OKA'S CONJECTURE ON IRREDUCIBLE PLANE SEXTICS. II

Alex Degtyarev

Abstract. We complete the proof of Oka's conjecture on the Alexander polynomial of an irreducible plane sextic. We also calculate the fundamental groups of irreducible sextics with a singular point adjacent to $\mathbf{J}_{10}$.

1. Introduction

This paper is a sequel to my paper [D7], where Oka's conjecture on the Alexander polynomial is settled for all irreducible sextics with simple singularities. Here, we complete the proof for the missing case of sextics with a non-simple singular point adjacent to $\mathbf{J}_{10}$ (a point of simple tangency of three smooth branches).

Recall that a plane sextic $C \subset \mathbb{P}^2$ is said to be of *torus type* if its equation can be represented in the form $p^3 + q^2 = 0$, where p and q are some homogeneous polynomials of degree 2 and 3, respectively (see Section 3.3 for details). Sextics of torus type are a major source of examples of plane curves with large fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus C)$. Historically, it was a six cuspidal sextic of torus type that was the first example of an irreducible plane curve with infinite fundamental group, see O. Zariski [Za]. (An irreducible quintic with infinite fundamental group was discovered much later in [D4].) All sextics of torus type have nontrivial Alexander polynomials (see below); hence, their fundamental groups are infinite.

The fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ is a powerful invariant of a plane curve, but it is extremely difficult to calculate. A much simpler invariant, capturing the abelinization of the first commutant of the group, is the so called *Alexander polynomial.* For an irreducible curve $C \subset \mathbb{P}^2$ of degree m, its Alexander polynomial $\Delta_C(t)$ can be defined as the characteristic polynomial of the deck translation automorphism of the vector space $H_1(X_m \smallsetminus C; \mathbb{C})$, where $X_m \to \mathbb{P}^2$ is the cyclic m-fold covering ramified at C. (Since C is assumed irreducible, such a covering is unique.) The Alexander polynomial is a purely algebraic invariant of the fundamental group; in particular, $\Delta_C(t) \neq 1$ if and only if the quotient $K/[K, K]$ is infinite, where K is the commutant of the group. For more details, alternative definitions, and basic properties of the Alexander polynomial see A. Libgober [L1] and [L2]. The Alexander polynomials of irreducible sextics are calculated in [D3]. For further references, see M. Oka's survey [Oka].

Note that the Alexander polynomial of a plane curve is subject to rather strong divisibility conditions, see [Za], [L1], and [D5]. In particular, six is the first degree where the polynomial of an irreducible curve may be nontrivial.

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Based on the known examples, Oka [EO] conjectured that *any irreducible sextic whose Alexander polynomial is nontrivial is of torus type*. (A similar conjecture on the fundamental group was disproved in [D7]; more counterexamples are given by Theorem 1.4 below.) Proof of Oka's conjecture constitutes the main result of the present paper. More precisely, the following statement holds.

1.1. Main Theorem. *For an irreducible plane sextic C, the following statements are equivalent*:

(1) *C is of torus type*;
(2) *the Alexander polynomial $\Delta_C(t)$ is nontrivial*;
(3) *the group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ factors to the reduced braid group $\mathbb{B}_3/\Delta^2$*;
(4) *the group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ factors to the symmetric group $\mathbb{S}_3$.*

Theorem 1.1 is proved in [D7] under the assumption that either all singular points of C are simple or C has a non-simple singular point adjacent to $\mathbf{X}_9$ (a quadruple point). It is also shown in [D7] that (1) implies (2)–(4) and that (2) implies (4). (Obviously, (3) implies (4) as well.) The only remaining case is the implication (4) $\Rightarrow$ (1) for a sextic with a triple non-simple singular point. All such points are adjacent to $\mathbf{J}_{10}$ (a semiquasihomogeneous singularity of type $(6,3)$); a sextic with such a point is called a **J**-*sextic*, see Definition 2.1. Thus, Theorem 1.1 is a consequence of the following statement, which is actually proved in the paper.

1.2. Theorem. *Let C be an irreducible* **J**-*sextic, and assume that the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ factors to $\mathbb{S}_3 = \mathbb{D}_6$. Then C is of torus type.*

In fact, Theorem 1.2 holds for reducible **J**-sextics as well, see Remark 3.8.

As in the case of sextics with simple singularities, see [D7], Theorem 1.2 admits a slightly more precise version (which is meaningful for the sets of singularities $\mathbf{J}_{2,0} \oplus 4\mathbf{A}_2$ and $\mathbf{J}_{2,3} \oplus 3\mathbf{A}_2$).

1.3. Theorem. *For an irreducible* **J**-*sextic C, there is a natural one to one correspondence between the set of quotients of $\pi_1(\mathbb{P}^2 \smallsetminus C)$ isomorphic to $\mathbb{D}_6$ and the set of torus structures of C.*

Throughout the paper, we use the notation of [AVG] for the types of singularities adjacent to $\mathbf{J}_{10} = \mathbf{J}_{2,0}$. One can use Table 1 below as a guide. Although, in the presence of non-simple singular points, a set of singularities is no longer determined by its resolution lattice, we keep using the lattice notation $\oplus$ in the listings. (In general, we refer to [D7] for the less common notation and terminology.)

As another application of the approach developed in the paper, we calculate the fundamental groups of most irreducible **J**-sextics (all except the two families mentioned in Theorem 1.5). The results are stated in Theorems 1.4–1.6 below.

1.4. Theorem. *There exist irreducible plane sextics with the following sets of singularities*: $\mathbf{J}_{2,0} \oplus 2\mathbf{A}_4$, $\mathbf{J}_{2,1} \oplus 2\mathbf{A}_4$, *and* $\mathbf{J}_{2,5} \oplus \mathbf{A}_4$. *For each such sextic C, one has $\pi_1(\mathbb{P}^2 \smallsetminus C) = \mathbb{D}_{10} \times (\mathbb{Z}/3\mathbb{Z})$, where $\mathbb{D}_{10}$ is the dihedral group of order* 10.

1.5. Theorem. *Let C be an irreducible* **J**-*sextic of torus type and with a set of singularities other than $\mathbf{J}_{2,0} \oplus 4\mathbf{A}_2$ or $\mathbf{J}_{2,3} \oplus 3\mathbf{A}_2$. Then $\pi_1(\mathbb{P}^2 \smallsetminus C)$ is the reduced braid group $\mathbb{B}_3/\Delta^2$.*

For the two exceptional sets of singularities listed in Theorem 1.5, the Alexander polynomial of the curve is $(t^2 - t + 1)^2$, see, *e.g.*, [D3]. Hence, the fundamental group must be much larger than $\mathbb{B}_3/\Delta^2$. As in Section 4.9, one can use [D4] and assert that this group is a quotient of

$$\langle a, b, c \mid aba = bab,\ bcb = cbc,\ abcb^{-1}a = bcb^{-1}abcb^{-1} \rangle.$$

1.6. Theorem. *Let C be an irreducible* **J***-sextic that is neither of torus type nor one of the curves listed in Theorem* 1.4*. Then the group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ is abelian.*

1.7. Contents of the paper. Proof of Theorem 1.2 follows the lines of [D3]: we construct an appropriate conic $Q = \{p = 0\}$ and cubic $K = \{q = 0\}$ and, using the Bézout theorem, show that the difference $\varphi - q^2$ should be divisible by p^3 (where $\{\varphi = 0\}$ is the original curve).

To simplify the calculation and minimize the number of cases to be considered, we start with reducing the problem to the study of the so called *trigonal models*, which are trigonal curves on the quadratic cone in $\mathbb{P}^3$. This is done in §2. We show that, in all cases of interest, trigonal models have simple singularities (and in rather small number: the set of singularities should admit an embedding to $\mathbf{E}_8$) and extend to such curves the results of [D6] and [D7] concerning plane sextic. An important result here is Proposition 2.3, which relates the fundamental groups of a **J**-sextic and its trigonal model, and its Corollary 2.4 stating that, to a certain extent, the fundamental group of a **J**-sextic does not depend on its non-simple singular point.

Theorems 1.2 and 1.3 are proved in §3. Again, the torus type condition for the original curve is reduced to a similar condition for its trigonal model, so that one needs to deal with simple singularities only. Note that, unlike the case of abundant *vs.* non-abundant curves in [D3], where the conic Q was obtained from the linear system calculating the Alexander polynomial of the curve, here the existence of Q follows from a simple dimension count.

In §4, we continue exploring properties of trigonal models and find the fundamental groups of most **J**-sextics. Theorems 1.4–1.6 are proved here. We use a simple consequence of van Kampen's method [vK] (avoiding any attempt to calculate the global braid monodromy) and reduce the few remaining cases to the results of [D4] dealing with plane quintics.

In concluding §5, we apply the results of previous sections to prohibit several **J**-sextics. Propositions 5.1 and 5.2 proved here are not new, see [D1], but we give new short proofs. It is remarkable that prohibited are precisely the curves that would contradict Theorems 1.2–1.6.

2. The trigonal model

In this section, we introduce the so called *trigonal model* of a **J**-sextic C, thus reducing the study of C to the study of a certain curve with simple singularities.

2.1. Definition. A **J***-sextic* is a reduced plane sextic $C \in \mathbb{P}^2$ with a distinguished triple singular point O adjacent to $\mathbf{J}_{10}$ and without linear components passing through O.

Let (C, O) be a **J**-sextic. Blow up O to obtain a Hirzebruch surface $\Sigma_1 = \mathbb{P}^2(O)$. The proper transform of C has a triple point O' in the exceptional section. Blow it up and blow down the fiber through O'. The result is a Hirzebruch surface Σ_2 (quadratic cone with the vertex blown up); we denote it by Z. It is a geometrically ruled surface with an exceptional section S_0 of selfintersection (-2). When speaking about *fibers* of Z, we mean fibers of the ruling.

The proper transform of C is a certain curve $B \subset Z$ disjoint from S_0; it is called the *trigonal model* of C. The trigonal model is equipped with a distinguished point $\bar{O}$ (image of the fiber through O') and distinguished fiber $F_0 \ni \bar{O}$ (image of the exceptional divisor over O'). It is important that $\bar{O}$ does not belong to either B or S_0. Below, we show that, as long as the fundamental group is concerned, $\bar{O}$ and F_0 are irrelevant (see Proposition 2.3).

Let F be a generic fiber of Z. As is well known, S_0 and F generate the semigroup of effective divisors on Z. One has $B \in |3S_0 + 6F|$; in particular, B *is* a trigonal curve, in the sense that it intersects each fiber of Z at three points. The *singular fibers* of B are the fibers of Z that are not transversal to B. Counting with multiplicities, B has twelve singular fibers. If B has simple singularities only, its singular fibers can be regarded as the singular fibers of the Jacobian elliptic surface $\tilde{X}$ obtained as the double covering of Z ramified at B and S_0, see below.

Note that B may be reducible; however, it cannot contain S_0 or a fiber of Z. Any curve $B \in |3S_0 + 6F|$ satisfying this condition is called a *trigonal model*. Any trigonal model B gives rise to a $\mathbf{J}$-sextic after a point $\bar{O} \in Z \setminus (B \cup S_0)$ is chosen.

2.2. Lemma. *If a* $\mathbf{J}$*-sextic* C *is irreducible, then all singular points of its trigonal model* B *are simple.*

Proof. The statement follows immediately from the genus formula, as a nonsingular curve in $|3S_0 + 6F|$ has genus 4 and any non-simple singular point takes off the genus at least 6. □

It is easy to see (*e.g.*, using the associated cubic, see Section 4.1 below) that the only non-simple singular point that B may have is $\mathbf{J}_{10}$. In this case, depending on the distinguished fiber F_0, the set of singularities of the original sextic C is either $2\mathbf{J}_{10}$ or $\mathbf{J}_{4,0}$, and C consists of three conics, which either have two common tangency points (the case $2\mathbf{J}_{10}$) or one common point of 4-fold intersection (the case $\mathbf{J}_{4,0}$). Note that both families are obviously of torus type.

From now on, we always assume that all singular points of B are simple. We identify a set of simple singularities of B with its resolution lattice, which is a direct sum of irreducible root systems (**A**–**D**–**E** lattices), one summand for each singular point of the same name. The relation between the types of the distinguished fiber F_0 and distinguished singular point O of C is given by Table 1, where for F_0 we use (one of) the standard notation for singular elliptic fibers.

Table 1. Singular fibers and singular points

Fiber F_0	Point O	Fiber F_0	Point O	Fiber F_0	Point O
$\tilde{\mathbf{A}}_0$	$\mathbf{J}_{2,0}$	$\tilde{\mathbf{A}}_2^*$	$\mathbf{E}_{14}$	$\tilde{\mathbf{A}}_p$, $p \geqslant 1$	$\mathbf{J}_{2,p+1}$
$\tilde{\mathbf{A}}_0^*$	$\mathbf{J}_{2,1}$	$\tilde{\mathbf{E}}_6$	$\mathbf{E}_{18}$	$\tilde{\mathbf{D}}_q$, $q \geqslant 4$	$\mathbf{J}_{3,q-4}$
$\tilde{\mathbf{A}}_0^{**}$	$\mathbf{E}_{12}$	$\tilde{\mathbf{E}}_7$	$\mathbf{E}_{19}$		
$\tilde{\mathbf{A}}_1^*$	$\mathbf{E}_{13}$	$\tilde{\mathbf{E}}_8$	$\mathbf{E}_{20}$		

2.3. Proposition. *For a* $\mathbf{J}$*-sextic* (C, O) *and its trigonal model* $B \subset Z$*, there is a canonical isomorphism* $\pi_1(\mathbb{P}^2 \setminus C) = \pi_1(Z \setminus (B \cup S_0))$.

Proof. Consider the surface $\tilde Z$ obtained from Z by blowing up the distinguished point $\bar O$. Denote by $\tilde{}$ the proper pull-backs of the curves involved, and let $\tilde L$ be the exceptional divisor over $\bar O$. There are obvious diffeomorphisms

$$\mathbb{P}^2 \smallsetminus C = \tilde Z \smallsetminus (\tilde B \cup \tilde S_0 \cup \tilde F_0),$$
$$\mathbb{P}^2 \smallsetminus (C \cup L) = \tilde Z \smallsetminus (\tilde B \cup \tilde S_0 \cup \tilde F_0 \cup \tilde L) = Z \smallsetminus (B \cup S_0 \cup F_0),$$

where $L \subset \mathbb{P}^2$ is the line tangent to C at O. Thus, $\pi_1(\mathbb{P}^2 \smallsetminus C)$ is obtained from the group $\pi_1(\tilde Z \smallsetminus (\tilde B \cup \tilde S_0 \cup \tilde F_0 \cup \tilde L))$ by adding the relation $[\partial\tilde\Gamma] = 1$, where $\tilde\Gamma \subset \tilde Z$ is a small analytic disc transversal to $\tilde L$ (and disjoint from the other curves). The projection $\Gamma \subset Z$ of $\tilde\Gamma$ is a small analytic disk transversal to F_0 at $\bar O$; since $\bar O$ does not belong to the union $B \cup S_0$, the relation $[\partial\Gamma] = 1$ is precisely the relation resulting from patching the distinguished fiber F_0. □

2.4. Corollary. *The fundamental group of a* **J***-sextic C obtained from a trigonal model B does not depend on the choice of a distinguished fiber F_0.* □

Let $p\colon X \to Z$ be the double covering of Z ramified along $B + S_0$, and let $\tilde X$ be the minimal resolution of singularities of X. It is well known that $\tilde X$ is a rational elliptic surface; its intersection lattice $H_2(\tilde X)$ is the only odd unimodular lattice of signature $(1, 9)$. Let E be the exceptional divisor contracted by the projection $\tilde X \to X$, and let $\tilde S_0$ and $\tilde B$ be the proper transforms of, respectively, S_0 and B in $\tilde X$. The copies of S_0 and B in X are identified with S_0 and B themselves.

Let $s_0 = [\tilde S_0] \in H_2(\tilde X)$, let $f \in H_2(\tilde X)$ be the class realized by the pull-back of a generic fiber of Z, and denote by T_B the sublattice spanned by s_0 and f, *i.e.*, $T_B = \mathbb{Z}s_0 + \mathbb{Z}f \subset H_2(\tilde X)$.

2.5. Lemma. *The sublattice $T_B \subset H_2(\tilde X)$ is an orthogonal summand, and one has $T_B^\perp \cong \mathbf{E}_8$.*

Proof. One has $e_0^2 = -1$, $f^2 = 0$, and $e_0 \cdot f = 1$. Hence, T_B is a unimodular lattice of signature $(1, 1)$. In particular, T_B is an orthogonal summand in any larger lattice. The orthogonal complement $T_B^\perp$ is a unimodular lattice of signature $(0, 8)$ and, to complete the proof, it remains to show that $T_B^\perp$ is even, *i.e.*, that T_B contains a characteristic vector of $H_2(\tilde X)$.

Replace $\tilde X$ with a surface X' obtained from X by a small perturbation; it can be regarded as the double covering of Z ramified at S_0 and a nonsingular curve B' obtained by a perturbation of B. There is a diffeomorphism $\tilde X \cong X'$ identical outside some regular neighborhoods of the singular points, see [Du]. Since Z is a Spin-manifold, the topological projection formula implies that the Stiefel-Whitney class $w_2(X')$ is given by $([B'] + [S_0]) \bmod 2 \in H_2(X'; \mathbb{Z}/2\mathbb{Z})$. On the other hand, using the fact that $H_2(X')$ is torsion free, one can see that $[B'] = 3s_0 + 3f$. Hence, $w_2(\tilde X) = f \bmod 2$, and the statement follows. □

Let Σ_B be the set of singularities of B. Recall that we identify a set of simple singularities with its resolution lattice, *i.e.*, Σ_B can be regarded as the sublattice in $H_2(\tilde X)$ spanned by the classes of the exceptional divisors in $\tilde X$ (those that are contracted in X). Obviously, $\Sigma_B \subset T_B^\perp \cong \mathbf{E}_8$. Denote $\mathcal{K}_B = \operatorname{Tors}(T_B^\perp/\Sigma_B)$.

Consider the canonical epimorphism $\kappa\colon \pi_1(Z \smallsetminus (B \cup S_0)) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$. (If B is irreducible, then κ is the only epimorphism to $\mathbb{Z}/2\mathbb{Z}$. Otherwise, κ can be defined as the map sending each van Kampen generator to $1 \in \mathbb{Z}/2\mathbb{Z}$. Alternatively, $\operatorname{Ker}\kappa \subset \pi_1(Z \smallsetminus (B \cup S_0))$ is the group of the covering $X \smallsetminus (B \cup S_0) \to Z \smallsetminus (B \cup S_0)$.) Let $K(B) = \operatorname{Ker}\kappa$ and denote by $\bar{K}(B)$ the abelinization of $K(B)$. One has

$$K(B) = \pi_1(X \smallsetminus (B \cup S_0)) \quad \text{and} \quad \bar{K}(B) = H_1(X \smallsetminus (B \cup S_0)).$$

Denote by tr the automorphism of $\bar{K}(B)$ given by $[a] \mapsto [\tilde{1}^{-1}a\tilde{1}]$, where $a \in K(B)$, $[a]$ stands for the class realized by a in $\bar{K}(B)$, and $\tilde{1} \in \pi_1(Z \smallsetminus (B \cup S_0))$ is a lift of the generator $1 \in \mathbb{Z}/2\mathbb{Z}$. Alternatively, tr is induced by the deck translation of the covering $X \smallsetminus (B \cup S_0) \to Z \smallsetminus (B \cup S_0)$.

Next three statements are analogs of similar statements for plane sextics with simple singularities, see [D6] and [D7]. Proofs are omitted; instead, we refer to the counterparts for plane sextics, whose proofs apply almost literally.

2.6. Proposition (*cf*. Theorem 4.3.1 in [D6]). *A trigonal model B (and, hence, a $\mathbf{J}$-sextic C) is irreducible if and only if the group $\mathcal{K}_B$ is free of 2-torsion.* □

2.7. Proposition (*cf*. Proposition 3.4.4 in [D7]). *Let B be an irreducible trigonal model. Then there is a splitting $\bar{K}(B) = \operatorname{Ker}(\operatorname{tr}-1) \oplus \operatorname{Ker}(\operatorname{tr}+1)$ and canonical isomorphisms $\operatorname{Ker}(\operatorname{tr}-1) = \mathbb{Z}/3\mathbb{Z}$ and $\operatorname{Ker}(\operatorname{tr}+1) = \operatorname{Ext}(\mathcal{K}_B, \mathbb{Z})$.* □

2.8. Corollary (*cf*. Corollary 3.4.6 in [D7]). *Let B be the trigonal model of an irreducible $\mathbf{J}$-sextic C. Then there is a canonical one to one correspondence between the set of normal subgroups $N \subset \pi_1(\mathbb{P}^2 \smallsetminus C)$ with $\pi_1(\mathbb{P}^2 \smallsetminus C)/N \cong \mathbb{D}_{2n}$, $n \geqslant 3$, and the set of subgroups of $\operatorname{Tor}(\mathcal{K}_B, \mathbb{Z}/n\mathbb{Z})$ isomorphic to $\mathbb{Z}/n\mathbb{Z}$.* □

Lemma 2.5 and Proposition 2.6 result in the following necessary condition for a set of simple singularities to be realized by the trigonal model of a $\mathbf{J}$-sextic.

2.9. Corollary. *Let Σ be a set of simple singularities of a trigonal model B. Then Σ admits an embedding to $\mathbf{E}_8$. If B is irreducible, then there is an embedding $\Sigma \hookrightarrow \mathbf{E}_8$ with $\mathbf{E}_8/\Sigma$ free of 2-torsion.* □

3. Proof of Theorems 1.2 and 1.3

As in the previous section, we fix a $\mathbf{J}$-sextic C and denote by $B \subset Z$ its trigonal model. We assume that all singular points of B are simple, see Lemma 2.2.

3.1. Lemma. *Let Σ be a root system embedded to $\mathbf{E}_8$, and let $\mathcal{K} = \operatorname{Tors}(\mathbf{E}_8/\Sigma)$. If $|\mathcal{K}|$ is odd (in particular, $\mathcal{K} \neq 0$), then Σ is one of the following lattices:*

$\mathcal{K} = (\mathbb{Z}/3\mathbb{Z})^2 : \ \Sigma = 4\mathbf{A}_2$;
$\mathcal{K} = \mathbb{Z}/3\mathbb{Z} : \ \Sigma = 3\mathbf{A}_2,\ 3\mathbf{A}_2 \oplus \mathbf{A}_1,\ \mathbf{A}_5 \oplus \mathbf{A}_2,\ \mathbf{A}_8$, *or* $\mathbf{E}_6 \oplus \mathbf{A}_2$;
$\mathcal{K} = \mathbb{Z}/5\mathbb{Z} : \ \Sigma = 2\mathbf{A}_4$.

Conversely, for each root system listed above there is a unique, up to isomorphism, embedding to $\mathbf{E}_8$.

Proof. The enumeration of the embeddings of root systems to $\mathbf{E}_8$ is a simple task. For example, one can use V. Nikulin's techniques [Ni] of lattice extensions and discriminant forms. We omit the details. □

3.2. Corollary. *The fundamental group of an irreducible $\mathbf{J}$-sextic C factors to the dihedral group $\mathbb{D}_6$ (respectively, $\mathbb{D}_{10}$) if and only if the set of singularities of the trigonal model of C is one of the following: $3\mathbf{A}_2 \oplus \ldots$, $\mathbf{A}_5 \oplus \mathbf{A}_2$, $\mathbf{A}_8$, or $\mathbf{E}_6 \oplus \mathbf{A}_2$ (respectively, $2\mathbf{A}_4$).*

Proof. The statement follows from Corollary 2.8 and Lemma 3.1. □

3.3. Torus structures. A plane sextic C is said to be of *torus type* if its equation can be represented in the form $p^3 + q^2 = 0$, where p and q are some homogeneous polynomials in (x_0, x_1, x_2) of degree 2 and 3, respectively. Any representation as above (considered up to rescaling) is called a *torus structure* of C. With the exception of a few very degenerate cases, a torus structure is determined by the conic $Q = \{p = 0\}$.

A sextic is of torus type if and only if it is the critical locus of a projection to $\mathbb{P}^2$ of a cubic surface $V \subset \mathbb{P}^3$; the latter is given by $3x_3^3 + 3x_3p + 2q = 0$.

Each point of intersection of the conic $Q = \{p = 0\}$ and cubic $K = \{q = 0\}$ is a singular point of C; such points are called *inner*, and the other singular points that C may have are called *outer*. The type of a simple inner singular point P is determined by the mutual topology of Q and K at P, whereas outer points occur in the family $(\alpha p)^3 + (\beta q)^2 = 0$ under some special values of parameters $\alpha, \beta \in \mathbb{C}^*$. Note that, in the case of non-simple singularities, one can speak about 'outer degenerations' of inner singular points. Thus, with Q and K fixed, an inner point of type $\mathbf{J}_{10} = \mathbf{J}_{2,0}$ may degenerate to $\mathbf{J}_{2,1}$ or $\mathbf{J}_{2,2}$.

3.4. Lemma. *A* $\mathbf{J}$*-sextic* C *is of torus type if and only if there exist sections* $p \in \Gamma(Z; \mathcal{O}_Z(S_0 + 2F))$ *and* $q \in \Gamma(Z; \mathcal{O}_Z(S_0 + 3F))$ *such that the trigonal model* B *of* C *is given by an equation of the form* $p^3 + s_0 q^2 = 0$*, where* $s_0 \in \Gamma(Z; \mathcal{O}_Z(S_0))$ *is a fixed section whose zero set is* S_0*.*

Proof. Let $\bar\varphi = 0$ be an equation of C, and let $\bar\varphi = \bar p^3 + \bar q^2$ be its torus structure. It is well known that the conic $Q = \{\bar p = 0\}$ at O is smooth and tangent to C, the cubic $K = \{\bar q = 0\}$ is singular at O, and the local intersection index of Q and K at O is at least 3. (Indeed, if both K and Q are singular, then C has a quadruple point at O. If K is nonsingular or the local intersection index of Q and K at O is 2, then C has a simple singular point at O.) Then, pulling back, one arrives at $s_0^3 f_0^6 \varphi = (s_0 f_0^2 p)^3 + (s_0^2 f_0^3 q)^2$, where $f_0 \in \Gamma(Z; \mathcal{O}_Z(F_0))$ is a section defining F_0, and the representation as in the statement is obtained by cancelling $s_0^3 f_0^6$.

Conversely, since S_0 is contracted, any representation as in the statement is pushed forward to a torus structure of C. $\square$

Consider a germ φ at a singular point P of type $\mathbf{A}_{3k-1}$ (respectively, $\mathbf{E}_6$) and fix local coordinates (x, y) in which φ is given by $x^{3k} + y^2$ (respectively, $x^4 + y^3$). Let, in the same coordinate system, p be a semiquasihomogeneous germ of type $(k, 1)$ (respectively, $(2, 1)$), and let q be adjacent to a semiquasihomogeneous germ of type $([\frac{1}{2}(3k + 1)], 1)$ (respectively, $(2, 2)$).

3.5. Lemma. *In the notation above, assume that* $\varphi - q^2 = ph_1$ *or* $\varphi - q^2 = p^2 h_2$*. Then* $(h_1 \cdot p)_P \geqslant \frac{1}{2}(3k + 1)$ *(*$(h_1 \cdot p)_P \geqslant 3$ *for* P *of type* $\mathbf{E}_6$*) or, respectively,* $(h_2 \cdot p)_P \geqslant k$ *(*$(h_2 \cdot p)_P \geqslant 2$ *for* P *of type* $\mathbf{E}_6$*), where* $(a \cdot b)_P$ *stands for the local intersection index at* P *of the curves* $\{a = 0\}$ *and* $\{b = 0\}$*.*

Proof. Under the assumptions, the Newton polygon of $\varphi - q^2$ at P is contained in that of φ, and a simple analysis gives a 'lower bound' for the Newton polygons of h_1 and h_2. If the singular point P of φ is of type $\mathbf{E}_6$, then h_1 and h_2 must be adjacent to semiquasihomogeneous germs of type $(3, 2)$ and $(2, 1)$, respectively. If P is of type $\mathbf{A}_{3k-1}$, then h_2 is adjacent to a semiquasihomogeneous germ of type $(k, 1)$, and the Newton polygon of h_1 is a subset of the minimal Newton polygon containing points $(0, 2)$, $([\frac{1}{2}(k+1)], 1)$, and $(2k, 0)$. Now, the estimates for the local intersection indices are immediate. $\square$

3.6. Lemma. *In the notation of Lemma* 3.5, *assume that P is of type $\mathbf{A}_2$ and that q is only adjacent to a semiquasihomogeneous germ of type* $(1,1)$. *Then $(h_1\cdot p)_P \geqslant 1$ and $(h_2 \cdot p)_P \geqslant 1$.*

Proof. The proof is similar to that of Lemma 3.5. Now, we can only assert that $\varphi - q^2$ is singular at P and, hence, h_1 vanishes at P. This proves the first estimate; for the second one, we need to show that h_2 vanishes at P as well. Assume the contrary. Then the singularity of φ at P is equivalent to that of $p^2 + q^2$, which is $\mathbf{A}_{2r-1}$, $r = (p \cdot q)_P$. This is a contradiction. □

3.7. Proof of Theorem 1.2. Fix an irreducible **J**-sextic C whose fundamental group factors to $\mathbb{D}_6$, and consider its trigonal model B. The set of singularities Σ_B is given by Corollary 3.2. If $\Sigma_B = 3\mathbf{A}_2 \oplus \ldots$, we pick and fix three cusps and ignore all other singular points of B.

One has $\dim|S_0 + 2F| = 3$. Hence, there exists a curve $Q = \{p = 0\} \in |S_0 + 2F|$ such that the germ of p at each singular point P of B is as in Lemma 3.5. This curve is necessarily irreducible and, hence, nonsingular. (Indeed, any reducible curve in $|S_0 + 2F|$ is a union of S_0 and two fibers, and such a curve cannot pass through all singular points of B with multiplicities prescribed above.) In particular, from the Bézout theorem it follows that the local intersection indices of Q and B at each point are exactly as in the lemma, *i.e.*, $(Q \cdot B)_P = 2k$ if P is of type $\mathbf{A}_{3k-1}$, and $(Q \cdot B)_P = 2$ if P is of type $\mathbf{E}_6$.

Next, one has $\dim|S_0 + 3F| = 5$, and unless $\Sigma_B = 3\mathbf{A}_2 \oplus \ldots$, there is a curve $K = \{q = 0\} \in |S_0+3F|$ such that the germ of q at each singular point P of B is as in Lemma 3.5. If $\Sigma_B = 3\mathbf{A}_2 \oplus \ldots$, we choose q as in Lemma 3.5 at two of the three cusps and as in Lemma 3.6 at the third one. One can see that K does not contain Q as a component; indeed, otherwise K would split into Q and a fiber F, and a curve of this form cannot satisfy all the conditions imposed. Since $Q \cdot K = 3$, the Bézout theorem implies that, at each singular point P of B, one has $(Q \cdot K)_P = k$ if P is of type $\mathbf{A}_{3k-1}$, and $(Q \cdot K)_P = 2$ if P is of type $\mathbf{E}_6$.

Let $\varphi \in \Gamma(Z; \mathcal{O}_Z(3S_0 + 6F))$ be a section whose zero set is B. Comparing the local intersection indices, one observes that the restrictions $\varphi|_Q$ and $s_0q^2|_Q$ have the same zero divisor. (We multiply q^2 by s_0 to make it a section of the same line bundle as φ; the restriction $s_0|_Q$ is a constant.) Hence, after an appropriate rescaling, the restriction $(\varphi - s_0q^2)|_Q$ is identically zero, *i.e.*, $\varphi - s_0q^2 = ph_1$ for some $h_1 \in \Gamma(Z; \mathcal{O}_Z(2S_0 + 4F))$. Then Lemmas 3.5 and 3.6 imply that $H_1 \cdot Q \geqslant 5$, where $H_1 = \{h_1 = 0\}$. Hence, H_1 contains Q as a component, *i.e.*, $\varphi - s_0q^2 = p^2h_2$, and, applying Lemmas 3.5 and 3.6 and the Bézout theorem once more, one concludes that the curve $H_2 = \{h_2 = 0\}$ coincides with Q. Thus, after another rescaling, $\varphi = p^3 + s_0q^2$, and the statement of Theorem 1.2 follows from Lemma 3.4. □

3.8. Remark. Theorem 1.2 holds for reducible **J**-sextics as well. With the exception of the two degenerate families mentioned after Lemma 2.2, the fundamental group of a reducible **J**-sextic C factors to $\mathbb{D}_6$ if and only if the set of singularities of the trigonal model B of C is $\Sigma_B = \mathbf{A}_5 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$: it is the only root system in $\mathbf{E}_8$ with $\mathbf{E}_8/\Sigma_B$ having both 2- and 3-torsion. (Formally, one would need to replace Corollary 2.8 with an analog of Theorem 3.5.1 in [D7], which would assert that, as in the irreducible case, the fundamental group factors to $\mathbb{D}_6$ if and only if $\mathbf{E}_8/\Sigma_B$ has 3-torsion.) Then, the proof given in Section 3.7 extends literally to $\Sigma_B = \mathbf{A}_5 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$: one just ignores the $\mathbf{A}_1$-point.

3.9. Proof of Theorem 1.2. Unless the set of singularities Σ_B is $4\mathbf{A}_2$, there is a unique quotient $\pi_2(Z \smallsetminus (B \cup S_0)) \twoheadrightarrow \mathbb{D}_6$, see Corollary 2.8 and Lemma 3.1, and Section 3.7 produces a unique torus structure. If $\Sigma_B = 4\mathbf{A}_2$, in the construction of Section 3.7 one can choose any three of the four cusps; then the fourth one is necessarily an outer singularity. Thus, one obtains four distinct torus structures. On the other hand, in $\operatorname{Tors}(\mathcal{K}_B, \mathbb{Z}/3\mathbb{Z}) = (\mathbb{Z}/3\mathbb{Z})^3$ there are exactly four subgroups isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The correspondence between the two sets is established as in [D7]: a cusp P is inner (outer) if and only if the composition $\pi_1(U_P \smallsetminus C) \to \pi_1(\mathbb{P}^2 \smallsetminus C) \to \mathbb{D}_6$ is (respectively, is not) an epimorphism (where U_P is a Milnor ball about P). In terms of the cubic surface $V \subset \mathbb{P}^3$ ramified at C, the three inner cusps are the cusps (Whitney pleats) of the projection $V \to \mathbb{P}^2$, whereas the outer one is the projection of an $\mathbf{A}_2$-singular point of V. □

4. Proof of Theorems 1.4–1.6

We start with a description of two additional models of **J**-sextics.

4.1. The associated cubic. Let B be a trigonal model, and assume that B has a triple singular point P. Blow it up, and blow down the fiber through P and the exceptional section S_0. The result is $\mathbb{P}^2$, and the proper transform of B is a cubic $\bar{D}_3 \subset \mathbb{P}^2$. The inverse transformation is determined by a point $\bar{P} \in \mathbb{P}^2 \smallsetminus \bar{D}_3$ (the image of S_0) and a line $\bar{L}$ through $\bar{P}$ (the image of the exceptional divisor over P). The triple $(\bar{D}_3, \bar{P} \in \bar{L})$ is called the *associated cubic* of B.

The fiber of Z through P contracts to a point; the other fibers are in a one to one correspondence with the lines through $\bar{P}$ other than $\bar{L}$.

4.2. The associated quartic. This construction is similar to the previous one, but now we start with a double singular point P of B. The proper transform of B is a quartic curve $\bar{D}_4 \subset \mathbb{P}^2$, and the inverse transformation is determined by a nonsingular point $\bar{P} \in \bar{D}_4$. (The exceptional divisor over P projects to the tangent $\bar{L}$ to $\bar{D}_4$ at $\bar{P}$.) The pair $(\bar{P} \in \bar{D}_4)$ is called the *associated quartic* of B. The fibers of Z other than that through P transform to the lines through $\bar{P}$ other than $\bar{L}$. Furthermore, the birational map establishes a diffeomorphism $Z \smallsetminus (B \cup S_0 \cup F_P) \cong \mathbb{P}^2 \smallsetminus (\bar{D}_4 \cup \bar{L})$, where F_P is the fiber of Z through P. As a consequence, there is an epimorphism

$$\pi_1(\mathbb{P}^2 \smallsetminus (\bar{D}_4 \cup \bar{L})) \twoheadrightarrow \pi_1(Z \smallsetminus (B \cup S_0)).$$

4.3. Lemma. *If a trigonal model B has a singular fiber of type $\tilde{\mathbf{A}}_0^{**}$ or $\tilde{\mathbf{A}}_1^*$, then the group $\pi_1(Z \smallsetminus (B \cup S_0))$ is abelian. If B has a singular fiber of type $\tilde{\mathbf{A}}_2^*$, then there is an epimorphism $\mathbb{B}_3 \twoheadrightarrow \pi_1(Z \smallsetminus (B \cup S_0))$.*

Proof. The fundamental group $\pi_1(Z \smallsetminus (B \cup S_0))$ can be found using van Kampen's method [vK] applied to the ruling of Z. Remove a nonsingular fiber F_0, pick another nonsingular fiber F', and pick a generic section S disjoint from S_0 and from the critical points of the projection $B \to \mathbb{P}^1$. Let $G = \pi_1(F' \smallsetminus (B \cup S_0), F' \cap S)$, and let α_1, α_2, α_3 be a standard set of generators of G. (Clearly, $F' \smallsetminus (B \cup S_0)$ is a real plane with three punctures.) Let $F_1, \ldots, F_r$ be the singular fibers of B. For each F_j, dragging F' about F_j and keeping the base point in S results in a certain automorphism $m_j \colon G \to G$, called the *braid monodromy* about F_j. Then, the group $\pi_1(Z \smallsetminus (B \cup S_0), F' \cap S)$ has a representation of the form

$$\bigl\langle \alpha_1, \alpha_2, \alpha_3 \bigm| m_j(\alpha_i) = \alpha_i,\ i = 1, 2, 3,\ j = 1, \ldots, r, \text{ and } [\gamma_0] = 1 \bigr\rangle,$$

where γ_0 is a small circle in S about F_0. (The class $[\gamma_0]$ can be expressed in terms of α_i, but this expression is irrelevant for our purposes.)

The same approach can be used to find the group $\pi_1(U_F \smallsetminus (B \cup S_0))$, where F is a fiber of Z, singular or not, and U_F is a tubular neighborhood of F. The resulting representation is $\langle \alpha_1, \alpha_2, \alpha_3, \mid m(\alpha_i) = \alpha_i,\ i = 1, 2, 3\rangle$, where m is the local braid monodromy about F. An immediate and well known consequence is the fact that the inclusion homomorphism $\pi_1(U_F \smallsetminus (B \cup S_0)) \to \pi_1(Z \smallsetminus (B \cup S_0))$ is onto.

The local monodromy can easily be found using model equations; for the fibers as in the statement, it is given by the following expressions:

$\tilde{\mathbf{A}}_0^{**}:\ \alpha_1 \mapsto \alpha_2,\quad \alpha_2 \mapsto \alpha_3,\quad \alpha_3 \mapsto \Pi\alpha_1\Pi^{-1},$
$\tilde{\mathbf{A}}_1^{*}:\ \alpha_1 \mapsto \alpha_3,\quad \alpha_2 \mapsto \alpha_3\alpha_2\alpha_3^{-1},\quad \alpha_3 \mapsto \Pi\alpha_1\Pi^{-1},$
$\tilde{\mathbf{A}}_2^{*}:\ \alpha_1 \mapsto \alpha_3,\quad \alpha_2 \mapsto \Pi\alpha_1\Pi^{-1},\quad \alpha_3 \mapsto \Pi\alpha_2\Pi^{-1},$

where $\Pi = \alpha_1\alpha_2\alpha_3$. Now, it is obvious that, in the first two cases, the group $\pi_1(U_F \smallsetminus (B \cup S_0))$ is free abelian (of rank one and two, respectively), and in the last case, $\pi_1(U_F \smallsetminus (B \cup S_0)) \cong \mathbb{B}_3$. □

4.4. Corollary. *If C is a plane sextic with a singular point of type $\mathbf{E}_{12}$ or $\mathbf{E}_{13}$, then the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ is abelian.*

Proof. Any sextic with a singular point of type $\mathbf{E}_{12}$ or $\mathbf{E}_{13}$ is a $\mathbf{J}$-sextic, and the statement follows from Proposition 2.3, Lemma 4.3, and Table 1. □

4.5. Lemma. *If an irreducible trigonal model B has a triple point and the set of singularities of B is not $\mathbf{E}_6 \oplus \mathbf{A}_2$, then the group $\pi_1(Z \smallsetminus (B \cup S_0))$ is abelian.*

Proof. Consider the associated cubic $(\bar{D}_3, \bar{P} \in \bar{L})$ and denote by $\bar{P}'$ the intersection of $\bar{L}$ and an inflection tangent to $\bar{D}_3$ other than $\bar{L}$. Moving $\bar{P}$ along $\bar{L}$ towards $\bar{P}'$ deforms B to a trigonal model B' with a singular fiber of type $\tilde{\mathbf{A}}_0^{**}$, and the statement of the lemma follows from Lemma 4.3. The only exception is the case when $\bar{L}$ is the only inflection tangent to $\bar{D}_3$; then $\bar{D}_3$ has a cusp and $\Sigma_B = \mathbf{E}_6 \oplus \mathbf{A}_2$. □

4.6. Lemma. *If an irreducible trigonal model B has a simple node $\mathbf{A}_1$ and the set of singularities of B is not $3\mathbf{A}_2 \oplus \mathbf{A}_1$, then the group $\pi_1(Z \smallsetminus (B \cup S_0))$ is abelian.*

Proof. Let $(\bar{P} \in \bar{D}_4)$ be the associated quartic constructed using a node P. Unless $\bar{D}_4$ is a three cuspidal quartic, it has an inflection point $\bar{P}'$, see [D2], and moving $\bar{P}$ towards $\bar{P}'$ deforms B to a trigonal model with a singular fiber of type $\tilde{\mathbf{A}}_1^{*}$. Now, the statement follows from Lemma 4.3. □

4.7. Proof of Theorem 1.4. The sextics mentioned in the statement are precisely those whose trigonal model has the set of singularities $2\mathbf{A}_4$. Let C be such a sextic. Denote by B its trigonal model and let $G = \pi_1(\mathbb{P}^2 \smallsetminus C) = \pi_1(Z \smallsetminus (B \cup S_0)$. Due to Corollary 3.2, G factors to $\mathbb{D}_{10}$; since also $G/[G,G] = \mathbb{Z}/6\mathbb{Z}$, in fact G must factor to $\mathbb{D}_{10} \times (\mathbb{Z}/3\mathbb{Z})$.

Take for P one of the two singular points and consider the associated quartic $(\bar{P} \in \bar{D}_4)$. The set of singularities of $\bar{D}_4$ is $\mathbf{A}_4 \oplus \mathbf{A}_2$, and the tangent $\bar{L}$ at $\bar{P}$ passes through the $\mathbf{A}_2$-point. Such pairs $(\bar{D}_4, \bar{L})$ do exist, see [D2]; this proves the existence of sextics. Furthermore, there is an epimorphism $\tilde{G} = \pi_1(\mathbb{P}^2 \smallsetminus (\bar{D}_4 \cup \bar{L})) \twoheadrightarrow G$, see Section 4.2, and, according to [D4], $\tilde{G}$ is the semidirect product given by the exact sequence

$$1 \to \mathbb{F}_5[t]/(t+1) \to \tilde{G} \to \mathbb{Z} \to 1,$$

where t is the conjugation by the generator of $\mathbb{Z}$. The largest quotient of $\tilde{G}$ whose abelinization is $\mathbb{Z}/6\mathbb{Z}$ is again $\mathbb{D}_{10} \times (\mathbb{Z}/3\mathbb{Z})$. This completes the proof. □

4.8. Proof of Theorem 1.6. Let C be a sextic as in the statement, let B be its trigonal model, and let $\Sigma_B \subset \mathbf{E}_8$ be its set of singularities. Then the group $\mathbf{E}_8/\Sigma$ has neither 2-torsion (since C is irreducible, Proposition 2.6), nor 3-torsion (since C is not of torus type, Theorem 1.2), nor 5-torsion (since C is not as in Theorem 1.4); thus, due to Lemma 2.5, the embedding $\Sigma_B \subset \mathbf{E}_8$ is primitive.

In view of Lemmas 4.5 and 4.6, one can assume that B has neither triple points nor nodes. (The exceptional cases in Lemmas 4.5 and 4.6 are both of torus type.) Thus, $\Sigma_B = \bigoplus \mathbf{A}_{p_i}$, $p_i \geqslant 2$, and applying Nori's theorem [No] to the irreducible sextic with the set of singularities $\mathbf{J}_{10} \oplus \Sigma$ (*i.e.*, considering a non-singular distinguished fiber F_0), one rules out all possibilities with $\sum(p_i+1) < 9$. After this, the only set of singularities left is $\Sigma_B = \mathbf{A}_4 \oplus \mathbf{A}_3$.

Let $\Sigma_B = \mathbf{A}_4 \oplus \mathbf{A}_3$. Take for P the $\mathbf{A}_3$-point and consider the associated quartic ($\bar{P} \in \bar{D}_4$). The set of singularities of $\bar{D}_4$ is $\mathbf{A}_4 \oplus \mathbf{A}_1$, and the tangent $\bar{L}$ at $\bar{P}$ passes through the $\mathbf{A}_1$-point. According to [D4], the group $\pi_1(\mathbb{P}^2 \smallsetminus (\bar{D}_4 \cup \bar{L}))$ is abelian; hence, so is $\pi_1(\mathbb{P}^2 \smallsetminus C)$, see the epimorphism in Section 4.2. □

4.9. Proof of Theorem 1.5. According to Theorem 1.2 and Corollary 3.2, an irreducible $\mathbf{J}$-sextic C satisfies the hypotheses of the theorem if and only if its trigonal model B has one of the following sets of singularities: $3\mathbf{A}_2$, $3\mathbf{A}_2 \oplus \mathbf{A}_1$, $\mathbf{A}_5 \oplus \mathbf{A}_2$, $\mathbf{A}_8$, or $\mathbf{E}_6 \oplus \mathbf{A}_2$.

First, let us show that there is an epimorphism $\mathbb{B}_3 \twoheadrightarrow \pi_1(Z \smallsetminus (B \cup S_0))$. Due to Lemma 4.3, it is the case whenever B has a singular fiber of type $\mathbf{A}_2^*$. Otherwise, take for P one of the $\mathbf{A}_2$-points (or the only $\mathbf{A}_8$-point) and consider the associated quartic ($\bar{P} \in \bar{D}_4$). Its set of singularities is Σ_B with one copy of $\mathbf{A}_2$ removed (respectively, $\mathbf{A}_6$), and the tangent $\bar{L}$ at $\bar{P}$ is a double tangent (respectively, passes through the $\mathbf{A}_6$-point). In each case, one has $\pi_1(\mathbb{P}^2 \smallsetminus (\bar{D}_4 \cup \bar{L})) \cong \mathbb{B}_3$, see [D4], and the desired epimorphism is that of Section 4.2.

One has $\mathbb{B}_3/[\mathbb{B}_3, \mathbb{B}_3] = \mathbb{Z}$, and the central element $\Delta^2 \in \mathbb{B}_3$ projects to $6 \in \mathbb{Z}$. Thus, the largest quotient of $\mathbb{B}_3$ whose abelinization is $\mathbb{Z}/6\mathbb{Z}$ is $\mathbb{B}_3/\Delta^2$, and one obtains epimorphisms

$$\mathbb{B}_3/\Delta^2 \twoheadrightarrow \pi_1(Z \smallsetminus (B \cup S_0)) \twoheadrightarrow \mathbb{B}_3/\Delta^2.$$

(The latter epimorphism is due to the fact that C is of torus type, see Theorem 1.1.) On the other hand, the group $\mathbb{B}_3/\Delta^2 = PSL(2,\mathbb{Z})$ is Hopfian (as it is obviously residually finite). Hence, the two epimorphisms above are isomorphisms. □

4.10. Remark. Alternatively, instead of referring to [D4], one can argue that, in all cases except $\Sigma_B = \mathbf{A}_8$, the trigonal model can be deformed into a curve with a singular fiber of type $\mathbf{A}_2^*$. This procedure corresponds to deforming the associated quartic $\bar{D}_4$ so that the double tangent $\bar{L}$ becomes a line intersecting $\bar{D}_4$ at a single point with multiplicity four. Such a deformation exists due to [D2].

5. An application to the classification

The deformation classification of $\mathbf{J}$-sextics appeared in [D1], but the proof has never been published. Most results of [D1] related to $\mathbf{J}$-sextics can be obtained by passing to the trigonal model B and then, to the associated cubic or associated quartic. (As a matter of fact, in most cases the results are stated in terms of the associated cubic or quartic, with a further reference to [D2].) Most difficult are the cases when the $\mathbf{J}$-sextic has a singular point of type $\mathbf{J}_{2,1}$ or $\mathbf{E}_{12}$, so that one needs to keep track of a singular fiber not passing through a singular point of B, see Table 1. In these cases, one should add the transform $\bar{L}'$ of the singular fiber in question to the associated quartic $\bar{D}_4$, thus reducing the problem to the deformation classification of reducible sextics $\bar{D}_4 + \bar{L} + \bar{L}'$ with simple singularities, see [D6].

Here, we present a simple proof of the 'non-existence' statements of [D1].

5.1. Proposition. *There are no $\mathbf{J}$-sextics with the following sets of singularities*: $\mathbf{J}_{2,i} \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$, $\mathbf{J}_{2,i} \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2$, $\mathbf{J}_{2,i} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$ $(i = 0, 1)$, *and* $\mathbf{J}_{2,1} \oplus 4\mathbf{A}_2$.

Proof. In the first three cases, the trigonal model B of the curve would have set of singularities $\mathbf{A}_3 \oplus 2\mathbf{A}_2$, $\mathbf{A}_4 \oplus 2\mathbf{A}_2$, or $\mathbf{A}_6 \oplus \mathbf{A}_2$. None of these lattices admits an embedding to $\mathbf{E}_8$, see Corollary 2.9. In the last case, B has four cusps, each cusp counting as a triple singular fiber. On the other hand, B always has twelve singular fibers (counted with multiplicity); hence, in the case of four cusps, there are no fibers of type $\tilde{\mathbf{A}}_0^*$, see Table 1. □

5.2. Proposition. *There are no $\mathbf{J}$-sextics with the sets of singularities $\mathbf{E}_{12} \oplus \Sigma$, where either Σ is one of the lattices listed in Lemma* 3.1 *or* $\Sigma = \mathbf{A}_3 \oplus 2\mathbf{A}_2$, $\mathbf{A}_4 \oplus 2\mathbf{A}_2$, *or* $\mathbf{A}_6 \oplus \mathbf{A}_2$.

Proof. If $\Sigma = \Sigma_B$ is one of the lattices listed in Lemma 3.1, the fundamental group of the curve has a dihedral quotient, see Corollary 3.2. This contradicts to Corollary 4.4. The other three sets of singularities do not admit an embedding to $\mathbf{E}_8$, see Corollary 2.9. □

5.3. Remark. For the sets of singularities $\Sigma = 3\mathbf{A}_2$, $\mathbf{A}_5 \oplus \mathbf{A}_2$, $\mathbf{A}_8$, and $2\mathbf{A}_4$, Proposition 5.2 can be interpreted as follows: a trigonal model B with one of these sets of singularities cannot be deformed so that two simplest singular fibers (of type $\tilde{\mathbf{A}}_0^*$) come together to form a singular fiber of type $\tilde{\mathbf{A}}_0^{**}$. (For the other sets of singularities listed in Lemma 3.1 this statement is obvious as the curve has at most one type $\tilde{\mathbf{A}}_0^*$ singular fiber.) Similarly, using the part of Corollary 4.4 concerning $\mathbf{E}_{13}$, one can see that, in the case $\Sigma_B = 3\mathbf{A}_2 \oplus \mathbf{A}_1$, the $\mathbf{A}_1$-point cannot join the remaining singular fiber of type $\tilde{\mathbf{A}}_0^*$ to form a singular fiber of type $\tilde{\mathbf{A}}_1^*$.

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Department of Mathematics, Bilkent University, 06800 Ankara, Turkey
E-mail address: degt@fen.bilkent.edu.tr