

PERIODIC SHARPNESS OF MIYAOKA'S BOUND FOR SMOOTH RATIONAL CURVES

ALEX DEGTYAREV AND SŁAWOMIR RAMS

ABSTRACT. We determine the maximal number of smooth rational degree d curves on a complex $K3$ -surface of degree $2n$ provided n is sufficiently large as compared to $d > 1$ (Theorem 1.3). We obtain precise characterization of configurations of rational degree d curves for which Miyaoka's bound is sharp for nd odd and $n \gg d$ (Corollary 1.5).

1. INTRODUCTION

Configurations of lines and conics on various classes of surfaces have been studied for over 170 years, with numerous partial results spread over the literature. Within the last decade our understanding of line configurations on polarized $K3$ -surfaces over various fields became almost complete (cf. [3, 6, 10, 12, 19, 18] and further references therein) and much has been shown for conics (see [4, 7, 5, 8, 9, 15, 17]).

Quite unexpectedly, one can use the Miyaoka–Yau–Sakai inequality [14] to give a very concise description of the behaviour of the maximal number of degree d (irreducible, but not necessarily smooth) rational curves on high-degree complex $K3$ -surfaces. Indeed, for fixed integers $d \geq 1$ and $n \geq 2$ let us define

$$\mathfrak{r}(2n, d) := \max\{r_d(X) \mid X \text{ is a degree } 2n \text{ complex } K3\text{-surface}\},$$

where $r_d(X)$ stands for the number of degree d (irreducible, but not necessarily smooth) rational curves on the surface X . Then, [15, Proposition A]) implies that

$$(1.1) \quad \mathfrak{r}(2n, d) \leq 24 \quad \text{for } n > 50d^2.$$

If we assume n sufficiently large as compared to d , these bounds are sharp for $d \geq 3$ (see [20, Theorem 1.1]) and $d = 2$ (see [17, Corollary 1.2]), whereas [3, Theorem 1.5] yields

$$(1.2) \quad \mathfrak{r}(2n, 1) \in \{21, 22, 24\} \quad \text{for } n \gg 0.$$

(Precise arithmetic conditions on the degree $2n$ of the polarization for Miyaoka's bound of 24 lines to be attained can be found in [3, Theorem 1.5].) Consequently, (1.1) fails to be sharp for infinitely many n only for lines.

Until recently, far less has been known about maximal configurations of *smooth* rational curves of small degrees (even for $d = 3$, i.e., for twisted cubics). The classical construction of $K3$ -surfaces with many degree d rational curves in [20, Theorem 1.1] always results in configurations of *irreducible* rational curves with

2010 *Mathematics Subject Classification*. Primary: 14J28; Secondary: 14N25.

Key words and phrases. $K3$ -surface, lattice, genus one fibration .

A.D. was partially supported by the TÜBİTAK grant 123F111.

S.R. was supported by the National Science Centre, Poland, Opus grant no. 2024/53/B/ST1/01413 and partially supported by TÜBİTAK BİDEB 2221 Visiting Scientist Fellowship Program.

singularities. In contrast, the recent construction from [17] yields configurations of smooth curves, but it works only when either n or d is even (and in the extra case $2n = 2d^2 \bmod 6$ - see [17, Proposition 6.1]).

In the present paper we complete the picture of the asymptotic behaviour of the maximal number of *smooth* degree d rational curves on $K3$ -surfaces of high degree. More precisely, we prove the following analogue of [3, Theorem 1.5]. In the statement, we need a very strange arithmetic condition; for a prime q and integer m , we denote by $\nu_q(m) := \max\{r \in \mathbb{N} \mid q^r \mid m\}$ the q -adic valuation of m . As usual, this extends to $\mathbb{Q}$ via $\nu_q(m/n) := \nu_q(m) - \nu_q(n)$.

Theorem 1.3 (see §5.1). *Let $d > 1$ be a fixed integer. If $n \gg d$ (i.e. $n \gg 0$ as compared to d) then the maximal number of smooth rational degree d curves on a complex $K3$ -surface of degree $2n$ equals*

- 24, if either $2nd = 0 \bmod 4$ or $2n = 2d^2 \bmod 3$;
- otherwise, 22 if $2n = 4d^2 \bmod 5$ and, for each prime $q = 3 \bmod 4$, either $\nu_q(n) \geq \nu_q(d)$ or $\nu_q(2n)$ is even;
- otherwise, 21.

Remark 1.4. The periodicity of the maximal number of smooth degree d rational curves depends on d . For example, if d is square free, the period is $30 = 2 \cdot 3 \cdot 5$. In general, it is

$$30 \prod q^{2\lfloor \nu_q(d)/2 \rfloor}, \quad q = 3 \bmod 4.$$

If $9 \mid d$, the coefficient 30 can be reduced down to 10.

As a consequence, we obtain the classification of the maximal configurations of irreducible rational degree d curves on $K3$ -surfaces of high degree $2n$ with both n and d odd, which in this case provides a complete answer to the question posed in [15, § 3] (see Remark 5.3 below for the discussion of the case of nd even).

Corollary 1.5 (see §5.2). *Let $n \gg d$, with nd odd, and let a complex $K3$ -surface of degree $2n$ contain 24 irreducible rational degree d curves. If $n \not\equiv d^2 \bmod 3$, then the curves in question form the configuration $24\tilde{\mathbf{A}}_0^*$. Otherwise, their configuration is either $8\tilde{\mathbf{A}}_2$ (and then the curves are smooth) or $24\tilde{\mathbf{A}}_0^*$.*

Observe also that a complex $K3$ -surface of degree $2n$ with the configurations given above exists for any pair of integers (n, d) that satisfy the assumptions of Corollary 1.5 (see [20, § 10] and Proposition 4.14 below).

The arguments used in [3] (and their generalizations in [20]) suggest that the maximal number of smooth rational degree d curves on a complex $K3$ -surface of degree $2n$ should be a periodic function of n , with the period proportional to the degree d of the curves in question (see Proposition 2.13 below), so its precise description for all values of d may be out of reach. In contrast to this expectation, Theorem 1.3 shows that the behaviour of the maximal configurations for most odd values of d is very much similar to the case $d = 1$ of lines.

At early stages of this project our goal was understanding the behaviour of the maximal configurations realizing Miyaoka's bound [15, Proposition A] for small odd integers d . There are four major reasons why one is able to prove an analogue of (1.2) for *every* odd d . Firstly, by [3] (see also [20, Corollary 6.4]), for n large, we can restrict our attention to the configurations of rational curves that correspond to the so-called parabolic graphs. Secondly, for each parabolic graph of interest we

are able to find a basis of the corresponding lattice (sometimes upon tensoring the latter by $\mathbb{Z}_p$) that enables us to control its discriminant forms for all pairs (n, d) . Thirdly, the fact that all exceptional classes in the discriminant group must be orthogonal to the classes of *both* the polarization and the fiber (see [Lemma 2.14](#)) lets us exclude most finite index extensions of the lattices associated to the graphs considered (cf. [Example 2.20](#)). Last but not least, the assumption that d and n are odd substantially simplifies the structure of the 2-torsion of the discriminant.

The paper is organized as follows. In [§2](#) we recall basic facts used and introduce the notation. We illustrate this by a very detailed study of the lattice associated to the parabolic graph $\Gamma := 12\tilde{\mathbf{A}}_1$ (see [Example 2.20](#)). Then, [§2.5](#) through [§4.5](#) are devoted to finding the degrees in which various parabolic graphs are geometric. Finally, in [§5](#) we give the proof of [Theorem 1.3](#).

Remark 1.6. One can ask whether [Theorem 1.3](#) holds for $K3$ -surfaces defined over any algebraically closed field of characteristic $p \geq 5$ (cf. [\[20, § 11\]](#)). We presume that some extra cases may appear for small $p > 0$ but, since our main inspiration was Miyaoka's paper [\[15\]](#) that deals with *complex* surfaces, we confine ourselves to the complex case.

Although GAP [\[13\]](#) was a useful tool helping us guess the statements and verify our findings, we managed to present the ultimate proofs in a human readable form, without a reference to software.

1.1. Acknowledgements. This paper was conceived during Alex Degtyarev's visit to the Jagiellonian University (Cracow, Poland). A.D. is profoundly grateful to this institution for the hospitality and excellent working conditions. Substantial part of the paper was written during Sławomir Rams' stay at the Department of Mathematics of Bilkent University (Ankara, Turkey) within the scope of TÜBİTAK "2221-Fellowships for visiting scientists" programme. S.R. would like to thank Bilkent University and TÜBİTAK for creating perfect working conditions and numerous inspiring discussions with members of the Department of Mathematics of Bilkent University. Both authors are would like to thank Matthias Schütt for a number of inspiring discussions.

2. PRELIMINARIES

In this note we work over the field $\mathbb{C}$ of complex numbers. Whenever we speak of a $K3$ -surface of degree $2n$, or a $(2n)$ -polarized $K3$ -surface, we mean a pair (X, h) , where X is a projective simply connected complex surface with trivial canonical bundle and $h \in \text{NS}(X)$ is *very ample* with $h^2 = 2n$. Sometimes, for the sake of brevity, we write "a $K3$ -surface" when we mean "a *polarized* $K3$ -surface".

We freely extend modular arithmetic to fractions with the denominator prime to the modulus, so that, e.g., $1/2 = 2 \bmod 3$ and $1/2 = 3 \bmod 5$, as in GAP [\[13\]](#).

2.1. Fano graph of an admissible lattice. All lattices considered in this note are even. We maintain the notation consistent with [\[3\]](#):

- $\mathbf{A}_p$, $p \geq 1$, $\mathbf{D}_q$, $q \geq 4$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$ are the *negative definite* root lattices generated by the indecomposable root systems of the same name (see [\[1\]](#));
- $\mathbf{U} := [0, 1, 0]$ is the unimodular even lattice of rank 2;

- $L(n)$ denotes the lattice obtained by the scaling of a given lattice L by a fixed integer $n \in \mathbb{Z}$ (occasionally, we use $n \in \mathbb{Q}$); this is opposed to nL that stands for the orthogonal direct sum of n copies of the lattice L ;
- $\mathbf{L} := 2\mathbf{E}_8 \oplus 3\mathbf{U}$ stands for the *K3-lattice*;
- for a lattice L , $L^\vee := \text{Hom}(L, \mathbb{Z})$ denotes the dual lattice;
- if L is non-degenerate, $\text{discr } L := L^\vee/L$ is the *discriminant group*, see [16]; it inherits from $L^\vee$ a $\mathbb{Q}/2\mathbb{Z}$ -valued quadratic form q (see §2.2 below);
- the inertia indices of the quadratic form $L \otimes \mathbb{R}$ are denoted by $\sigma_{\pm,0}(L)$;
- a lattice S with $\sigma_+ S = 1$ is called *hyperbolic*.

Furthermore, we follow the standard lattice-theoretic notation as in, e.g., [2, 16].

Recall that a $2n$ -polarized lattice $S \ni h$ is defined as a non-degenerate hyperbolic lattice S equipped with a distinguished vector h of positive square $h^2 = 2n$, called the *degree*. To simplify our exposition (cf. [11, Lemma 2.27.3]), we assume that

$$n > 4.$$

For a polarized lattice $S \ni h$ and $d \in \mathbb{N}$, we define

$$\text{root}_d(S, h) := \{r \in S \mid r^2 = -2, r \cdot h = d\}.$$

Definition 2.1 (cf. [11, § 2]). A polarized lattice $S \ni h$ is called *admissible* if none of the vectors $e \in S$ satisfies either of the following conditions

- (1) $e^2 = -2$ and $e \cdot h = 0$ (*exceptional divisor*);
- (2) $e^2 = 0$ and $e \cdot h = 2$ (*2-isotropic vector*).

An admissible lattice is *geometric* if it admits a primitive isometry to $\mathbf{L}$.

More restrictively, given an admissible (resp. geometric) lattice $S \ni h$ and a subset $\Gamma \subset \text{root}_d(S, h)$, we say that S is Γ -*admissible* (resp. Γ -*geometric*) if, for each $1 \leq r < d$, there is no vector $e \in \text{root}_r(S, h)$ such that

- (3) $e \cdot v < 0$ for some $v \in \Gamma$ (Γ -*inadmissible root*).

For an admissible lattice $S \ni h$ (so that, in particular, $\text{root}_0(S, h) = \emptyset$), we put $\{\Delta^\sharp\}_1 := \text{root}_1(S, h)$ and, for $d > 1$, we define recursively (cf. [11, § 2.2])

$$\{\Delta^\sharp\}_d := \{r \in \text{root}_d(S, h) \mid r \cdot e \geq 0 \text{ for all } e \in \{\Delta^\sharp\}_r, 1 \leq r < d\}.$$

Definition 2.2. We define the (plain) *degree d Fano graph* of an admissible lattice $S \ni h$ as the set of vertices

$$(2.3) \quad \text{Fn}^d(S, h) := \{\Delta^\sharp\}_d,$$

with two vertices $l_1 \neq l_2$ connected by an edge of multiplicity $l_1 \cdot l_2$.

For example, the degree 1 Fano graph $\text{Fn}^1(S, h)$ is $\text{Fn}_\emptyset(S, h) = \text{root}_1(S, h)$ in the sense of [11, (2.6)].

Remark 2.4. It follows immediately from the general theory of groups generated by reflections that any root $e \in \text{root}_d(S, h)$, $d > 0$, is a *positive* linear combination

$$e = \sum_i n_i e_i, \quad n_i \in \mathbb{N}, \quad e_i \in \bigcup_{r=0}^d \text{Fn}^r(S, h).$$

Geometrically (cf. the discussion below), this means that any (-2) -curve of positive degree splits into a union of smooth rational components (clearly, of smaller degrees unless the original curve is smooth). It follows that

$$\text{Fn}_d(S, h) = \{r \in \text{root}_d(S, h) \mid r \cdot e \geq 0 \text{ for all } e \in \text{root}_r(S, h), 1 \leq r < d\},$$

i.e., there is no need to compute the intermediate sets $\text{Fn}_r(S, h)$. Furthermore, the Γ -admissibility condition in [Definition 2.1](#) merely means that $\Gamma \subset \text{Fn}_d(S, h)$.

2.2. Primitive isometries into $\mathbf{L}$ (see [\[16\]](#)). Consider a non-degenerate even lattice N and extend the bilinear form to a $\mathbb{Q}$ -valued form on $N \otimes \mathbb{Q}$. The dual group $N^\vee$ can be identified with the subgroup

$$N^\vee = \{x \in N \otimes \mathbb{Q} \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in N\} \supset N,$$

and the *discriminant group* (denoted q_N in [\[16\]](#)) is defined as the finite quotient

$$\text{discr } N := N^\vee / N.$$

This group is equipped with a $(\mathbb{Q}/\mathbb{Z})$ -valued symmetric bilinear form and its $(\mathbb{Q}/2\mathbb{Z})$ -valued quadratic extension:

$$(x \bmod N) \otimes (y \bmod N) \mapsto x \cdot y \bmod \mathbb{Z}, \quad x \bmod N \mapsto x^2 \bmod 2\mathbb{Z}.$$

We often abbreviate

$$\mathcal{N} := \text{discr } N \quad \text{and} \quad \mathcal{N}_p := \text{discr}_p N := \mathcal{N} \otimes \mathbb{Z}_p \text{ for a prime } p$$

and use the shorthand

$$[\xi] := \xi \bmod N \in \mathcal{N}$$

for the discriminant class of a vector $\xi \in N^\vee$. We also use the notation $\langle x \rangle \subset \mathcal{N}$ for the subgroup generated by $x \in \mathcal{N}$, and we abbreviate $\langle \xi \rangle := \langle [\xi] \rangle$ for $\xi \in N^\vee$.

Obviously, $\mathcal{N} = \bigoplus_p \mathcal{N}_p$ over all primes p . The discriminant form on $\mathcal{N}$ is non-degenerate in the sense that the map

$$\mathcal{N} \rightarrow \text{Hom}(\mathcal{N}, \mathbb{Q}/\mathbb{Z}), \quad x \mapsto (y \mapsto x \cdot y \bmod \mathbb{Z}),$$

is an isomorphism. Besides, one has $|\text{discr } N| = |\det N|$.

Remark 2.5. To find the group $\text{discr}_p N$, we indicate a subset therein, whereupon it suffices to compute the Gram matrix G and check that $\nu_p(\det G \cdot \det N) = 0$ (cf. [Example 2.20](#) below): the non-degeneracy would imply that the set is a basis.

For $p = 2$, we distinguish between *even* (*i.e.*, such that $x^2 = 0 \bmod \mathbb{Z}$ for each element $x \in \mathcal{N}_2$ of order 2) and *odd* (otherwise) forms. In other words, the form $\mathcal{N}_2$ is odd if and only if it has an element of square $\pm \frac{1}{2} \bmod 2\mathbb{Z}$ (necessarily of order 2 and necessarily an orthogonal direct summand).

The *length* $\ell(\mathcal{N})$ of a finite abelian group $\mathcal{N}$ is the minimal number of generators. We usually describe a discriminant form by its *Gram matrix* $[a_{ij}]$ in a minimal basis $x_1, x_2, \dots$: the diagonal entries $a_{ii} := x_{ii}^2$ are defined modulo $2\mathbb{Z}$, whereas all other entries $a_{ij} := x_i \cdot x_j$, $i \neq j$, are only defined modulo $\mathbb{Z}$.

If $p > 2$ or $p = 2$ and $\mathcal{N}_2$ is even, one can speak about the *determinant* $\det \mathcal{N}_p$:

$$|\mathcal{N}_p| \det \mathcal{N}_p \in \mathbb{Z}_p^\times, \quad \text{well defined modulo } (\mathbb{Z}_p^\times)^2.$$

In [\[16\]](#), this is defined as the determinant (divided by $|\mathcal{N}_p|^2$, which is irrelevant modulo squares) of the only $\mathbb{Z}_p$ -lattice K_p such that

$$\text{rk } K_p = \ell(\mathcal{N}_p), \quad \text{discr } K_p \simeq \mathcal{N}_p.$$

(Such a lattice exists and, unless $p = 2$ and $\mathcal{N}_2$ is odd, is unique up to isomorphism.) Alternatively, this is just the determinant of the Gram matrix in a carefully chosen minimal basis for $\mathcal{N}_2$: *e.g.*, it suffices to require that any two basis elements of distinct orders must be orthogonal. Indeed, such a basis immediately gives us K_p : we merely change all p^{-k} to p^k , $k \in \mathbb{N}$, in the matrix.

If $\mathcal{N}_2$ is odd, then $\det \mathcal{N}_2$ is only defined modulo 5 in $\mathbb{Z}_2^\times$, making it redundant in Theorem 2.6 below.

The determinant is multiplicative, $\det(\mathcal{N} \oplus \mathcal{M}) = (\det \mathcal{N})(\det \mathcal{M})$. Each non-degenerate finite quadratic form splits into orthogonal direct sum of the following elementary blocks: for a prime $p > 2$,

$$\mathcal{W} = \left[\frac{2m}{p^k} \right], \quad \text{g.c.d.}(m, p) = 1 \text{ and only } p^k \det \mathcal{W} = \left(\frac{2m}{p} \right) \in \{\pm 1\} \text{ matters,}$$

and for $p = 2$,

$$\mathcal{W} = \left[\frac{m}{2^k} \right], \quad m \text{ is odd and only } 2^k \det \mathcal{W} = m \bmod 8 \in \{\pm 1, \pm 5\} \text{ matters,}$$

or

$$\mathcal{U}_{2^k} := \begin{bmatrix} 0 & 1/2^k \\ 1/2^k & 0 \end{bmatrix}, \quad \mathcal{V}_{2^k} := \begin{bmatrix} 1/2^{k-1} & 1/2^k \\ 1/2^k & 1/2^{k-1} \end{bmatrix};$$

we have

$$2^{2k} \det \mathcal{N} = \begin{cases} -1 \bmod 8, & \text{if } \mathcal{N} = \mathcal{U}_{2^k}, \\ 3 \bmod 8, & \text{if } \mathcal{N} = \mathcal{V}_{2^k}. \end{cases}$$

Theorem 2.6. (see [16, Theorem 1.12.2]) *A non-degenerate hyperbolic lattice N admits a primitive isometry into the K3-lattice $\mathbf{L} := 2\mathbf{E}_8 \oplus 3\mathbf{U}$ if and only if the following conditions are satisfied simultaneously:*

- (1) *one has $r := 22 - \text{rk } N \geq 2$;*
- (2) *for each prime $p > 2$, either*
 - $\ell(\mathcal{N}_p) \leq r - 1$, *or*
 - $\ell(\mathcal{N}_p) = r$ *and* $|\mathcal{N}| \det \mathcal{N}_p = 1 \bmod (\mathbb{Z}_p^\times)^2$;
- (3) *for $p = 2$, either*
 - $\ell(\mathcal{N}_2) \leq r - 2$, *or*
 - $\ell(\mathcal{N}_2) = r$ *and the form $\mathcal{N}_2$ is odd, or*
 - $\ell(\mathcal{N}_2) = r$, *the form $\mathcal{N}_2$ is even, and* $|\mathcal{N}| \det \mathcal{N}_2 = \pm 1 \bmod 8$. $\triangleleft$

2.3. Lattice associated to a graph. Let Γ be a loop-free (multi-)graph. Then, as in [3], we put $\mathbb{Z}\Gamma$ to denote the lattice freely generated by the vertices of Γ , with the bilinear form defined by $v^2 = -2$ and $u \cdot v = r$ for $u, v \in \Gamma$ connected by an r -fold edge (no edge meaning a 0-fold edge).

Recall that Γ is called *elliptic* (resp. *parabolic*, resp. *hyperbolic*) iff $\mathbb{Z}\Gamma$ is negative definite (resp. $\sigma_+(\mathbb{Z}\Gamma) = 0$ and $\sigma_0(\mathbb{Z}\Gamma) > 0$, resp. $\sigma_+(\mathbb{Z}\Gamma) = 1$).

Given a pair of integers $n, d > 0$, we mimic [3, § 2.4] and associate to the graph Γ the $2n$ -polarized lattice

$$(2.7) \quad \mathfrak{F}_{2n}^d(\Gamma) := (\mathbb{Z}\Gamma + \mathbb{Z}h) / \ker, \quad h^2 = 2n, \quad h \cdot v = d \text{ for } v \in \Gamma,$$

where $\ker := \ker(\mathbb{Z}\Gamma + \mathbb{Z}h) = (\mathbb{Z}\Gamma + \mathbb{Z}h)^\perp$ is the radical of the bilinear form.

Remark 2.8. More generally, one can consider a colored graph with the coloring function $\deg: \Gamma \rightarrow \{0, \dots, d\}$ and let $h \cdot v = \deg(v)$ in (2.7). Most definitions below extend to this case in an obvious way. In this paper, most of the time we assume that $\deg(v) = d$ is a constant fixed in advance.

By [16, Proposition 1.4.1], there is a bijection between the (isomorphism classes of) even finite index extensions $N' \supset N$ and isotropic subgroups (*a.k.a. kernels*) $\mathcal{K} \subset \mathcal{N}$: one has

$$\mathcal{K} := N'/N \subset \text{discr } N, \quad \text{discr } N' = \mathcal{K}^\perp / \mathcal{K}.$$

Thus, one can pick a kernel $\mathcal{K} \subset \text{discr } \mathfrak{F}_{2n}^d(\Gamma)$ and consider the extension $\mathfrak{F}_{2n}^d(\Gamma, \mathcal{K})$ of $\mathfrak{F}_{2n}^d(\Gamma)$ by $\mathcal{K}$. The kernel $\mathcal{K}$ is called Γ -*admissible* if so is $\mathfrak{F}_{2n}^d(\Gamma, \mathcal{K})$.

Definition 2.9. We call a graph Γ *admissible* in degree $2n$ (with respect to d) if the lattice $\mathfrak{F}_{2n}^d(\Gamma)$ is Γ -admissible. An admissible graph Γ is called *subgeometric* if there is a Γ -geometric finite index extension $\mathfrak{F}_{2n}^d(\Gamma, \mathcal{K})$; the kernel $\mathcal{K}$ of such an extension is called a *geometric kernel* for Γ . If such a geometric kernel $\mathcal{K}$ can be chosen so that $\Gamma = \text{Fn}_d \mathfrak{F}_{2n}^d(\Gamma, \mathcal{K})$, then the graph Γ is called *geometric*.

Occasionally, we speak about a *partial* or, more precisely, $(\text{mod } p)$ -*geometric kernel* $\mathcal{K}_p \subset \text{discr}_p \mathfrak{F}_{2n}^d(\Gamma)$, meaning that $\mathcal{K}_p$ is Γ -admissible and the p -torsion part $(\mathcal{K}_p^\perp / \mathcal{K}_p)_p$ satisfies respective condition (2) or (3) in Theorem 2.6.

Let Σ be a connected parabolic graph (*a.k.a.* affine Dynkin diagram). The radical $\ker \Sigma$ has a unique minimal positive generator

$$(2.10) \quad k_\Sigma = \sum_{v \in \Sigma} n_v v, \quad n_v \in \mathbb{Z}_+,$$

and the *degree* $\deg(\Sigma)$ is defined as the coefficient sum $\sum n_v$. We use the notation $\Sigma^+ \subset \Sigma$ for an associated (elliptic) Dynkin diagram obtained from Σ by removing (any) one vertex $v \in \Sigma$ with $n_v = 1$ in (2.10). (This operation is not quite a functor as a vertex v that can be removed is typically not unique.) In general, for a parabolic graph Γ , we denote by Γ^+ the graph obtained by replacing each parabolic component Σ with Σ^+ (and keeping all elliptic components intact).

If Γ is a parabolic graph, one has $\sigma_+(\mathbb{Z}\Gamma + \mathbb{Z}h) = 1$ if and only if all parabolic components of Γ have the same degree; this common value $\deg(\Gamma)$ is called the *degree* of Γ . In this case, we can define

$$(2.11) \quad k := k_\Sigma \in \mathfrak{F}_{2n}^d(\Gamma), \quad \varkappa := \frac{1}{\deg(\Sigma)} k_\Sigma \in \mathfrak{F}_{2n}^d(\Gamma)^\vee,$$

where $\Sigma \subset \Gamma$ is any parabolic component: all images in $\mathfrak{F}_{2n}^d(\Gamma)^\vee$ are equal.

Lemma 2.12. *Let Γ be a parabolic graph. Then any kernel $\langle qk \rangle \subset \text{discr } \mathfrak{F}_{2n}^d(\Gamma)$, $q \in \mathbb{Q} \setminus \mathbb{Z}$, contains an exceptional divisor or a Γ -inadmissible root.*

Proof. Pick a parabolic component $\Sigma \subset \Gamma$ and let $f := \deg(\Sigma) = \deg(\Gamma)$. We recall the following less known property of the elliptic Dynkin diagram Σ^+ : the set of values taken by the coefficient sum $\sum n_e$ over all positive roots $r = \sum n_v v$, $v \in \Sigma^+$, $n_v \in \mathbb{N}$, is precisely the set $\{1, \dots, f-1\}$.

We can assume that $0 < q \leq 1/2$ (and, clearly, $fdq \in \mathbb{Z}$). Then $0 < fq \leq f-1$ and, letting $m := \lceil fq \rceil$, we can find a positive root $r \in \mathbb{Z}\Sigma^+$ with the coefficient sum m . As another property of Dynkin diagrams, $r \cdot v < 0$ for some $v \in \Sigma^+$. Thus, $\llbracket -qk \rrbracket$ contains the root $e := -qk + r$ of degree $0 \leq e \cdot h = (m - qf)d < d$ and such that $e \cdot v < 0$ for some $v \in \Gamma$. $\square$

The following statement would explain the $d \deg(\Gamma)$ -periodicity of the degree d curve counts, but it is not strong enough; therefore, we omit the proof.

Proposition 2.13 (cf. [3, Proposition 3.7]). *Let Γ be a parabolic graph, and let $f := \deg(\Gamma)$. If $2n', 2n'' \geq (df)^2 + df$ and $2n' = 2n'' \pmod{df}$, then Γ is geometric in degree $2n'$ if and only if so it is in degree $2n''$.*

Lemma 2.14. *Let Γ be a parabolic graph and $d > 0$ a fixed integer. Then, for any integer $n \gg d$, any vector $e \in \mathfrak{F}_{2n}^d(\Gamma)^\vee$ that is either*

- (1) an exceptional divisor, as in [Definition 2.1\(1\)](#), or
- (2) a 2-isotropic vector, as in [Definition 2.1\(2\)](#), or
- (3) more generally, an r -root, i.e., $e^2 = -2$ and $r := e \cdot h \leq d$

lies in $k^\perp$. In case (2), this implies that $e = 2\kappa/d$ is a multiple of k .

Conversely, any root $e \in k^\perp \subset \mathfrak{F}_{2n}^d(\Gamma)^\vee$ such that $0 \leq r := e \cdot h < d$ generates a Γ -inadmissible finite index extension of $\mathfrak{F}_{2n}^d(\Gamma)$.

Essentially, in items (1) and (3) we assert that any smooth rational curve of low degree is a fiber component of the elliptic pencil in question, which is well known. Though, the presence of such a curve most likely means that the pencil is not what we think it is, cf. [Example 4.22](#) below.

Proof. We compute the Gram matrix G of h, k, e and assert that $\det G \geq 0$, since the lattice is hyperbolic. In cases (1) (letting $r = 0$) and (3) this results in

$$2x(df r - nx) + 2(df)^2 \geq 0,$$

where $x := e \cdot k \in \mathbb{Z}$; in case (2), we obtain

$$2x(2df - nx) \geq 0.$$

In all cases, since $x \in \mathbb{Z}$, we conclude that $x = 0$ for $n \gg 0$.

For the converse statement, consider the root lattice

$$\mathbb{Z}\Gamma^+ = \mathbb{Z}\Gamma / \ker;$$

in this lattice, the vertices of Γ^+ constitute a standard Dynkin basis (*a.k.a.* Weyl chamber). By [Lemma 2.12](#), we can assume that $e \notin \llbracket qk \rrbracket$, $q \in \mathbb{Q}$; hence, the root $\bar{e} := e \bmod (\ker \mathbb{Z}\Gamma) \in (\mathbb{Z}\Gamma^+)^\vee$ represents a non-trivial discriminant class. The root lattice

$$(\mathbb{Z}\Gamma^+ + \mathbb{Z}\bar{e}) / \ker \supset \mathbb{Z}\Gamma^+$$

is a proper finite index extension; hence, it has strictly smaller Weyl chambers, so it cannot be Γ -admissible. $\square$

Remark 2.15. Let Γ be a parabolic graph and $n \gg 0$. [Lemma 2.14](#) implies that any exceptional divisor ($r = 0$) or degree r root $e \in \mathfrak{F}_{2n}^d(\Gamma)^\vee$ is of the form

$$e = \sum_{v \in \Gamma} q_v v, \quad \text{where } q_v \in \mathbb{Q}, \quad \sum_{v \in \Gamma} q_v = \frac{r}{d}, \quad e^2 = -2, \quad e \cdot u \in \mathbb{Z} \text{ for all } u \in \Gamma.$$

De facto, we are working with the discriminant groups of root systems, which are very well understood. In particular, we can speak about shortest representatives of the discriminant classes. It is this fact and the periodicity of the relevant part of the discriminant (rather than that of the lattice itself) that explains period $\deg(\Sigma)$ (rather than $d \deg(\Sigma)$ given by [Proposition 2.13](#)) in most statements.

Definition 2.16. With [Lemma 2.14](#) in mind, an isotropic class $\alpha \in \text{discr } \mathfrak{F}_{2n}^d(\Gamma)$ is called *exceptional* if it is represented by a root $e \in \mathfrak{F}_{2n}^d(\Gamma)^\vee$ orthogonal to k and such that $0 \leq e \cdot h < d$. According to [Remark 2.15](#), such classes are independent of n . An isotropic subgroup $\mathcal{K} \subset \text{discr } \mathfrak{F}_{2n}^d(\Gamma)$ is called *exceptional* if it contains an exceptional class. This terminology extends, in the obvious way, to the $(\text{Aut } \Gamma)$ -orbits of isotropic classes/subgroups. We can also speak about *minimal* exceptional subgroups and *minimal* exceptional classes as those generating such subgroups.

2.4. Degree d smooth rational curves on $K3$ -surfaces. A *degree $2n$ polarized $K3$ -surface* is a pair (X, h) , where X is a $K3$ -surface and $h \in \text{NS}(X)$ is a very ample divisor, $h^2 = 2n$; cf. [3, 20, 17]. Usually, the polarization h is considered part of the structure and omitted from the notation.

We imitate [3, 20] (cf. also [11, 21]) and define the Fano graph of smooth rational degree d curves as the set

$$(2.17) \quad \text{Fn}^d(X, h) := \{\text{smooth rational curves } C \subset X \text{ with } C \cdot h = d\},$$

where two vertices C, C' are connected by an edge of multiplicity $C \cdot C'$. We focus on curves of a fixed degree d ; hence, contrary to [11, 20], we treat $\text{Fn}^d(X, h)$ as a plain graph; in other words, we assign a fixed color d to all vertices.

We say that the Néron–Severi lattice $\text{NS}(X)$ is *spanned by degree d curves* if it is a finite index extension of its sublattice generated by the classes of *smooth rational* degree d curves on X and the polarization h .

Torelli theorem for $K3$ -surfaces and the theory of lattice-polarized $K3$ -surfaces immediately yield the following version of [11, Theorem 3.9]

Theorem 2.18. *Let $d > 0$ and $n > 4$ be fixed integers. A graph Γ is geometric in degree $2n$ with respect to d if and only if $\Gamma \simeq \text{Fn}^d(X, h)$ for a degree $2n$ polarized $K3$ -surface (X, h) such that $\text{NS}(X)$ is spanned by degree d curves.* $\triangleleft$

Remark 2.19. If $d > 1$, the realizability criterion for a graph of degree d curves *must* appeal to curves of lower degree. For example, famous Schur’s quartic [22] has (obviously admissible) Néron–Severi lattice spanned by conics, but most of these conics are reducible, see [9]. In the parabolic context, see Example 4.22 below.

In particular, to lift the condition that $\text{NS}(X)$ should be spanned by d -curves, one would have to consider the full Fano graph of smooth rational curves of degree up to d , cf. Remark 2.8 and (5.2) below. Clearly, the requirement that the graph should be *sub-geometric* is necessary for its realizability in any case.

Theorem 2.18 justifies the fact that, once the integers d and n have been fixed, one can identify geometric graphs with configurations of smooth rational degree d curves on $K3$ -surfaces. In particular, when we speak of a *pencil* (resp. its *fiber*), we mean a parabolic graph (resp. connected parabolic subgraph).

2.5. Simplest applications: type $\tilde{\mathbf{A}}_1$ pencils. We start with a simple example illustrating the machinery described in this section.

Example 2.20. Fix an *odd* integer $d > 1$. We claim that, for an *odd* integer $n \gg d$, there is no $K3$ -surface of degree $2n$ with $\text{Fn}^d(X, h) \simeq 12\tilde{\mathbf{A}}_1$. By Theorem 2.18 it suffices to prove that

$$(2.21) \quad \Gamma := 12\tilde{\mathbf{A}}_1 \text{ fails to be subgeometric in degree } 2n \text{ with respect to } d.$$

A basis for $\mathfrak{F}_{2n}^d(\Gamma)$ is

$$h, k, a_1, \dots, a_{12},$$

where a_i, a'_i , $i = 1, \dots, 12$, are the fiber components and $k = a_i + a'_i$ is as in (2.10). Obviously,

$$\det \mathfrak{F}_{2n}^d(\Gamma) = -2^{14} d^2,$$

as can be seen from the orthogonal decomposition over $\mathbb{Q}$. Then, by [Remark 2.5](#), $\text{discr}_2 \mathfrak{F}_{2n}^d(\Gamma) \simeq (\mathbb{Z}/2)^{10} \oplus (\mathbb{Z}/4)^2$ (recall that d is odd) is generated by

$$(2.22) \quad \alpha_i := \frac{1}{2}(a_i - a_{i+1}), \quad i = 1, \dots, 10, \quad \eta := \frac{1}{2}h + \frac{1}{4} \sum_{i=1}^{12} a_i, \quad \kappa_4 := \frac{1}{4}k - \frac{1}{2}a_{12}.$$

with the Gram matrix

$$(2.23) \quad \mathbf{A}_{10}(\tfrac{1}{2}) \oplus \begin{bmatrix} s & 1/4 \\ 1/4 & -1/2 \end{bmatrix}, \quad s := 3d + \frac{1}{2}n - \frac{3}{2}.$$

Thus,

$$\ell(\text{discr}_2 \mathfrak{F}_{2n}^d(\Gamma)) + \text{rk} \mathfrak{F}_{2n}^d(\Gamma) = 26 = \text{rk}(\mathbf{L}) + 4,$$

i.e., we need a finite index extension of $\mathfrak{F}_{2n}^d(\Gamma)$ by at least two isotropic vectors (see [Theorem 2.6](#)). The $O_h(\mathfrak{F}_{2n}^d(\Gamma))$ -orbits of isotropic vectors in $\text{discr}_2 \mathfrak{F}_{2n}^d(\Gamma)$ are represented by

$$\begin{aligned} \llbracket 2\kappa_4 \rrbracket = \llbracket \varkappa \rrbracket &\ni \frac{1}{2}(a'_{12} - a_{12}), & \llbracket \alpha_1 + \alpha_3 \rrbracket &\ni \frac{1}{2}(a_1 - a_2 + a_3 - a_4), \\ \llbracket \alpha_1 + \alpha_3 + \alpha_5 + \alpha_7 \rrbracket &\ni \frac{1}{2} \sum_{i=1}^8 a_i, & \llbracket 2\eta \rrbracket &\ni \frac{1}{2} \sum_{i=1}^{12} a_i; \end{aligned}$$

Since $2n := h^2 = 4m + 2$, we have $s = 3d + m - 1 \in \mathbb{Z}$ in [\(2.23\)](#); hence, there also is an orbit of order 4 classes $\llbracket \lambda \rrbracket$ such that $\llbracket 2\lambda \rrbracket = \llbracket 2\eta \rrbracket$. The first two orbits are exceptional. Now, it is straightforward that, up to $O_h(\mathfrak{F}_{2n}^d(\Gamma))$, the only admissible index 4 extension $N' \supset \mathfrak{F}_{2n}^d(\Gamma)$ that *might* embed to the $K3$ -lattice $\mathbf{L}$ is that by

$$(2.24) \quad \frac{1}{2} \sum_{i=1}^8 a_i \quad \text{and} \quad \frac{1}{2} \sum_{i=5}^{12} a_i,$$

with the new discriminant

$$(2.25) \quad \text{discr}_2 N' = \mathcal{K}^\perp / \mathcal{K} \simeq \mathcal{V}_2^3 \oplus \begin{cases} \mathcal{V}_4, & \text{if } h^2 = 0 \pmod{4}, \\ \mathcal{U}_4, & \text{if } h^2 = 2 \pmod{4} \end{cases}$$

of length $8 = \text{rk} \mathbf{L} - \text{rk} \mathfrak{F}_{2n}^d(\Gamma)$, *cf.* [Remark 2.26](#) below. The last case $\mathcal{V}_2^3 \oplus \mathcal{U}_4$ with n odd violates Nikulin's existence criterion, *viz.* [Theorem 2.6\(3\)](#). $\square$

Remark 2.26. The isomorphism type of the form [\(2.25\)](#) the group $(\mathbb{Z}/2)^6 \oplus (\mathbb{Z}/4)^2$ is uniquely determined by its Brown invariant $\text{Br} = 4 \pmod{8}$ and the number of order 4 classes of square $0 \pmod{4}$; clearly, the latter gets divided by 4 when passing to $\mathcal{K}^\perp / \mathcal{K}$. However, in the sequel we choose to refrain from the discussion of various shortcuts that can be used: all forms can be computed explicitly.

Thanks to [\[17\]](#), we are not concerned with the *existence* of type $\tilde{\mathbf{A}}_1$ pencil for d or n even. Therefore, we confine ourselves to the following negative statement.

Proposition 2.27. *Let $h \gg d > 1$ be odd integers. Then neither of the graphs*

$$12\tilde{\mathbf{A}}_1, 11\tilde{\mathbf{A}}_1$$

is subgeometric in degree $2n$ with respect to d .

Proof. It suffices to imitate the argument of [Example 2.20](#) to show that the graph $\Gamma := 11\tilde{\mathbf{A}}_1$ fails to be subgeometric. This time, $\text{discr}_2 \mathfrak{F}_{2n}^d(\Gamma) \simeq (\mathbb{Z}/2)^{10} \oplus (\mathbb{Z}/8)$ is generated by $\alpha_1, \dots, \alpha_{10}$ as in (2.22) and

$$\bar{\eta} := \frac{1}{2}h + \frac{1}{8}k + \frac{1}{4} \sum_{i=1}^{11} a_i, \quad \bar{\eta} \cdot \alpha_i = 0, \quad \bar{\eta}^2 = 3d + \frac{1}{2}n - \frac{11}{8}.$$

Up to $O_h(\mathfrak{F}_{2n}^d(\Gamma))$, the only admissible extension $N' \supset \mathfrak{F}_{2n}^d(\Gamma)$ is that of index 2 by the first vector in (2.24). We have

$$\text{discr}_2 N' = \mathcal{U}_2^3 \oplus \mathcal{V}_2 \oplus [\bar{\eta}^2],$$

which fails to satisfy [Theorem 2.6\(3\)](#) unless $2n = 0 \pmod{4}$. $\square$

3. TYPE $\tilde{\mathbf{A}}_3$ PENCILS

In this section, we also assume that d is odd and mainly prove the *non*-existence of $K3$ -surfaces of degree $2n = 2 \pmod{4}$ with several $\tilde{\mathbf{A}}_3$ -configurations of smooth rational curves of odd degree d . We index the $\tilde{\mathbf{A}}_3$ type components of the graph in question and denote by $a_i^0, a_i^1, a_i^2, a_i^3$ the vertices (numbered consecutively along the cycle $\tilde{\mathbf{A}}_3$) of the i -th component. We also use the polarization h and isotropic vector $k = a_i^0 + \dots + a_i^3 = \text{const}(i)$, see (2.10).

3.1. The pencil $6\tilde{\mathbf{A}}_3$. The following proposition holds.

Proposition 3.1. *Let $n \gg d > 1$ be odd integers. Then the graph $\Gamma := 6\tilde{\mathbf{A}}_3$ fails to be subgeometric in degree $2n$ with respect to d .*

Proof. For the basis of $\mathfrak{F}_{2n}^d(\Gamma)$, we take

$$h, k, a_i^r, \quad i = 1, \dots, 6, \quad r = 1, 2, 3.$$

Obviously, we have $\det \mathfrak{F}_{2n}^d(\Gamma) = -2^{16}d^2$ and, letting

$$\bar{a}_i := a_i^1 + 2a_i^2 + 3a_i^3, \quad a_i^* := 3a_i^1 + 4a_i^2 + 3a_i^3, \quad i = 1, \dots, 6,$$

the group $\text{discr}_2 \mathfrak{F}_{2n}^d(\Gamma) \simeq (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/4)^4 \oplus (\mathbb{Z}/8)^2$ is generated by

$$(3.2) \quad \begin{aligned} \eta_2 &:= \frac{1}{2}h + \frac{1}{2} \sum_{i=1}^6 (a_i^1 + a_i^2), \quad \beta_6 := \frac{1}{2}(a_6^1 - a_6^3), \\ \alpha_i &:= \frac{1}{4}(\bar{a}_i - \bar{a}_{i+1}), \quad i = 1, \dots, 4, \end{aligned}$$

$$\eta_8 := \frac{d}{4}h + \frac{1-n}{8}k + \frac{1}{8} \sum_{i=1}^6 a_i^*, \quad \kappa_8 := \frac{1}{8}k + \frac{1}{4} \sum_{i=1}^5 \bar{a}_i.$$

In this basis, the matrix of the quadratic form is

$$(3.3) \quad \begin{bmatrix} s_2 & 1/2 \\ 1/2 & 1 \end{bmatrix} \oplus \mathbf{A}_4(\frac{1}{4}) \oplus \begin{bmatrix} s_8 & 1/8 \\ 1/8 & 1/4 \end{bmatrix}, \quad s_2 := \frac{1}{2}n - 1, \quad s_8 := \frac{1}{8} - \frac{1}{8}nd^2.$$

Thus, we find that the form $\text{discr}_2 \mathfrak{F}_{2n}^d(\Gamma)$ depends on the residue $n \pmod{4}$ only. (For n odd, the apparent dependence of s_8 on the *odd* residue $d \pmod{8}$ is irrelevant due to the obvious isomorphism of the last summand to $\mathcal{U}_8$.) By [Lemma 2.14](#), neither n nor d affect the minimal exceptional classes; up to $O_h(\mathfrak{F}_{2n}^d(\Gamma))$ they are

$$(3.4) \quad \llbracket 4\kappa_8 \rrbracket = \llbracket 2\alpha \rrbracket \ni \frac{1}{2}(a_1^0 + a_1^1 - a_1^2 - a_1^3), \quad \llbracket 2\alpha_1 \rrbracket \ni \frac{1}{2}(a_1^1 + a_1^3 - a_2^1 - a_2^3).$$

We conclude that, for $n \gg d$, the existence of the $K3$ -surface does not depend on d provided that the latter is odd; hence, as in the case $d = 1$ of lines (see [3]), Γ cannot be subgeometric unless $2n = 4 \bmod 8$. $\square$

Remark 3.5. We do not discuss the *existence* of surfaces with a pencil $6\tilde{\mathbf{A}}_3$ of degree d curves as we do not need it for the proof of Theorem 1.3. It appears that, for d odd, apart from $2n = 4 \bmod 8$ (cf. [20, §11.1]) one should also impose a q -balancing condition, cf. Lemma 4.12 and Proposition 4.21 below.

3.2. The pencil $5\tilde{\mathbf{A}}_3 + 2\mathbf{A}_1$. Below we prove the following statement.

Proposition 3.6. *Let $n \gg d > 1$ be odd integers. Then the graph $\Gamma := 5\tilde{\mathbf{A}}_3 + 2\mathbf{A}_1$ fails to be subgeometric in degree $2n$ with respect to d .*

Proof. We consider the following basis of $\mathfrak{F}_{2n}^d(\Gamma)$:

$$h, k, a_i^r, b_1, b_2, \quad i = 1, \dots, 6, \quad r = 1, 2, 3,$$

where b_1, b_2 are the two $\mathbf{A}_1$ fibers. The group $\text{discr}_2 \mathfrak{F}_{2n}^d(\Gamma) \simeq (\mathbb{Z}/4)^5 \oplus (\mathbb{Z}/8)^2$ is generated by $\alpha_1, \dots, \alpha_4$ as in (3.2) and

$$\begin{aligned} \bar{\kappa}_4 &:= \frac{1}{4}k + \frac{1}{4} \sum_{i=1}^5 \bar{a}_i + \frac{1}{2}b_1, & \bar{\kappa}_8 &:= \frac{7}{8}k + \frac{1}{2} \sum_{i=1}^5 \bar{a}_i + \frac{3}{2}b_1, \\ \bar{\eta}_8 &:= \frac{d}{4}h + \frac{3-n}{8}k + \frac{1}{8} \sum_{i=1}^5 (5a_i^1 + a_i^3) + \frac{5}{8}b_1 + \frac{1}{8}b_2; \end{aligned}$$

the reduced intersection matrix is

$$(3.7) \quad \mathbf{A}_4\left(\frac{1}{4}\right) \oplus \begin{bmatrix} 7 \\ 4 \end{bmatrix} \oplus \begin{bmatrix} 1/2 & 3/4 \\ 3/4 & s_8 \end{bmatrix},$$

where s_8 is as in (3.3). As in §3.1, $\text{discr}_2 \mathfrak{F}_{2n}^d(\Gamma)$ depends on $n \bmod 4$ only. The minimal exceptional classes are $\llbracket 4\bar{\kappa}_8 \rrbracket = \llbracket 2\mathcal{K} \rrbracket$ and $\llbracket 2\alpha_1 \rrbracket$ as in (3.4) and

$$\llbracket 4\bar{\eta}_8 - 2\alpha_1 - 2\alpha_3 \rrbracket \ni \frac{1}{2}(a_5^r + a_5^{r+2} - b_1 - b_2),$$

where $r = 0$ or 1 for n even or odd, respectively. Thus, as in the case $d = 1$ of lines, see [3], Γ is not subgeometric unless n is even. $\square$

3.3. The pencil $5\tilde{\mathbf{A}}_3 + \mathbf{A}_2$. The last $\tilde{\mathbf{A}}_3$ pencil to be considered is $5\tilde{\mathbf{A}}_3 + \mathbf{A}_2$.

Proposition 3.8. *Let $n \gg d > 1$ be odd integers. Then the graph $\Gamma := 5\tilde{\mathbf{A}}_3 + \mathbf{A}_2$ fails to be subgeometric in degree $2n$ with respect to d .*

Proof. We take for the basis

$$(3.9) \quad h, k, a_i^r, b_1, b_2, \quad i = 1, \dots, 6, \quad r = 1, 2, 3,$$

where b_1, b_2 are the two curves in the $\mathbf{A}_2$ fiber. The discriminant group

$$\text{discr}_2 \mathfrak{F}_{2n}^d(\Gamma) \simeq (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/4)^4 \oplus (\mathbb{Z}/16)$$

is generated by $\alpha_1, \dots, \alpha_4$ as in (3.2) and

$$\begin{aligned} \bar{\eta}_2 &:= \frac{1}{2}h + \frac{1}{2} \sum_{i=1}^5 (a_i^1 + a_i^2) + \frac{1}{2}(b_1 + b_2), & \beta_1 &:= \frac{1}{2}(a_1^1 - a_1^3), \\ \bar{\eta}_{16} &:= \frac{d}{4}h - \frac{1+2n}{16}k + \frac{1}{8} \sum_{i=1}^5 (5a_i^1 + a_i^2) + \frac{5}{4}(b_1 + b_2). \end{aligned}$$

The reduced intersection matrix in the basis $\bar{\eta}_2, \beta_1, \alpha_1, \dots, \alpha_4, \bar{\eta}_{16}$ is

$$\begin{bmatrix} s_2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 3/4 & 0 & 0 & 0 \\ 0 & 0 & 3/4 & 1/2 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 3/4 & 1/2 & 3/4 & 0 \\ 0 & 0 & 0 & 0 & 3/4 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & s_{16} \end{bmatrix}, \quad s_{16} := -\frac{3}{16} - \frac{1}{8}nd^2,$$

where s_2 is as in (3.3). As above, $\text{discr}_2 \mathfrak{F}_{2n}^d(\Gamma)$ depends on $n \bmod 4$ only. The minimal exceptional classes are $\llbracket 8\bar{\eta}_{16} \rrbracket = \llbracket 2\mathcal{K} \rrbracket$ and $\llbracket \alpha_2 \rrbracket$ as in (3.4) and, as in the case of lines, Γ is not subgeometric unless $2n = 4 \bmod 8$. $\square$

Remark 3.10. In fact, in the *non*-existence statements in this section, we do not use the assumption $n \gg d$. The reference to Lemma 2.14 *limits* the set of exceptional classes, *i.e.*, for n small the situation can only become *worse*.

4. TYPE $\tilde{\mathbf{A}}_{p-1}$ PENCILS FOR ODD PRIME p

We consider graphs of the type

$$\Gamma := m\tilde{\mathbf{A}}_{p-1} \oplus s\mathbf{A}_1, \quad p = 3, 5, \quad m \geq 4, \quad s \leq 3.$$

The vertices of the i -th $\tilde{\mathbf{A}}_{p-1}$ -subgraph of Γ are denoted as $a_i^0, \dots, a_i^{p-1}$, where $i = 1, \dots, m$, and the vertices of the $\mathbf{A}_1$ -components are $b_1, \dots, b_s$. We put

$$k := \sum_{r=0}^{p-1} a_i^r = \text{const}(i), \quad \text{cf. (2.10)}, \quad b := b_1 + \dots + b_s$$

and define

$$\begin{aligned} a_i^* &:= a_i^1 + a_i^2, & \alpha_i &:= \frac{1}{3}(a_i^1 - a_i^2) & \text{for } p = 3, \\ a_i^* &:= 2a_i^1 + 3a_i^2 + 3a_i^3 + 2a_i^4, & \alpha_i &:= \frac{1}{5}(a_i^1 + 2a_i^2 + 3a_i^3 + 4a_i^4) & \text{for } p = 5. \end{aligned}$$

Note that $-a_i^*$ is the *intrinsic polarization* of the component in the sense of [3].

4.1. Preliminaries. Apart from the lattice $\mathfrak{F}_{2n}^d(\Gamma)$ generated by

$$h, k, a_i^r, b_j, \quad i = 1, \dots, m, \quad r = 1, \dots, p-1, \quad j = 1, \dots, s,$$

we consider the auxiliary lattice

$$(4.1) \quad \mathfrak{G}_{2n}^d(\Gamma) := (m\mathbf{A}_{p-1} \oplus s\mathbf{A}_1) \oplus (\mathbb{Z}k + \mathbb{Z}h), \quad \text{where } h := h + d \sum_{i=1}^m a_i^* + \frac{d}{2}b,$$

with the Gram matrix

$$m\mathbf{A}_{p-1} \oplus s\mathbf{A}_1 \oplus \begin{bmatrix} 0 & pd \\ pd & h^2 \end{bmatrix}.$$

Obviously, for $s = 0$ we have $\mathfrak{G}_{2n}^d(\Gamma) = \mathfrak{F}_{2n}^d(\Gamma)$. In general,

- the discriminants of the two lattices are of the same order, *i.e.*,

$$(4.2) \quad |\text{discr } \mathfrak{F}_{2n}^d(\Gamma)| = |\text{discr } \mathfrak{G}_{2n}^d(\Gamma)| = p^{2+m} \cdot 2^s \cdot d^2$$

(indeed, this is an immediate consequence of the definition of h);

- for each prime $q > 2$ we have $\mathfrak{F}_{2n}^d(\Gamma) \otimes \mathbb{Z}_q = \mathfrak{G}_{2n}^d(\Gamma) \otimes \mathbb{Z}_q$, resulting in

$$(4.3) \quad \text{discr}_q \mathfrak{F}_{2n}^d(\Gamma) = \text{discr}_q \mathfrak{G}_{2n}^d(\Gamma);$$

- one has

$$O_{\hbar}(\mathfrak{G}_{2n}^d(\Gamma)) = \text{stab } \hbar \subset O_{\hbar}(\mathfrak{F}_{2n}^d(\Gamma));$$

- the orthogonal complements $k^{\perp}$ in the two lattices are equal; in particular, we can use the concepts of shortest representatives (cf. [Remark 2.15](#)) and exceptional classes/orbits (cf. [Definition 2.16](#)).

The splitting (4.1) results in the direct sum decomposition

$$(4.4) \quad \text{discr}_q(\mathfrak{F}_{2n}^d(\Gamma)) = \mathcal{D}_q \oplus \mathcal{N}_q \quad \text{for } q > 2$$

with “the standard part” $\mathcal{D}_q := \text{discr}_q(m\mathbf{A}_2 \oplus s\mathbf{A}_1)$ and the group

$$\mathcal{N}_q := \text{discr}_q(\mathbb{Z}k + \mathbb{Z}\hbar)$$

that will play a crucial rôle in our arguments. For $q = p$ (4.4) reads

$$(4.5) \quad \text{discr}_p \mathfrak{F}_{2n}^d(\Gamma) = \bigoplus_{i=1}^m \langle \alpha_i \rangle \oplus \mathcal{N}_p, \quad \alpha_i^2 = \frac{p-1}{p} \bmod 2\mathbb{Z}.$$

We start with a few useful observations. By the definition, we have (still $q > 2$)

$$(4.6) \quad \ell(\mathcal{N}_q) \leq 2.$$

This group is nontrivial if and only if $\nu_q(pd) > 0$, and it is cyclic, i.e., $\ell(\mathcal{N}_q) = 1$, if and only if $\nu_q(\hbar^2) = 0$.

Lemma 4.7. *Let $q > 2$ be a prime. If $\mathcal{N}_q$ is cyclic, i.e., $\ell(\mathcal{N}_q) = 1$, then*

$$\det \mathcal{N}_q = (-1) \cdot \hbar^2 \bmod (\mathbb{Z}_q^{\times})^2.$$

In this case, all isotropic classes in $\mathcal{N}_q$ are those of rational multiples of k , and they are all exceptional due to [Lemma 2.12](#). $\triangleleft$

We need to make use of the discriminant class of $\hbar/q$, which is well defined upon tensoring by $\mathbb{Z}_q$. However, to be on the safe side, we introduce the vectors

$$\varkappa_q := \frac{1}{q}k, \quad \eta_q := \frac{q+1}{q}\hbar \in \mathfrak{F}_{2n}^d(\Gamma)^{\vee};$$

certainly, at this point we assume that $\nu_q(pd) > 0$ and $\nu_q(\hbar^2) > 0$ to make sure that the two vectors are indeed in the dual lattice.

Lemma 4.8. *Let $q > 2$ be a prime. If $\ell(\mathcal{N}_q) = 2$, then*

$$\det \mathcal{N}_q = (-1) \bmod (\mathbb{Z}_q^{\times})^2.$$

In this case, the order q classes in $\mathcal{N}_q$ are of the form

$$u\varkappa_q + v\eta_q, \quad u, v \in \mathbb{Z}/p.$$

An isotropic class of this form is exceptional if and only if $v = 0$. $\triangleleft$

In the sequel we will use the following general lemma on group forms.

Lemma 4.9. *Let $\mathcal{K} \subset \mathcal{M}_q$ be an isotropic subgroup of order q generated by a certain element $\alpha \in \mathcal{M}_q$. If there exists another order q element $\beta \in \mathcal{M}_q$ such that $\alpha \cdot \beta \neq 0 \bmod \mathbb{Z}$, then*

$$\ell(\mathcal{K}^{\perp}/\mathcal{K}) = \ell(\mathcal{M}_q) - 2 \quad \text{and} \quad \det(\mathcal{K}^{\perp}/\mathcal{K}) = -\det \mathcal{M}_q \bmod (\mathbb{Z}_q^{\times})^2.$$

Proof. Abstractly, α and β generate $(\mathbb{Z}/q)^2$ with the Gram matrix

$$\begin{bmatrix} 0 & \varepsilon/q \\ \varepsilon/q & * \end{bmatrix}, \quad \varepsilon \not\equiv 0 \pmod{q},$$

of determinant $-1 \pmod{(\mathbb{Z}_q^\times)^2}$. Since the form is obviously nondegenerate, α and β are independent in $\mathcal{M}_q$ and generate an orthogonal direct summand. It is this summand that disappears in $\mathcal{K}^\perp/\mathcal{K}$. $\square$

Observation 4.10. Let $q = p$. If $\mathcal{N}_p$ is cyclic, as in Lemma 4.7, the orbits of the action of $O_h(\mathfrak{F}_{2n}^d(\Gamma))$ on the isotropic order p subgroups of $\text{discr}_p(\mathfrak{F}_{2n}^d(\Gamma))$ are

- the exceptional orbit of $\langle \kappa \rangle$ and,
- typically, two orbits of $\langle \omega_1 \rangle, \langle \omega_2 \rangle \subset \mathcal{D}_p$, one exceptional and one not.

(This fact is to be established on the case-by-case basis. Note that $\kappa + \omega_i \sim \omega_i$ and that any isotropic class of order $p^r > p$ is exceptional, see Lemma 2.12.) Hence, with one exception (see §4.3 below), each Γ -admissible kernel $\mathcal{K}$ must be cyclic of order p . Lemmata 4.9 and 4.7 imply that

$$\ell(\text{discr}_p \mathfrak{F}_{2n}^d(\Gamma, \mathcal{K})) = \ell(\text{discr}_p \mathfrak{F}_{2n}^d(\Gamma)) - 2$$

and $\text{discr}_p \mathfrak{F}_{2n}^d(\Gamma, \mathcal{K})$ satisfies condition (2) of Theorem 2.6 if and only if

$$pm + s < 21 \quad \text{or} \quad pm + s = 21 \quad \text{and} \quad h^2 = (-1)^m 2^s \pmod{p}.$$

Observation 4.11. Still $q = p$. If $\mathcal{N}_p$ is not cyclic, as in Lemma 4.8, in addition to the orbits of $\langle \kappa \rangle, \langle \omega_1 \rangle, \langle \omega_2 \rangle$ we also have that of $\langle \eta_p + \beta \rangle$ for some $[\beta] \in \mathcal{D}_p$. Since we assume $m \geq 4$, we can redecompose

$$\mathcal{D}_p = \underbrace{\begin{bmatrix} 2 \\ p \end{bmatrix} \oplus \begin{bmatrix} -2 \\ p \end{bmatrix}}_{\mathcal{D}'} \oplus \underbrace{\begin{bmatrix} 2 \\ p \end{bmatrix} \oplus \begin{bmatrix} -2 \\ p \end{bmatrix}}_{\mathcal{D}''} \oplus \dots$$

so that $[\omega_2] \in \mathcal{D}'$. Thus, no matter what $\eta_p^2 \pmod{2\mathbb{Z}}$ is, we can find $[\beta], [\gamma] \in \mathcal{D}''$ (obviously orthogonal to $[\omega_2]$) with the property that $(\eta_p + \beta)^2 = 0 \pmod{2\mathbb{Z}}$ and $(\eta_p + \beta) \cdot \gamma \neq 0 \pmod{\mathbb{Z}}$. Then, the kernel $\mathcal{K} := \langle [\omega_2], [\eta + \beta] \rangle$ is Γ -admissible and, applying Lemma 4.9 twice and Lemma 4.8, we find that

$$\ell(\text{discr}_p \mathfrak{F}_{2n}^d(\Gamma, \mathcal{K})) = \ell(\text{discr}_p \mathfrak{F}_{2n}^d(\Gamma)) - 4$$

and a *sufficient* condition for $\text{discr}_p \mathfrak{F}_{2n}^d(\Gamma, \mathcal{K})$ to satisfy Theorem 2.6(2) is

$$pm + s < 22 \quad \text{or} \quad pm + s = 22 \quad \text{and} \quad (-1)^{m+1} = 2^s \pmod{p}.$$

The last relation holds in all applications below; therefore, we do not need to engage into a study of isotropic elements of higher order (cf. Lemma 4.12).

4.2. Other torsion in the discriminant. A necessary realizability condition that cannot be achieved by a finite index extension of the lattice is Theorem 2.6(1). Therefore, we always assume that

$$(p-1)m + s \leq 18.$$

If $q \neq p$ is an odd prime, then $\text{discr}_q \mathfrak{F}_{2n}^d(\Gamma) = \mathcal{N}_q$ and $\ell(\mathcal{N}_q) \leq 2$, see (4.6). We need the following arithmetic condition: given a prime q , a pair (h, d) of rational numbers is called *q-balanced* if either

$$\nu_q(h) \geq \nu_q(d) \quad \text{or} \quad \nu_q(h) \text{ is even,}$$

cf. the statement of Theorem 1.3.

Lemma 4.12. *Let $n \gg d > 0$ and $q \neq p$ a prime. Then Γ has a $(\bmod q)$ -geometric kernel if and only if either*

- $q = 1 \bmod 4$, or
- $(p-1)m + s \leq 17$, or
- the pair $(\hbar^2, d)$ is q -balanced.

Proof. The case $(p-1)m + s \leq 17$ is trivial (as then we need $\ell(\mathcal{N}_q) \leq 3$), and so is the case $q = 1 \bmod 4$ (as then $-1 \bmod q$ is a quadratic residue). Therefore, in the rest of the proof we assume $(p-1)m + s = 18$ and $q = 3 \bmod 4$. Condition (2) of Theorem 2.6 is immediately satisfied whenever

- $\mathcal{N}_q = 0$, i.e., $\nu_q(d) = 0$, or
- $\ell(\mathcal{N}_q) = 1$, i.e., $\nu_q(\hbar^2) = 0 < \nu_q(d)$.

Note that, in the last two cases, the pair $(\hbar^2, d)$ is q -balanced.

Thus, assume that $u := \nu_q(d) > 0$ and $v := \nu_q(\hbar^2) > 0$ and let $t := \min\{u, v\}$. Then,

$$\mathcal{N}_q = \left[-\frac{2\varepsilon}{q^{2u-t}} \right] \oplus \left[\frac{2\varepsilon}{q^t} \right],$$

generated by some classes x, y . If $t = u$, this group has at least two cyclic isotropic subgroups $\mathcal{K}$ of order u , viz. those generated by $x \pm y$, and at least one of them does not contain $\varkappa_q$ and, hence, is Γ -admissible. In this case, $\mathcal{K}^\perp/\mathcal{K} = 0$.

If $t = v < u$ is even, $t = 2r$, we can take $y = \llbracket \hbar/q^t \rrbracket$ to see that the isotropic subgroup $\mathcal{K}$ generated by $q^r y$ is Γ -admissible and $\ell(\mathcal{K}^\perp/\mathcal{K}) = 1$.

Finally, assume that $t = v < u$ is odd, $t = 2r + 1$, i.e., the only case where the pair $(\hbar^2, d)$ is not q -balanced. The subgroup

$$\mathcal{K} = \mathcal{N}_q^r := \{z \in \mathcal{N}_q \mid q^r z = 0\}$$

is isotropic, but it does not suffice: we have $\mathcal{K}^\perp/\mathcal{K} \simeq (\mathbb{Z}/q^{2u-2r}) \oplus (\mathbb{Z}/q)$. Hence, we need at least one isotropic class γ of order q^{r+1} . Modulo $\mathcal{N}_q^r$, all such classes are multiples of x ; since, on the other hand, all rational multiples of k are isotropic, any “suitable” kernel must contain the exceptional class $\llbracket \varkappa_q \rrbracket = \llbracket q^r x \rrbracket$. $\square$

Observation 4.13. We can also compute the 2-torsion. If d is odd, then

$$\text{discr}_2 \mathfrak{F}_{2n}^d(\Gamma) = s \left[-\frac{1}{2} \right], \quad \text{generated by } \beta_i := \frac{1}{2}(k + b_i)$$

by (4.2) and Remark 2.5. It is odd and, since $s \leq 3$, has no isotropic subgroups. Thus, condition (3) in Theorem 2.6 is satisfied if and only if

$$(p-1)m + 2s \leq 20.$$

If d is even, then (4.1) is a valid change of basis over $\mathbb{Z}$ and we can use (4.4) for $q = 2$. We conclude that $\text{discr}_2 \mathfrak{F}_{2n}^d(\Gamma)$ is of length $s + 2$ and odd (unless $s = 0$ and $\nu_2(\hbar^2) \geq 2$) and the inequality

$$(p-1)m + 2s \leq 18$$

is sufficient for condition (3) in Theorem 2.6. For some values of n, d it may not be necessary; the tedious analysis of isotropic subgroups is left to the reader, cf. the proof of Lemma 4.12.

4.3. **The pencil $8\tilde{\mathbf{A}}_2$.** For $\Gamma := 8\tilde{\mathbf{A}}_2$ we have

$$\hbar^2 = 2n + 16d^2,$$

see (4.1). The following claim strengthens [17, Proposition 6.1]; at the same time, it serves as an example of the machinery explained in this section.

Proposition 4.14. *For integers $n \gg d > 1$, the statements below are equivalent:*

- the graph $\Gamma := 8\tilde{\mathbf{A}}_2$ is geometric in degree $2n$ with respect to d ;
- the discriminant group $\mathcal{N}_3 := \text{discr}_3(\mathbb{Z}k + \mathbb{Z}\hbar)$ is not cyclic;
- $\hbar^2 = 0 \pmod{3}$;
- $2n = 2d^2 \pmod{3}$.

Proof. The last three statements are obviously equivalent.

We have $\text{rk } \mathfrak{F}_{2n}^d(\Gamma) = 18$ and $\ell(\text{discr}_3 \mathfrak{F}_{2n}^d(\Gamma)) = 8 + \ell(\mathcal{N}_3) \geq 9$.

If $\mathcal{N}_3$ is cyclic, i.e., $\hbar^2 \neq 0 \pmod{3}$, cf. Observation 4.10, then, apart from $\langle \varkappa \rangle$, there are two $O_h(\mathfrak{F}_{2n}^d(\Gamma))$ -orbits of isotropic order-3 subgroups, with representatives generated by the classes of

$$(4.15) \quad \omega_1 := \alpha_1 + \alpha_2 + \alpha_3 \quad \text{or} \quad \omega_2 := \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6.$$

(We freely use the notation of §4.1.) The former is exceptional, the latter is not, as its shortest representatives have square (-4) . This is the only case where we can have a Γ -admissible kernel $\mathcal{K}$ of length 2, e.g., the one generated by

$$[\omega_2] \quad \text{and} \quad [\omega'_2] \ni \alpha_3 + \alpha_4 - \alpha_5 - \alpha_6 + \alpha_7 + \alpha_8.$$

Hence, we can reduce $\ell(\text{discr}_3 \mathfrak{F}_{2n}^d(\Gamma)) = 9$ down to 5, which is still not enough for condition (2) in Theorem 2.6.

If $\mathcal{N}_3$ is not cyclic, $\hbar^2 = 0 \pmod{3}$, then, arguing as in Observation 4.11, we can find a Γ -admissible kernel $\mathcal{K}$ of length 3, viz. the one generated by $[\omega_2]$, $[\omega'_2]$, and an element of the form $[\eta_3 + \beta]$, $\beta \in \mathcal{D}_3$, and reduce $\ell(\text{discr}_3 \mathfrak{F}_{2n}^d(\Gamma)) = 10$ by six units, down to 6. Since Lemma 4.9 is applied *three* times, Lemma 4.8 yields

$$(4.16) \quad \det_3(\mathcal{K}^\perp/\mathcal{K}) = 1 \pmod{(\mathbb{Z}_3^\times)^2},$$

which fits condition (2) in Theorem 2.6.

For the other primes $q \neq 3$, see Lemma 4.12 and Observation 4.13. □

Remark 4.17. We do not assert the uniqueness of a family of $K3$ -surfaces that contain an $8\tilde{\mathbf{A}}_2$ configuration we just constructed. The primes $q \mid d$, $q \neq 3$, often result in several admissible extensions.

The assumption $\hbar^2 \gg 0$ may be relevant as occasionally there are (-2) -vectors orthogonal to $\hbar$ but not to k .

4.4. **Other type $\tilde{\mathbf{A}}_2$ pencils.** The other three type $\tilde{\mathbf{A}}_2$ pencils that we need for the proof of Theorem 1.3 are straightforward: for $\text{discr}_q \mathfrak{F}_{2n}^d(\Gamma)$, we use

- the identity $\text{discr}_q \mathfrak{F}_{2n}^d(\Gamma) = \mathcal{N}_q$ and Lemma 4.12 for $q > 3$,
- Observations 4.10, 4.11 and the list (4.15) for $q = 3$, and
- Observation 4.13 for $q = 2$.

For the readers convenience, we compute

$$\hbar^2 = 2n + \left(2m + \frac{s}{2}\right) d^2 \quad \text{for } \Gamma := m\tilde{\mathbf{A}}_2 + s\mathbf{A}_1,$$

whereupon we merely state the ultimate results.

Proposition 4.18. *For integers $n \gg d > 1$, the statements below are equivalent:*

- the graph $\Gamma := 7\tilde{\mathbf{A}}_2$ is geometric in degree $2n$ with respect to d ;
- $\hbar^2 \not\equiv 2 \pmod{3}$;
- $2n \not\equiv d^2 - 1 \pmod{3}$. □

Proposition 4.19. *For integers $n \gg d > 1$, the statements below are equivalent:*

- the graph $\Gamma := 7\tilde{\mathbf{A}}_2 \oplus \mathbf{A}_1$ is geometric in degree $2n$ with respect to d ;
- the discriminant group $\mathcal{N}_3 := \text{discr}_3(\mathbb{Z}k + \mathbb{Z}\hbar)$ is not cyclic;
- $\hbar^2 \equiv 0 \pmod{3}$;
- $2n \equiv 2d^2 \pmod{3}$. □

Proposition 4.20. *For integers $n \gg d > 1$, the statements below are equivalent:*

- the graph $\Gamma := 6\tilde{\mathbf{A}}_2 \oplus 3\mathbf{A}_1$ is geometric in degree $2n$ with respect to d ;
- $\hbar^2 \not\equiv 1 \pmod{3}$;
- $2n \not\equiv 1 \pmod{3}$. □

4.5. The pencil $4\tilde{\mathbf{A}}_4 + 2\mathbf{A}_1$. In this section, we do have to assume that d is odd, cf. [Observation 4.13](#). We keep using the notation of [§4.1](#).

Proposition 4.21. *Let $n \gg d > 1$ be integers with d odd. Then, the following statements are equivalent:*

- the graph $\Gamma := 4\tilde{\mathbf{A}}_4 \oplus 2\mathbf{A}_1$ is geometric in degree $2n$ with respect to d ;
- the discriminant group $\mathcal{N}_5 := \text{discr}_5(\mathbb{Z}k + \mathbb{Z}\hbar)$ is not cyclic and the pair $(\hbar^2, d)$ is q -balanced for each prime $q \equiv 3 \pmod{4}$;
- $\hbar^2 \equiv 0 \pmod{5}$ and $(\hbar^2, d)$ is q -balanced for each prime $q \equiv 3 \pmod{4}$;
- $2n \equiv 4d^2 \pmod{5}$ and $(2n, d)$ is q -balanced for each prime $q \equiv 3 \pmod{4}$.

Proof. We have

$$\hbar^2 = 2n + 41d^2,$$

and it is clear that the last three statements are equivalent.

The 5-torsion is handled by [Observations 4.10](#) and [4.11](#): shortest representatives in $\llbracket \pm\alpha_i \rrbracket$ and $\llbracket \pm 2\alpha_i \rrbracket$ have squares $-4/5$ and $-6/5$, respectively, and the two orbits in [Observation 4.10](#) are those of

$$\omega_1 := \alpha_1 + 2\alpha_2, \quad \omega_2 := \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4;$$

the former is exceptional, the latter is not. We conclude that a $(\text{mod } 5)$ -geometric kernel exists if and only if $\mathcal{N}_5$ is not cyclic.

For the other primes $q \neq 5$, we use [Lemma 4.12](#) and [Observation 4.13](#): since $\text{rk } \mathfrak{F}_{2n}^d(\Gamma) = 20$ is maximal, the q -balancing condition does play a rôle. □

Example 4.22. Let $d = 3$. Consider the pencil $\Gamma := 4\tilde{\mathbf{A}}_4 \oplus 2\mathbf{A}_1$ and the lattice $N := \mathfrak{F}_{2n}^3(\Gamma, \mathcal{K})$, where $\mathcal{K}$ is a geometric kernel containing $\lambda := k/3$, see [\(2.10\)](#). The latter vector is represented by the roots of the form

$$l_i^r := -\lambda + a_i^{2r} + a_i^{2r+1} \in \text{root}_1(N, h), \quad i = 1, \dots, 4, \quad r = 0, \dots, 4.$$

(Here and below, the upper index should be understood mod 5. We assume that $n \gg 0$ and, hence, there are no other vectors of interest.) Thus, we do obtain a $2n$ -polarized $K3$ -surface X with an elliptic pencil with four type $\tilde{\mathbf{A}}_4$ fibers, but the fiber components are the lines l_i^r , whereas the anticipated original cubics are split:

$$a_i^r = l_i^{3r} + l_i^{3r+1} + l_i^{3r+2}.$$

Of course, the two type $\mathbf{A}_1$ fibers remain cubics.

In this particular case, there is another, geometric kernel; we chose this example because it is more transparent geometrically. To illustrate the importance of the q -balancing condition, one would have to take, *e.g.*, $d = 9$ and $2n = 24 \bmod 90$: instead of degree 9 curves, each fiber would *have* to consist of five twisted cubics.

5. PROOF OF THEOREM 1.3 AND COROLLARY 1.5

5.1. Proof of Theorem 1.3. Let $n \gg d$. By [15, Proposition A], the maximal number of smooth rational degree d curves on a $2n$ -polarized $K3$ -surface does not exceed 24. If either d or n is even, the existence of surfaces with 24 smooth rational degree d curves follows from [17, Theorem 1.1]. Thus, we can assume that both n and d are odd.

For all $n \gg d > 0$, Propositions 4.18 and 4.20 give us surfaces with 21 curves; therefore, we concentrate on those with at least 22 curves, in which case (still assuming that $n \gg d$), all smooth rational curves of degree *at most* d must be fiber components of an elliptic pencil, see [20, Theorem 1.1(ii)] or [3]. In other words, to construct a $K3$ -surface with $N = 22, 23$, or 24 smooth rational curves, we need a graph $\tilde{\Gamma}$ appearing on Shimada's list [23] of elliptic pencils, with a certain number $N \leq F \leq 24$ vertices, and such that, upon assigning color d to N vertices and a color *other than* d to the remaining $\delta := F - N$ vertices, the *weighted degree* $\sum_{v \in \Sigma} n_v \deg(v)$, *cf.* (2.10), of the components $\Sigma \subset \tilde{\Gamma}$ is constant. Then our graph Γ is obtained from $\tilde{\Gamma}$ by removing all vertices of degree other than d .

Denote by f the “expected” plain degree $\deg(\Gamma)$, *i.e.*, the common degree of the components of $\tilde{\Gamma}$ that are to remain parabolic.

If $\delta = 0$, then $\deg(\Sigma) = f$ for *each* component $\Sigma \subset \tilde{\Gamma}$, leaving but four pencils:

$$(5.1) \quad \Gamma = \tilde{\Gamma} = 12\tilde{\mathbf{A}}_1, \quad 11\tilde{\mathbf{A}}_1, \quad 8\tilde{\mathbf{A}}_2, \quad \text{or} \quad 6\tilde{\mathbf{A}}_3,$$

see Propositions 2.27, 4.14, and 3.1.

If $\delta = 1$, then $\tilde{\Gamma}$ has one distinguished component Σ' of $\deg(\Sigma') < d$ (as otherwise the vertex removed would have to have the same degree d), and then Γ is obtained from $\tilde{\Gamma}$ by “stripping” the $\sim$ from Σ' :

$$\Gamma = 7\tilde{\mathbf{A}}_2 \oplus \mathbf{A}_1 \quad \text{or} \quad 5\tilde{\mathbf{A}}_3 \oplus \mathbf{A}_2,$$

see Propositions 4.19 and 3.8.

Finally, if $\delta = 2$, then either $\tilde{\Gamma}$ has two components Σ', Σ'' of $\deg(\Sigma^*) < d$:

$$\Gamma = 5\tilde{\mathbf{A}}_3 \oplus 2\mathbf{A}_1 \quad \text{or} \quad 4\tilde{\mathbf{A}}_4 \oplus 2\mathbf{A}_1,$$

see Propositions 3.6 and 4.21, or $\tilde{\Gamma}$ has one distinguished component Σ' , of any degree (assuming $d > 1$), with $3 \leq |\Sigma'| \leq f + 1$:

$$(5.2) \quad \Gamma = 7\tilde{\mathbf{A}}_2 \oplus (\tilde{\mathbf{A}}_2 \mapsto \mathbf{A}_1) \quad \text{or} \quad 5\tilde{\mathbf{A}}_3 \oplus (\tilde{\mathbf{A}}_3 \mapsto \mathbf{A}_2 \text{ or } 2\mathbf{A}_1).$$

It is this last case where we would have to use a version of Theorem 2.18 that does not assume that $\text{NS}(X)$ is spanned by degree d curves (*cf.* Remark 2.19). Fortunately, all three configurations have just been discussed as obtained by other means, satisfying the assumptions of Theorem 2.18.

Summarizing, Proposition 4.14 gives us an extra family (in addition to the case $nd = 0 \bmod 2$, see [17, Theorem 1.1]) of pairs (n, d) realized by a $K3$ -surface with 24 curves, whereupon Propositions 2.27, 3.6, 3.8, and 4.19 assert that the other type $\tilde{\mathbf{A}}_p$ graphs Γ , $p = 2, 3, 4$, with $|\Gamma| \geq 22$ do not give us anything new. The

only case where 22, but not more, curves can be obtained is $4\tilde{\mathbf{A}}_4 \oplus 2\mathbf{A}_1$ that is considered in [Proposition 4.21](#). $\square$

5.2. Proof of Corollary 1.5. Assume that a complex $K3$ -surface contains 24 rational degree d curves and $2n \gg d$. By [\[20, Theorem 1.1\(ii\)\]](#) (see also [\[3\]](#)) the rational curves are components of singular fibers of a genus one fibration on X . Since the topological Euler–Poincaré characteristic of X is 24 and general fiber of the fibration is smooth, *all* components of singular fibers must be irreducible rational degree d curves (see *e.g.* [\[20, \(5.4\)\]](#)).

Thus, if one singular fiber is irreducible, then so are the others (all fibers are of degree d). This results in the configuration $24\tilde{\mathbf{A}}_0^*$. This configuration exists for all pairs (n, d) provided that $d > 2$, see [\[20, § 10\]](#).

Otherwise, all singular fibers are reducible and, hence, all 24 curves are smooth. In view of [\(5.1\)](#), [Propositions 2.27, 4.14](#), and [3.1](#) complete the proof. $\square$

Remark 5.3 (*cf.* [Corollary 1.5](#)). Let $n \gg d$ with nd even. For completeness, we collect all known (to us) facts about the existence of configurations of 24 irreducible rational degree d curves on a complex $K3$ -surface of degree $2n$ in this case.

- $24\tilde{\mathbf{A}}_0^*$ exists for all (n, d) provided that $d > 2$, see [\[20, § 10\]](#);
- $12\tilde{\mathbf{A}}_1$ exists for all (n, d) by [\[17, Theorem 1.1\]](#);
- $8\tilde{\mathbf{A}}_2$ exists if and only if $n = d^2 \bmod 3$ by [Proposition 4.14](#);
- we conjecture that $6\tilde{\mathbf{A}}_3$ exists if and only if $n = 2d^2 \bmod 4d$, *cf.* [\[20, § 11.1\]](#).

The proof of the last conjecture, which seems too technical for the present paper, will appear elsewhere.

REFERENCES

1. Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley. MR 1890629 (2003a:17001)
2. J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 290, Springer, 1988, With contributions by E. Bannai, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. MR 920369 (89a:11067)
3. Alex Degtyarev, *Lines on Smooth Polarized $K3$ -Surfaces*, Discrete Comput. Geom. **62** (2019), no. 3, 601–648. MR 3996938
4. ———, *800 conics on a smooth quartic surface*, Journal of Pure and Applied Algebra **226** (2022), no. 10, 107077.
5. ———, *Conics in sextic $K3$ -surfaces in $\mathbb{P}^4$* , Nagoya Math. J. **246** (2022), 273–304. MR 4425289
6. ———, *Tritangents to smooth sextic curves*, Ann. Inst. Fourier (Grenoble) **72** (2022), no. 6, 2299–2338.
7. ———, *Conics on Kummer quartics*, Tohoku Math. J. (2) **75** (2023), no. 3, 395–421. MR 4646369
8. ———, *Conics on Barth–Bauer octics*, Science China Mathematics **67** (2024), no. 7, 1507–1524.
9. ———, *Conics on smooth quartic surfaces*, 2024, Submitted, [arXiv:2407.00493](#).
10. Alex Degtyarev, Ilia Itenberg, and Ali Sinan Sertöz, *Lines on quartic surfaces*, Math. Ann. **368** (2017), no. 1–2, 753–809. MR 3651588
11. Alex Degtyarev and Sławomir Rams, *Counting lines with Vinberg’s algorithm*, To appear in Rev. Mat. Iberoam., 2025.
12. ———, *Lines on $K3$ -quartics via triangular sets*, Discrete Comput. Geom. **73** (2025), 785–814. MR 3996938
13. The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.15.1*, 2025.

14. Yoichi Miyaoka, *The orbibundle Miyaoka-Yau-Sakai inequality and an effective Bogomolov-McQuillan theorem*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 403–417 (English).
15. ———, *Counting lines and conics on a surface*, Publ. Res. Inst. Math. Sci. **45** (2009), no. 3, 919–923. MR 2569571
16. V. V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177, 238, English translation: Math USSR-Izv. **14** (1979), no. 1, 103–167 (1980). MR 525944 (80j:10031)
17. Sławomir Rams and Matthias Schütt, *Miyaoka's bound for conics on K3 surfaces and beyond*, Nagoya Math. J. ?? ?? (??), ???
18. ———, *112 lines on smooth quartic surfaces (characteristic 3)*, Q. J. Math. **66** (2015), no. 3, 941–951. MR 3396099
19. ———, *64 lines on smooth quartic surfaces*, Math. Ann. **362** (2015), no. 1-2, 679–698. MR 3343894
20. ———, *24 rational curves on K3 surfaces*, Commun. Contemp. Math. **25** (2023), no. 6, Paper No. 2250008, 23.
21. ———, *Low degree rational curves on quasi-polarized K3 surfaces*, J. Pure Appl. Algebra **229** (2025), no. 2, 21 (English), Id/No 107904.
22. Friedrich Schur, *Ueber eine besondere Classe von Flächen vierter Ordnung*, Math. Ann. **20** (1882), no. 2, 254–296. MR 1510168
23. Ichiro Shimada, *On elliptic K3 surfaces*, Michigan Math. J. **47** (2000), no. 3, 423–446. MR 1813537

DEPARTMENT OF MATHEMATICS, BILKENT UNIVERSITY, 06800 ANKARA, TURKEY

Email address: `degt@fen.bilkent.edu.tr`

THEORETICAL COMPUTER SCIENCE DEPARTMENT, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, JAGIELLONIAN UNIVERSITY, UL. ŁOJASIEWICZA 6, 30-348 KRAKÓW, POLAND

Email address: `slawomir.rams@uj.edu.pl`