1, \ldots, k$, be all the singular points of B, and $d(O_i) = d_{5/6}(O_i)$ be the integers introduced in the previous section.

6.1. Lemma. *For an irreducible curve B of degree 6 one has $\widetilde{\Delta}_B(t) = (t^2 - t + 1)^h$ and $\widetilde{\Delta}_B^{\min}(t) = (t^2 - t + 1)^m$, where $h = \dim \mathcal{L}_5(B) + \sum d(O_i) - 5$, and $m = \max\{0, \sum d(O_i) - 6\}$. In particular, B is abundant if and only if $\dim \mathcal{L}_5(B) > 5 - \sum d(O_i)$.*

Proof. There are five linear systems, $\mathcal{L}_1(B)$ through $\mathcal{L}_5(B)$, which contribute to $\widetilde{\Delta}_B(t)$ and $\widetilde{\Delta}_B^{\min}(t)$. Three of them, $\mathcal{L}_2(B)$, $\mathcal{L}_3(B)$, and $\mathcal{L}_4(B)$, correspond to the eigenvalues $\exp(2\pi i/3)$, $\exp(\pi i)$, and $\exp(4\pi i/3)$ respectively, which neither of the two polynomials can have among its roots according to the following theorem by Zariski:

6.2. Theorem (Zariski [Z]). *Let $B \subset \mathbb{C}\mathrm{p}^2$ be an irreducible curve. Then no root of $\widetilde{\Delta}_B(t)$ can be of the form $\exp(2\pi ik/r)$, where $r = p^a$ is a power of a prime integer.*

Now, since the spectrum of any simple singular point O lies above $1/2$, $d_{1/6}(O) = 0$ for such a point, and, hence, $\mathcal{L}_1(B)$ is a *complete* linear system of degree -2, and $\dim \mathcal{L}_1(B) = \operatorname{vdim} \mathcal{L}(B) = -1$. Thus, the only system which contributes to either of the two polynomials is $\mathcal{L}_5(B)$, and the statement follows. $\square$

Thus, in what follows we should only consider the system $\mathcal{L}_5(B)$. To shorten the formulas, let us put $\mathcal{L} = \mathcal{L}_5(B)$ and denote by $\mathcal{K}$ the corresponding eigensheaf of $p_*\omega_{\tilde X}$.

Given a singular point O of B, denote by $i(O)$ the integer $\min\{(C \circ B)_O\}$, where $(\ \circ\ )_O$ means the local intersection index at O, and C runs over all the germs in $\mathcal{K}\big|_O$.

6.3. Lemma. *All the prescribed base points of $\mathcal{L}$ are simple (i.e., 1-fold), and one has*

$$\begin{aligned} d(A_p) &= \lfloor (p+1)/3 \rfloor, & i(A_p) &= 2\lfloor (p+1)/3 \rfloor, \\ d(D_q) &= \lfloor q/3 \rfloor, & i(D_q) &= 2\lfloor q/3 \rfloor + 1, \\ d(E_6) &= 2, & i(E_6) &= 4, \\ d(E_7) &= d(E_8) = 2, & i(E_7) &= i(E_8) = 5. \end{aligned}$$

In particular, always $i(O) \geqslant d(O)$, the inequality being sharp only for singular points of the types A_p and E_6.

Proof consists in a straightforward calculation using Proposition 4.4. □

6.4. Lemma. *B is abundant if and only if* $\operatorname{vdim}\mathcal{L} = -1$ *and* $\dim\mathcal{L} = 0$*. In this case:* (1) *all the singular points O_i of B are of the types A_p or E_6, and $\sum d(O_i) = 6$; and* (2) *$\mathcal{L}$ contains a unique conic C, which meets B only in its singular points O_i so that $(C \circ B)_{O_i} = i(O_i) = 2d(O_i)$.*

Proof. $\mathcal{L}$ is a linear system of conics with certain simple prescribed base points (possibly, infinitely near). Let $D = \sum d(O_i)$ be the total number of the prescribed base points, and $I = \sum i(O_i)$. Using the well known properties of the linear systems of plane conics, consider the following cases:

(1) $D \leqslant 3$. Then $\dim\mathcal{L} = \operatorname{vdim}\mathcal{L}$;
(2) $D = 4$ or 5. In this case $\dim\mathcal{L} > \operatorname{vdim}\mathcal{L}$ if and only if there is a line L through at least four of the prescribed base points. But then $L \circ B \geqslant 8$, and, hence, $L \subset B$, which contradicts to the assumption that B is irreducible;
(3) $D = 6$ ($\operatorname{vdim}\mathcal{L} = -1$). Suppose that $\mathcal{L} \neq \varnothing$ and pick some $C \in \mathcal{L}$. We have $12 = C \circ B \geqslant I \geqslant 2D = 12$, and, hence, $i(O_i) = 2d(O_i)$ for all the singular points O_i of B. According to Lemma 6.3, all the O_i's must be of the types A_p or E_6. The usual arguments with linear combinations of equations show that, if there are at least two distinct curves $C_1, C_2 \in \mathcal{L}$, then one can find another conic $C \in \mathcal{L}$ with $C \circ B \geqslant 13$, which is impossible;
(4) $D \geqslant 7$. If there is a conic $C \in \mathcal{L}$, then $C \circ B \geqslant I \geqslant 2D \geqslant 14$, which is impossible. □

From now on, we suppose that all the singular points of B are of the types A_p, $1 \leqslant p \leqslant 19$, or E_6. Suppose that in some local coordinates (x,y) about such a point O the germ of B is given by either $\{b(x,y) = x^2 + y^{3d+\varepsilon} = 0\}$, $\varepsilon = 0,1,2$ (type $A_{3d+\varepsilon-1}$), or $\{b(x,y) = x^3 + y^4 = 0\}$ (type E_6). We will say that another germ, say, h, at O satisfies the condition $\mathcal{C}_d^q$, $d = d(O)$, $0 \leqslant q \leqslant (d+1)/2$, if its Newton polygon belongs to that of $x + y^{d+q}$ (respectively, $x^{1+q} + y^2$ if O is of the type E_6). (In fact, $\mathcal{C}_d^0$ is the local condition for $H = \{h = 0\}$ to belong to $\mathcal{L}$, and $\mathcal{C}_d^q$ requires that, besides, $(H \circ B)_O \geqslant 2(d+q)$.) Let c, c_1, and c_2 be some other germs at O whose Newton polygons *coincide* with those of $x+y^d$, $x+y^{d-1}$, and $x+y$ respectively.

6.5. Lemma. *Let h satisfy $\mathcal{C}_d^q$ at O. Then certain divisibility conditions for $f = b - h^2$ imply the following estimates for the local intersection indices: (Capital letters stand for the curves given by the germs denoted by the corresponding small letters)*

if $f = c\varphi$ *then* $(\Phi \circ C)_O \geqslant d + q$,
if $f = c^2\varphi$ *then* $(\Phi \circ C)_O \geqslant \min\{d, 2q\}$,
if $f = c_1\varphi$ *then* $(\Phi \circ C_2)_O \geqslant 2$,
if $f = c_1 g_2 \varphi$ *then* $(\Phi \circ C_1)_O \geqslant d + q - 1$,
if $f = c_1^2\varphi$ *then* $(\Phi \circ C_2)_O \geqslant 1$,
if $f = c_1^2 g_2^2 \varphi$ *then* $(\Phi \circ C_1)_O \geqslant \min\{d-1, 2q\}$.

If $q = 0$, the second inequality can be improved: $(\Phi \circ G)_O \geqslant 1$.

Proof. All the displayed estimates can easily be obtained by a direct calculation using the standard resolution of the singular point O of B. To prove the improved version of the second inequality, just note that, if Φ does not pass through O, then $B = \{g^2\varphi_2 + h^2 = 0\}$ has singularity of the type A_{2d-1}, which contradicts to the assumption. $\square$

Let $C = \{c(x,y) = 0\} \in \mathcal{L}$ be the unique conic provided by Lemma 6.4. Let us prove that there is a cubic $H = \{h(x,y) = 0\}$ such that the equation of the original abundant curve B is represented as $b = h^2 + c^3 = 0$. To construct H, pick some integers q_i, one for each singular point O_i of B, such that $\sum q_i = 3$ and $(d_i - 1)/2 \leqslant q_i \leqslant (d_i + 1)/2$, where $d_i = d(O_i)$. Then, for the dimension reason, there is a cubic H satisfying $\mathcal{C}_{d_i}^{q_i}$ at each singular point. We will prove that $f = b - h^2$ is c^3 (after an appropriate choice of the coefficient in h). Consider the following cases: (We still denote curves and their equations by, respectively, capital and small letters of the same name)

Case 1: *C is irreducible.* Then H does not contain C. (Otherwise, if $H = C + L$ for some line L, then $L \circ C \geqslant 3$.) Since C is a rational curve and the restrictions $b\big|_C$ and $h^2\big|_C$ have the same roots, they are proportional, and, after multiplication of h by a constant, c divides the difference $b - h^2$.

Let $b - h^2 = c\varphi_1$. Then, according to Lemma 6.5, $\Phi_1 \circ C \geqslant \sum d_i + \sum q_i = 9$, and, hence, $\varphi_1 = c\varphi_2$. Similarly, $\Phi_2 \circ C \geqslant 5$, and, hence, $\Phi_2 = C$.

Case 2: *C splits into two lines $C_1 \neq C_2$, whose intersection point is non-singular for B.* Let $d^{(s)} = \sum d_i$ and $q^{(s)} = \sum q_i$, summation over all the singular points of B which lie in C_s. Then, clearly, $d^{(1)} = d^{(2)} = 3$, and we can choose the q_i's so that $q^{(1)} = 2$, $q^{(2)} = 1$. Now, similar to Case 1, one can prove that C_s are not components of H, and there is a splitting $b - h^2 = c_1 c_2 \varphi_1$. Then Lemma 6.5 (the first two inequalities) yields:

$$\begin{aligned}
&\Phi_1 \circ C_1 \geqslant 5 \implies \varphi_1 = c_1\varphi_2;\\
&\Phi_2 \circ C_2 \geqslant 4 \implies \varphi_2 = c_2\varphi_3;\\
&\Phi_3 \circ C_1 \geqslant 3 \implies \varphi_3 = c_1\varphi_4;\\
&\Phi_4 \circ C_2 \geqslant 2 \implies \varphi_4 = c_2.
\end{aligned}$$

Case 3: *C splits into two lines $C_1 \neq C_2$, which intersect in a singular point O of B.* Suppose that $(C_1 \circ B)_O \geqslant (C_2 \circ B)_O$. According to Bézout's theorem, O must be of the type A_p, $5 \leqslant p \leqslant 13$. (Indeed, if O is of the type A_p, $p \leqslant 4$, or E_6, then $(C \circ B)_O > 2d(O)$. If O is of the type A_p, $p \geqslant 14$, then $(C_1 \circ B)_O > 6$.) Now one starts with the splitting $b - h^2 = c_1\varphi_1$, which follows from the fact that the restrictions $b\big|_{C_1}$ and $h^2\big|_{C_1}$ are proportional, and then, using Lemma 6.5, consequently split C_2, C_1, C_2, C_1, and C_2.

Case 4: *$C = 2C_1$ is a double line.* In this case all the singular points of B are of the type A_p, $p = 1$ or $5 \leqslant p \leqslant 13$, and a direct calculation shows that $C_1 \in \mathcal{L}_4(B)$, while $\operatorname{vdim} \mathcal{L}_4(B) = -1$. Hence, $\widetilde{\Delta}_B(t)$ is divisible by (t^2+t+1), and, according to Theorem 6.2, B is reducible.

Thus, we have proved that the equation of an abundant curve B is of the form $h^2 + c^3 = 0$, and its singularities are some points of the type A_p, $2 \leqslant p \leqslant 20$, or E_6 on the conic $C = \{c = 0\}$, and, possibly, some points A_1 not on this conic. On the other hand, considering all the possibilities for the mutual location of C and $H = \{h = 0\}$, one can see that only singularities of the types A_{3d-1}, E_6, or those adjacent to J_{10} can actually occur on C.

It is well-known that a sextic B can be given by $h^2 + c^3 = 0$ if and only if it is the branch curve of a projection to $\mathbb{C}\mathrm{p}^2$ of a cubic surface $W \subset \mathbb{C}\mathrm{p}^3$ (namely, the surface $\{z^3 + 3c(x,y)z + h(x,y) = 0\}$), and the correspondence between the singularities of B and W is the following: (1) a singular point of B of the type A_p, E_6, or adjacent to J_{10} which lies on C is the projection of, respectively, a non-singular point of W, a non-degenerated double point, or a degenerated (i.e., more complex then A_1) singular point of W; (2) a singular point of B which does not lie on C is the projection of a stably equivalent singular point of W. Combined with the summary of the previous paragraph, this completes the proof. □

7. Further results and problems

In this section I would like to give an account of known results and state some problems concerning classification of irreducible curves of degree 6 up to rigid isotopy (i.e., isotopy in the class of such curves).

7.1. Proposition (see [D2]). *An irreducible curve of degree 6 with at least one non-simple singular point is determined up to rigid isotopy by its set of singularities.*

As to the curves with only simple singular points, only possible sets of singularities of such curves are known (see Urabe [U]). Existence of abundant curves shows that Proposition 7.1 does not directly extend to this case. From the main theorem it follows that any abundant curve has a singular set of the form $aA_1 + \sum b_d A_{3d-1} + cE_6$, where $\sum db_d + 2c = 6$, and the genus formula gives the estimate $a \leqslant a_{\max} = 10 - \sum b_d \lfloor 3d/2 \rfloor - 3c$. I know the following partial results about existence of abundant and non-abundant curves with such sets of singularities:

7.2. Proposition.

(1) *for every $a \leqslant 4$ there is exactly one rigid isotopy class of abundant sextics with the singular set $aA_1 + 6A_2$;*

(2) *for every $a \leqslant 3$ there is at least one non-abundant sextic with the singular set $aA_1 + 6A_2$;*

(3) *any two irreducible sextics with the singular set $4A_1 + 6A_2$ are rigidly isotopic* (*and, hence, every such curve is abundant*).

Proof. Statement (1): there exists a cubic surface in $\mathbb{C}\mathrm{p}^3$ with a singular points of the type A_1 ($a \leqslant 4$). The branch curve of a generic projection of this surface to $\mathbb{C}\mathrm{p}^2$ is a sextic in question. Since the construction is *generic,* any two such curves are rigidly isotopic. Statement (2) is proved by Zariski [Z]: the curve dual to a non-singular cubic is a sextic with 9 cusps. One can perturb three of them to nodes so that the remaining six do not lie in a conic, and then perturb any number of the nodes keeping the cusps. Statement (3) follows from the fact that all such curves are dual to quartics with three nodes. □

7.3. Proposition.

(1) *there are exactly two rigid isotopy classes* (*abundant and non-abundant*) *of irreducible sextics with the singular set $3E_6$;*

(2) *any two irreducible sextics with the singular set $A_1 + 3E_6$ are rigidly isotopic; these curves are abundant.*

Proof. Let B be a sextic with the singular set $aA_1 + 3E_6$ ($a \leqslant 1$). Apply to it the triangular transformation with centers at the points of type E_6. The result is an irreducible cubic B' with a singular points of the type A_1, and the inverse transformation is determined by three inflection points P_1, P_2, P_3 of B' — the images of the points E_6 of B. The original curve is abundant if and only if P_1, P_2, P_3 are concurrent. This is always the case if $a = 1$ (B' has only three inflection points). If $a = 0$, there are two isotopy classes of configurations $(B'; P_1, P_2, P_3)$, which correspond to abundant and non-abundant sextics. (The easiest way to see this is to consider the group low on B'. Details are left to the reader.) □

7.4. Proposition. *Any two abundant sextics without nodes (i.e., with the singular set $\sum b_d A_{3d-1} + cE_6$) are rigidly isotopic.*

Proof. The statement is almost obvious. An abundant curve B is given by $h^2 + c^3 = 0$, and the singularities of B along $C = \{c = 0\}$ and $H = \{h = 0\}$ are determined by the isotopy type of the pair (C, H). This pair being fixed, a generic curve given by the above equation does not have any extra singular point. □

Propositions 7.1–7.4 put a basis for the following conjectures:

7.5. Conjecture. *Consider irreducible sextics with a fixed singular set of the form $aA_1 + \sum b_d A_{3d-1} + cE_6$, $\sum db_d + 2c = 6$, and put $a_{\max} = 10 - \sum b_d \lfloor 3d/2 \rfloor - 3c$. Then:*

(1) *any two sextics with $a = \alpha_{\max}$ are isotopic to each other and are abundant;*
(2) *if $a < a_{\max}$, then there are exactly two isotopy classes, one abundant and one non-abundant.*

7.6. Problem. *Are there any other (rigid) isotopy invariants (in addition to the set of singularities and Alexander polynomial) of irreducible sextics?*

References

[AVG] V. I. Arnold, A. N. Varchenko, and S. M. Gusein-Zade, *Singularities of Differentiable Maps.* I, Nauka, Moscow, 1982; II, Nauka, Moscow, 1984. (Russian)

[D1] A. Degtyarev, *Topology of complex plane projective algebraic curves*, Ph.D. thesis, Leningrad State University, 1987. (Russian)

[D2] ———, *Classification of surfaces of degree four having a non-simple singular point*, Izv. Akad. Nauk SSSR, Ser. mat. **53** (1989), no. 6, 1269–1290 (Russian); English transl. in Math. USSR-Izv. **35** (1990), no. 3, 607–627.

[D3] ———, *Isotopy classification of complex plane projective curves of degree 5*, Algebra i Analis **1** (1989), no. 4, 78–101 (Russian); English transl. in Leningrad Math. J.

[Di] A. Dimca, *Singularities and Topology of Hypersurfaces*, Universitex, Springer, 1992.

[E] H. Esnault, *Fibre de Milnor d'un cône sur une courbe plane singulière*, Invent. Math. **68** (1982), 477–496. (French)

[H1] F. Hirzebruch, *Über vierdimensionale Riemannsche Flächen mehrdeufiger analitischer Functionen von zwei komplexen Veränderlichen*, Math. Ann. **126** (1953), no. 1, 1–22. (German)

[H2] ———, *The signature of ramified coverings*, Global Analysis. Papers in honor of K. Kodaira, 1969, pp. 253–265.

[K] J. Kóllar, *Higher direct images of dualizing sheaves*, Ann. Math. **123** (1986), no. 1, 11–42.

[L1] A. Libgober, *Alexander polynomial of plane algebraic curves and cyclic multiple planes*, Duke Math. J. **49** (1982), no. 4, 833–851.

[L2] ———, *Alexander modules of plane algebraic curves*, Contemporary Math. **20** (1983), 231–247.

[L3] ———, *Alexander invariants of plane algebraic curves*, Proc. Symp in Pure Math., Vol XXXX, part 2, 1983, pp. 135–143.

[L4] ———, *Homotopy groups of the complement to singular hypersurface*, Bul. AMS **13** (1985), no. 1, 49–51.

[Li] J. Lipman, *Introduction to resolution of singularities*, Proc. Symp. in Pure Math., Vol. XXIX, Alg. Geom., Arcata, 1974.

[Lo] F. Loser and M. Vaquie, *Le polynome d'Alexander d'une courbe plane projective*, Topology **29** (1990), 163–173.

[U] T. Urabe, *Elementary transformations of Dynkin graphs and singularities of quartic surfaces*, Invent. Math. **87** (1987), no. 3, 549–572.

[V] A. Varchenko, *Asymptotic Hodge structure in vanishing cohomology*, Izv. AN SSSR, Ser. mat. **45** (1981), 540–591. (Russian)

[Z] O. Zariski, *On the irregularity of cyclic multiple planes*, Ann. Math. **32** (1931), 485–511.

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