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FINITENESS AND QUASI-SIMPLICITY FOR SYMMETRIC $K3$-SURFACES

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Abstract. We compare the smooth and deformation equivalence of actions of finite groups on $K3$-surfaces by holomorphic and anti-holomorphic transformations. We prove that the number of deformation classes is finite and, in a number of cases, establish the expected coincidence of the two equivalence relations. More precisely, in these cases we show that an action is determined by the induced action in the homology. On the other hand, we construct two examples to show that, first, in general the homological type of an action does not even determine its topological type, and second, that $K3$-surfaces X and $\bar X$ with the same Klein action do not need to be equivariantly deformation equivalent even if the induced action on $H^{2,0}(X)$ is real, *i.e.*, reduces to multiplication by ±1.

1. Introduction

1.1. Questions. In this paper, we study equivariant deformations of complex $K3$-surfaces with symmetry groups, where by a symmetry we mean an either holomorphic or anti-holomorphic transformation of the surface. Although the automorphism group of a particular $K3$-surface may be infinite, we confine ourselves to finite group actions and address the following two questions (see 1.4–1.6 for precise definitions):

finiteness: whether the number of actions, counted up to equivariant deformation and isomorphism, is finite, and

quasi-simplicity: whether the differential topology of an action determines it up to the above equivalence.

The response to the second question, in the way that it is posed, is obviously in the negative. For example, given an action on a surface X, the same action on the complex conjugate surface $\bar X$ is diffeomorphic to the original one but often not deformation equivalent to it. Thus, we pose this question in a somewhat weaker form:

weak quasi-simplicity: does the differential topology of an action determine it up to equivariant deformation and (anti-)isomorphism?

Up to our knowledge, these questions have never been posed explicitly, and, moreover, despite numerous related partial results, they both remained open.

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One may notice a certain ambiguity in the statement of the above questions, especially in what concerns the quasi-simplicity: we do not specify whether we consider diffeomorphic actions on true $K3$-surfaces or, more generally, diffeomorphic actions on surfaces diffeomorphic to a $K3$-surface. Fortunately, a surface diffeomorphic to a $K3$-surface is a $K3$-surface, see [FM2], and the two versions turn out to be equivalent. Thus, we confine ourselves to true $K3$-surfaces and respond to both the finiteness and (to great extend) weak quasi-simplicity questions (see 1.7).

1.2. A brief retrospective of the method. Following the founding work by I. Piatetski-Shapiro and I. Shafarevich [PSS], we base our study on the global Torelli theorem. When combined with Vik. Kulikov's theorem on surjectivity of the period map [K], this fundamental result essentially reduces the finiteness and quasi-simplicity questions to certain arithmetic problems. It is this approach that was used by V. Nikulin in [N2] and [N3], where he established (partially implicitly) the finiteness and quasi-simplicity results for polarized $K3$-surfaces with symplectic actions of finite abelian groups and for those with real structures. (Partial preliminary results, based on the injectivity of the period map, are found in [N1], for symplectic actions, and in [Kh], for real structures.) In [DIK], we extended these results to real Enriques surfaces (which can be regarded as $K3$-surfaces with certain actions of $\mathbb{Z}_2 \times \mathbb{Z}_2$). In fact, [N2], [N3], and [DIK] give a complete deformation classification of respective surfaces. It is while studying real Enriques surfaces that we got interested in the above questions and obtained our first results in this direction.

In all cases above one starts with using the global Torelli theorem to show that the deformation class of a surface is determined by the induced action in its 2-homology and thus reduce the problem to (sometimes quite elaborate) study of the induced action. One of our principal results (Theorem 1.7.2) extends this statement to a wide class of actions, thus making it possible to complete the classification in many cases. (For example, G. Xiao's paper [X] seems very promising in classifying $K3$-surfaces with symplectic finite group actions; eventually, it reduces the study of the induced actions to the study of certain definite sublattices in the homology of the orbit space, which is also a $K3$-surface in this case.) On the other hand, in 6.1.1 we construct an example of an action of a relatively simple group (the dihedral group of order six) whose deformation and topological types cannot be read from the homology. The study of such actions would require new tools that would let one enumerate the walls in the period space that do matter.

1.3. Related results. One can find a certain similarity between our finiteness results and the finiteness in theory of moduli of complex structures on 4-manifolds, which states (see [FM1] and [F]) that the moduli space of Kählerian complex structures on a given underlying differentiable compact 4-manifold has finitely many components. (By Kählerian we mean a complex structure admitting a Kähler metric. In the case of surfaces this is a purely topological restriction: the complex structures on a given compact 4-manifold X are Kählerian if and only if the first Betti number $b_1(X;\mathbb{Q})$ is even.) Moreover, the moduli space is connected as soon as there is a Kählerian representative of Kodaira dimension $\leqslant 0$ (as it is the case for $K3$-surfaces and complex 2-tori); for Kodaira dimension one, there are at most two deformation classes, which are represented by X and $\bar{X}$, see [FM1]. Examples of general type surfaces X not deformation equivalent to $\bar{X}$ are found in [KK] and [Ca].

The principle results of our paper can be regarded as an equivariant version of the above statements for $K3$-surfaces. The finiteness theorem 1.7.1 is closely related to a series of results from theory of algebraic groups that go back to C. Jordan [J]. The original Jordan theorem states that $SL(n, \mathbb{Z})$ contains but a finite number of conjugacy classes of finite subgroups. A. Borel and Harish-Chandra, see [BH] and [B], generalized this statement to any arithmetic subgroup of an algebraic group; further recent generalizations are due to V. Platonov, see [Pl]. Note that, together with the global Torelli theorem, these Jordan type theorems (applied to the 2-cohomology lattice of a $K3$-surface) imply that the number of different finite groups acting faithfully on $K3$-surfaces is finite. A complete classification of finite groups acting symplectically (*i.e.*, identically on holomorphic forms) on $K3$-surfaces is found in Sh. Mukai [Mu] (see also Sh. Kondō [Ko1] and G. Xiao [X]; the abelian groups where first classified by Nikulin [N3]; unlike Mukai, who only listed the groups, Nikulin gave a description of the homological actions (*cf.* 1.2) and their moduli spaces and showed that the latter are connected). A sharp bound on the order of a group acting holomorphically on a $K3$-surface is given by Kondō [Ko2]; it is based on Nikulin's bound on the restriction of the induced action to the group of transcendental cycles. Here, as in the study of the components of the moduli space, the crucial starting point is a thorough analysis of the transcendental part of the action over $\mathbb{Q}$ (*cf.* almost geometric actions in 2.6); it was originated in [N3].

Among other related finiteness results found in the literature, we would like to mention a theorem by Piatetski-Shapiro and Shafarevich [PSS] stating that the automorphism group of an algebraic $K3$-surface is finitely generated, our [DIK] generalization of this theorem to all $K3$-surfaces, and H. Sterk's [St] finiteness results on the classes of irreducible curves on an algebraic $K3$-surface. Note that all these results deal with individual surfaces rather than with their deformation classes. They are related to the finiteness of the number of conjugacy classes of finite subgroups in the group of Klein automorphisms of a given variety. As a special case, one can ask whether the number of conjugacy classes of real structures on a given variety is finite. For the latter question, the key tool is the Borel-Serre [BS] finiteness theorem for Galois cohomology of finite groups; as an immediate consequence, it implies finiteness of the number of conjugacy classes of real structures on an abelian variety. In [DIK] we extended this statement to all surfaces of Kodaira dimension $\geqslant 1$ and to all minimal Kähler surfaces. Remarkably, finiteness of the number of conjugacy classes of real structures on a given rational surface is still an open question.

Unlike finiteness, the quasi-simplicity question does not make much sense for individual varieties. In the past, it was mainly studied for deformation equivalence of real structures: given a deformation family of complex varieties, is a real variety within this family determined up to equivariant deformation by the topology of the real structure? The first non trivial result in this direction, concerning real cubic surfaces in $\mathbb{P}^3$, was discovered by F. Klein and L. Schläfli (see, *e.g.*, the survey [DK1]). At present, the answer is known for curves (essentially due to F. Klein and G. Weichold, see, *e.g.*, the survey [Na]), complex tori (essentially due to A. Comessatti [Co]), rational surfaces (A. Degtyarev and V. Kharlamov [DK2]), ruled surfaces (J.-Y. Welschinger [Wel]), $K3$-surfaces (essentially due to Nikulin [N2]), Enriques surfaces (see [DIK]), hyperelliptic surfaces (F. Catanese and P. Frediani [CF]), and some sporadic surfaces of general type (*e.g.*, so called *Bogomolov-Miayoka-Yau surfaces*, see Kharlamov and Kulikov [KK]).

Note that for the above classes of special surfaces topological invariants that determine the deformation class are known. Together with the quasi-simplicity, this implies finiteness (as the invariants take values in finite sets). Finiteness also holds for varieties of general type (in any dimension), as for such varieties the Hilbert scheme is quasi-projective.

1.4. Terminology conventions. Unless stated otherwise, all complex varieties are supposed to be nonsingular, and differentiable manifolds are C^∞. A *real variety* (X, conj) is a complex variety X equipped with an anti-holomorphic involution conj. In spite of the fact that we work with anti-holomorphic transformations as well, we reserve the term *isomorphism* for bi-holomoprhic maps, whereas using *anti-isomorphism* for bi-anti-holomorphic ones.

When working with the period spaces, it is convenient to equip a $K3$-surface X with the fundamental class γ_X of a Kähler structure on X. We call γ_X a *polarization* of X. Strictly speaking, since we do not assume γ_X ample (nor even do we assume that X is algebraic or γ_X is an integral class), it would probably be more appropriate to invent a different term (quasi-polarization, K-polarization, Kählerization, ...). However, as in this paper it does not lead to a confusion, we decided to avoid awkward language and use a familiar term in a slightly more general sense.

1.5. Augmented groups and Klein actions. An *augmented group* is a finite group G supplied with a homomorphism $\kappa\colon G \to \{\pm 1\}$. (We do not exclude the case when κ is trivial.) Denote the kernel of κ by G^0. A *Klein action* of a group G on a complex variety X is a group action of G on X by both holomorphic and anti-holomorphic maps. Assigning $+1$ (respectively, -1) to an element of G acting holomorphically (respectively, anti-holomorphically), one obtains a natural augmentation $\kappa\colon G \to \{\pm 1\}$. An action is called *holomorphic* (respectively, *properly Klein*) if $\kappa = 1$ (respectively, $\kappa \neq 1$).

Replacing the complex structure J on a complex variety X with its conjugate $(-J)$, one obtains another complex variety, commonly denoted by $\bar{X}$, with the same underlying differentiable manifold. An automorphism of X is as well an automorphism of $\bar{X}$; it can also be regarded as an anti-holomorphic bijection between X and $\bar{X}$. Thus, a Klein G-action on X can as well be regarded as a Klein action on $\bar{X}$, with the same augmentation $\kappa\colon G \to \{\pm 1\}$ and the same subgroup G^0. These two actions are obviously diffeomorphic, but they do not need to be isomorphic.

A Klein action of a group G on a complex variety X gives rise to the induced action $G \to \operatorname{Aut} H^*(X)$, $g \mapsto g^*$, in the cohomology ring of X. Since we deal with $K3$-surfaces, which are simply connected, and since all elements of G are orientation preserving in this dimension, the induced action reduces essentially to the action on the group $H^2(X)$, regarded as a lattice via the intersection index form. For our purpose, it is more convenient to work with the *twisted induced action* $\theta_X\colon G \to \operatorname{Aut} H^2(X)$, $g \mapsto \kappa(g) g^*$. The latter, considered up to conjugation by lattice automorphisms, is called the *homological type* of the original Klein action on X. Clearly, it is a topological invariant.

1.6. Smooth deformations. A (*smooth*) *family*, or *deformation*, of complex varieties is a proper submersion $p\colon X \to S$ with differentiable, not necessarily compact or complex, manifolds X, S supplied with a fiberwise integrable complex structure on the bundle $\operatorname{Ker} dp$. The varieties $X_s = p^{-1}(s)$, $s \in S$, are called *members* of the family. Given a group G, a family $p\colon X \to S$ is called *G-equivariant* if it is supplied with a smooth fiberwise G-action that restricts to a Klein action on each fiber.

Two complex varieties X, Y supplied with Klein actions of a group G are called *equivariantly deformation equivalent* if there is a chain $X = X_0, X_1, \dots, X_k$ of complex varieties X_i with Klein actions of G such that for each $i = 0, \dots, k-1$ the varieties X_i and X_{i+1} are G-isomorphic to members of a G-equivariant smooth family. (By a G-isomorphism we mean a bi-holomoprhic map ϕ such that $\phi g = g\phi$ for any $g \in G$.)

Clearly, the equivariant deformation equivalence is an equivalence relation, G-equivariantly deformation equivalent varieties are G-diffeomorphic, and the homological type of a G-action is a deformation invariant.

1.7. The principal results. Let X be a $K3$-surface with a Klein action of a finite group G. Then G^0 acts on the subspace $H^{2,0}(X) \cong \mathbb{C}$, which gives rise to a natural representation $\rho\colon G^0 \to \mathbb{C}^*$. If G is finite, the image of ρ belongs to the unit circle $S^1 \subset \mathbb{C}^*$. We will refer to ρ as the *fundamental representation* associated with the original Klein action. It is a deformation but, in general, not topological invariant of the action. A typical example is the same Klein action on $\bar{X}$; its associated fundamental representation is the conjugate $\bar{\rho}\colon g \mapsto \overline{\rho g} \in \mathbb{C}^*$.

As shown below (see 4.3.1), in the case of finite group actions on a $K3$-surface X the twisted induced action θ_X determines the subgroup G^0 and 'almost' determines the fundamental representation $\rho\colon G^0 \to S^1$: from θ_X, one can recover a pair ρ, $\bar{\rho}$ of complex conjugate fundamental representations.

1.7.1. Finiteness Theorem. *The number of equivariant deformation classes of $K3$-surfaces with faithful Klein actions of finite groups is finite.*

1.7.2. Quasi-simplicity Theorem. *Let X and Y be two $K3$-surfaces with finite group G Klein actions of the same homological type. Assume that either*

(1) *the action is holomorphic, or*
(2) *the associated fundamental representation ρ is real*, i.e., $\rho = \bar{\rho}$.

Then either X or $\bar{X}$ is G-equivariantly deformation equivalent to Y. If the associate fundamental representation is trivial, then X and $\bar{X}$ are G-equivariantly deformation equivalent.

Remark. If ρ is non-real, the deformation classes of X and $\bar{X}$ are distinguished by the fundamental representation (ρ and $\bar{\rho}$). In 6.4.1 we give an example when X and $\bar{X}$ are not deformation equivalent even though ρ is real.

Remark. In 6.1.1 we discuss another example, that of a properly Klein action of the dihedral group $\mathbb{D}_3$ whose deformation class is not determined by its homological type and associated fundamental representation. However, the actions constructed differ by their topology. Thus, they do not constitute a counter-example to quasi-simplicity of $K3$-surfaces (in its weaker form), and the problem still remains open.

Note that this phenomenon is somewhat unusual and unexpected for $K3$-surfaces, as in all examples known before, such as (real) $K3$-surfaces, (real) Enriques surfaces, $K3$-surfaces with an involution, the deformation class (and, hence, the topological type of the action) can be read from the induced action on the homology. Though, all these examples are covered by Theorem 1.7.2.

A real variety (X, conj) with a real (*i.e.*, commuting with conj) holomorphic G^0-action can be regarded as a complex variety with a Klein action of the extended group $G = G^0 \times \mathbb{Z}_2$, the $\mathbb{Z}_2$-factor being generated by conj. Note that, if X

is a $K3$-surface with a real holomorphic G^0-action, the associated fundamental representation $\rho\colon G^0 \to \mathbb{C}^*$ is real.

1.7.3. Corollary. *Let X and Y be two real $K3$-surfaces with real holomorphic G^0-actions, so that the extended Klein actions of $G = G^0 \times \mathbb{Z}_2$ have the same homological type. Then X and Y are G-equivariantly deformation equivalent.* □

The methods used in the paper can as well be applied to the study of finite group Klein actions on 2-dimensional complex tori. (The corresponding version of global Torelli theorem was first discovered by Piatetski-Shapiro and Shafarevich [PSS] and then corrected by T. Shioda [Shi]). The analogs of 1.7.1 and 1.7.2 for 2-tori are Theorems A.1.1 (finiteness) and A.1.2 (quasi-simplicity) proved in Appendix A. For holomorphic actions preserving a point this is a known result; it is contained in the classification of finite group actions on 2-tori by A. Fujiki [Fu], where a complete description of the moduli spaces is also given. (The results for holomorphic actions on Jacobians go back to F. Enriques and F. Severi [ES], and on general abelian surfaces, back to G. Bagnera and M. de Franchis [BdF].) We give a short proof not using the classification, extend the results to nonlinear Klein actions, and compare the complex conjugated actions. As a straightforward consequence, we obtain analogous results for hyperelliptic surfaces. A number of tools used in Appendix A are close to those used by Fujuki in his study of the relation between symplectic actions and root systems.

Note that Theorem A.1.2 is stronger than its counterpart 1.7.2 for $K3$-surfaces: one does not need any additional assumption on the action. On the other hand, we show that, in quite a number of cases, a 2-torus X is not equivariantly deformation equivalent to $\bar{X}$ (see A.4).

Together, Theorems 1.7.1, 1.7.2 and A.1.1, and A.1.2 give finiteness and quasi-simplicity results for $K3$-surfaces, Enriques surfaces, 2-tori, and hyperelliptic surfaces, *i.e.*, for all Kähler surfaces of Kodaira dimension 0.

Among other results, not directly related to the proofs, worth mentioning is Theorem 5.2.1, which compares the homological types of Klein actions on a singular $K3$-surface and on close nonsingular ones. There also is a generalization that applies to any surface provided that the singularities are simple.

1.8. Idea of the proof. As it has already been mentioned, our study is based on the global Torelli theorem. As is known, in order to obtain a good period space, one should *mark* the $K3$-surfaces, *i.e.*, fix isomorphisms $H^2(X) \to L = 2E_8 \oplus 3U$ (see 1.10 for the notation). Technically, it is more convenient to deal with the period space $K\Omega_0$ of marked polarized $K3$-surfaces, which, in turn, is a sphere bundle over the period space Per_0 of marked Einstein $K3$-surfaces (see 4.1 for details). According to Kulikov [K], one has $\operatorname{Per}_0 = \operatorname{Per} \smallsetminus \Delta$, where Per is a contractible homogeneous space (the space of positive definite 3-subspaces in $L \otimes \mathbb{R}$) and Δ is the set of the subspaces orthogonal to roots of L.

Now, we fix a finite group G and an action $\theta\colon G \to \operatorname{Aut} L$. This gives rise to the equivariant period spaces $K\Omega_0^G$ and $\operatorname{Per}_0^G = \operatorname{Per}^G \smallsetminus \Delta$ of marked $K3$-surfaces with the given homological type of Klein G-action. Note that we are only interested in *geometric* actions, *i.e.*, those for which the spaces Per_0^G or $K\Omega_0^G$ are non-empty. Given a $K3$-surface, its markings compatible with θ differ by elements of the group $\operatorname{Aut}_G L$ of the automorphisms of L commuting with G. Thus, the finiteness and the (weak) quasi-simplicity problems reduce essentially to the study of the set of connected components of the orbit space $\mathfrak{M}^G = \operatorname{Per}_0^G/\operatorname{Aut}_G L$. In fact, the desired result (connectedness or finiteness of the number of connected components) can be obtained with a smaller group $A \subset \operatorname{Aut}_G L$, depending on the nature of the action. (A description of such 'underfactorized' moduli spaces is given in 4.4.2–4.4.7.) Furthermore, the quotient space Per_0^G/A can be replaced with a subspace $\operatorname{Int}\Gamma \smallsetminus \Delta$, where Γ is an appropriate convex (hence, connected) fundamental domain of the action of A on Per^G, and it remains to enumerate the *walls* in $\operatorname{Int}\Gamma$, *i.e.*, the strata of $\Delta \cap \operatorname{Int}\Gamma$ of codimension 1.

1.9. Contents of the paper. In Section 2 we give the basic definitions and cite some known results on lattices and group actions on them. In 2.6 we introduce the notion of *almost geometric* actions. This notion can be regarded as the '$\mathbb{Z}$-independent' (*i.e.*, defined over $\mathbb{R}$) part of the necessary condition for an action to be realizable by a $K3$-surface. We study the invariant subspaces of an almost geometric action and show, in particular, that such an action determines the augmentation of the group and, up to complex conjugation, the associated fundamental representation.

In Section 3 we introduce and study *geometric* actions, which we define in arithmetical terms. The main goal of this section are Theorems 3.1.2 and 3.1.3, which establish certain connectedness and finiteness properties of appropriate fundamental domains of groups of automorphisms of the lattice preserving a given geometric action.

In Section 4 we introduce the equivariant period and moduli spaces and show that an action on the lattice is geometric (in the sense of Section 3) if and only if it is realizable by a $K3$-surface. We give a detailed description of certain 'underfactorized' moduli spaces and use it to prove the main results.

Section 5 deals with equivariant degenerations of $K3$-surfaces: we discuss the behaviour of the twisted induced action along the walls of the period space.

In Section 6 we discuss two examples to show that, in general, the deformation type of a Klein action is not determined by its homological type and associated fundamental representation.

In Appendix A we treat the case of 2-tori.

1.10. Common notation. We freely use the notation $\mathbb{Z}_n$ and $\mathbb{D}_n$ for the cyclic group of order n and dihedral group of order $2n$, respectively. We use A_n, D_n, E_6, E_7, and E_8 for the even **negative** definite lattices generated by the root systems of the same name, and U, for the hyperbolic plane (indefinite unimodular even lattice of rank 2). All other non-standard symbols are explained in the text.

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2. Actions on lattices

2.1. Lattices. An (*integral*) *lattice* is a free abelian group L of finite rank supplied with a symmetric bilinear form $b\colon L \otimes L \to \mathbb{Z}$. We usually abbreviate $b(v, w) = v \cdot w$ and $b(v, v) = v^2$. For any ring $\Lambda \supset \mathbb{Z}$ we use the same notation b (as well as

$v \cdot w$ and v^2) for the linear extension $(v \otimes \lambda) \otimes (w \otimes \mu) \mapsto (v \cdot w)\lambda\mu$ of b to $L \otimes \Lambda$. A lattice L is called *even* if $v^2 = 0 \bmod 2$ for all $v \in L$; otherwise, L is called *odd*. Let $L^\vee = \operatorname{Hom}(L, \mathbb{Z})$ be the dual abelian group. The lattice L is called *nondegenerate* (*unimodular*) if the *correlation homomorphism* $L \to L^\vee$, $v \mapsto b(v, \cdot)$, is a monomorphism (respectively, isomorphism). The cokernel of the correlation homomorphism is called the *discriminant group* of L and denoted by $\operatorname{discr} L$. The group $\operatorname{discr} L$ is finite (trivial) if and only if L is nondegenerate (respectively, unimodular).

The assignment $(x \bmod L, y \bmod L) \mapsto (x \cdot y) \bmod \mathbb{Z}$, $x, y \in L^\vee$ is a well defined bilinear form $b\colon \operatorname{discr} L \otimes \operatorname{discr} L \to \mathbb{Q}/\mathbb{Z}$. If L is even, there also is a quadratic extension $q\colon \operatorname{discr} L \to \mathbb{Q}/2\mathbb{Z}$ of b given by $x \bmod L \mapsto x^2 \bmod 2\mathbb{Z}$.

Given a lattice L, we denote by $\sigma_+ L$ and $\sigma_- L$ its inertia indexes and by $\sigma L = \sigma_+ L - \sigma_- L$, its signature. We call a nondegenerate lattice L *elliptic* (respectively, *hyperbolic*) if $\sigma_+ L = 0$ (respectively, $\sigma_+ L = 1$). The terminology is not quite standard: we change the sign of the forms, and we treat a positive definite lattice of rank 1 as hyperbolic. This is caused by the fact that our lattices are related (explicitly or implicitly) to the Neron-Severi groups of complex surfaces.

A sublattice $M \subset L$ is called *primitive* if the quotient L/M is torsion free. Given a sublattice $M \subset L$, we denote by $M\hat{}$ its *primitive hull* in L, *i.e.*, the minimal primitive sublattice containing M: $M\hat{} = \{v \in L \mid kv \in M \text{ for some } k \in \mathbb{Z},\ k \neq 0\}$.

An element $v \in L$ of square (-2) is called a *root*.[1] A *root system* is a lattice generated (over $\mathbb{Z}$) by roots. Recall that any elliptic root system decomposes, uniquely up to order of the summands, into orthogonal sum of irreducible elliptic root systems, *i.e.*, those of type A_n, D_n, E_6, E_7, or E_8.

2.2. Automorphisms. An *isometry* (*dilation*) of a lattice L is an automorphism $a\colon L \to L$ preserving the form (respectively, multiplying the form by a fixed number $\neq 0$.) All isometries of L constitute a group; we denote it by $\operatorname{Aut} L$. If L is nondegenerate, there is a natural representation $\operatorname{Aut} L \to \operatorname{Aut} \operatorname{discr} L$. Denote its kernel $\operatorname{Aut}^0 L$. It is a finite index normal subgroup of $\operatorname{Aut} L$ consisting of the 'universally extensible' automorphisms. More precisely, an automorphism a of L belongs to $\operatorname{Aut}^0 L$ if and only if a extends to any suplattice $L' \supset L$ identically on $L^\perp$.

Given a vector $v \in L$, $v^2 \neq 0$, denote by $\mathfrak{s}_v$ the reflection against the hyperplane orthogonal to v, *i.e.*, the isometry of $L \otimes \mathbb{R}$ defined by $x \mapsto x - ((x \cdot v)/v^2)v$. If $\mathfrak{s}_v(L) \subset L$ (which is always the case when $v^2 = \pm 1$ or ± 2), we use the same notation for the induced automorphism of L. The subgroup $W(L) \subset \operatorname{Aut} L$ generated by the reflections against the hyperplanes orthogonal to roots of L is called the *Weil group* of L. Clearly, $W(L)$ is a normal subgroup of $\operatorname{Aut} L$ and $W(L) \subset \operatorname{Aut}^0 L$.

We recall a few facts on automorphisms of root systems; details can be found, *e.g.*, in [Bou]. Let R be an elliptic root system. The hyperplanes orthogonal to roots in R divide the space $R \otimes \mathbb{R}$ into several connected components, called *cameras* of R, and the Weil group $W(R)$ acts transitively on the set of cameras. For each camera C of R there is a canonical semi-direct product decomposition $\operatorname{Aut} R = W(R) \rtimes S_C$, where $S_C \subset O(R \otimes \mathbb{R})$ is the group of symmetries of C. (As an abstract group, S_C can be identified with the group of symmetries of the Dynkin diagram of R.) In particular, if an element $g \in \operatorname{Aut} R$ preserves C, one has $g \in S_C$. More generally, if g preserves a face $C' \subset C$, then in the decomposition $g = sw$,

[1] Traditionally, the roots are the elements of square (-2) or (-1). We exclude the case of square (-1) as we only consider even lattices.

$s \in S_C$, the element w belongs to the Weil group of the root system generated by the roots of R orthogonal to C'.

2.3. Actions. Let G be a group. A G-action on a lattice L is a representation $\theta\colon G \to \operatorname{Aut} L$. In what follows we always assume G finite. Given a ring $\Lambda \supset \mathbb{Z}$, we use the same notation θ for the extension $g \mapsto \theta g \otimes \mathrm{id}_\Lambda$ of the action to $L \otimes \Lambda$. Denote by $\operatorname{Aut}_G(L \otimes \Lambda)$ the group of G-equivariant Λ-isometries of $L \otimes \Lambda$, *i.e.*, the centralizer of θG in $\operatorname{Aut}(L \otimes \Lambda)$, and let $W_G(L) = W(L) \cap \operatorname{Aut}_G L$ and $\operatorname{Aut}^0_G L = \operatorname{Aut}^0 L \cap \operatorname{Aut}_G L$.

A submodule $M \subset L \otimes \Lambda$ is called *G-invariant* if $\theta g(M) \subset M$ for any $g \in G$; it is galled *G-characteristic* if $a(M) \subset M$ for any $a \in \operatorname{Aut}_G(L \otimes \Lambda)$.

Let $\mathbb{K} \subset \mathbb{C}$ be a field. For an irreducible $\mathbb{K}$-linear representation ξ of G, we denote by $L_\xi(\mathbb{K})$ the ξ-isotypic subspace of $L \otimes \mathbb{K}$, *i.e.*, the maximal invariant subspace of $L \otimes \mathbb{K}$ that is a sum of irreducible representations isomorphic to ξ. Given a subfield $\Bbbk \subset \mathbb{K}$, denote by $L_\xi(\Bbbk)$ the minimal $\Bbbk$-subspace of $L \otimes \Bbbk$ such that $L_\xi(\Bbbk) \otimes_\Bbbk \mathbb{K} \supset L_\xi(\mathbb{K})$, and for a subring $\mathfrak{O} \subset \Bbbk$, $\mathfrak{O} \ni 1$, let $L_\xi(\mathfrak{O}) = L_\xi(\Bbbk) \cap (L \otimes \mathfrak{O})$. Clearly, $L_\xi(\Bbbk)$ is the space of an isotypic $\Bbbk$-representation of G, and $L_\xi(\mathfrak{O})$ is G-invariant and G-characteristic. If $\Bbbk$ is an algebraic number field and $\mathfrak{O}$ is an order in $\Bbbk$, then $L_\xi(\mathfrak{O})$ is a finitely generated abelian group and $L_\xi(\Bbbk) = L_\xi(\mathfrak{O}) \otimes_\mathfrak{O} \Bbbk$.

We use the shortcut L^G for $L_1(\mathbb{Z}) = \{x \in L \mid gx = x \text{ for all } g \in G\}$.

2.4. Extending automorphisms. Below, we recall a few simple facts on extending automorphisms of lattices. All the results still hold if the lattices involved are supplied with an action of a finite group G and the automorphisms are G-equivariant. One can also consider lattices defined over an order in an algebraic number field.

2.4.1. Lemma. *Let M be a nondegenerate lattice and $M' \subset M$ a sublattice of finite index. Then the groups $\operatorname{Aut} M$ and $\operatorname{Aut} M'$ have a common finite index subgroup.* $\square$

2.4.2. Lemma. *Let M be a lattice and $M' \subset M$ a nondegenerate sublattice. Then the group of automorphisms of M' extending to M has finite index in $\operatorname{Aut} M'$.* $\square$

2.4.3. Lemma. *Let M be a nondegenerate lattice and A a group acting by isometries on $M \otimes \mathbb{Q}$. Assume that there is a finite index sublattice $M' \subset M$ such that $a(M') \subset M$ for any $a \in A$. Then A has a finite index subgroup acting on M.*

Proof. It suffices to apply 2.4.1 to the A-invariant sublattice $\sum_{a \in A} a(M') \subset M$. $\square$

2.4.4. Corollary. *Let M^+ and M^- be two nondegenerate lattices and $J\colon M^- \to M^+$ a dilation invertible over $\mathbb{Q}$. Then there exists a finite index subgroup $A^+ \subset \operatorname{Aut} M^+$ such that the correspondence $a \mapsto a \oplus J^{-1}aJ$ restricts to a well defined homomorphism $A^+ \to \operatorname{Aut}(M^+ \oplus M^-)$.* $\square$

2.5. Fundamental polyhedra. Given a real vector space V with a nondegenerate quadratic form, we denote by $\mathcal{H}(V)$ the space of maximal positive definite subspaces of V. Note that $\mathcal{H}(V)$ is a contractible space of non positive curvature. If $\sigma_+ V = 1$ (*i.e.*, V is hyperbolic), one can define $\mathcal{H}(V)$ as the projectivization $\mathcal{C}(V)/\mathbb{R}^*$ of the positive cone $\mathcal{C}(V) = \{x \in V \mid x^2 > 0\}$.

Fix an algebraic number field $\Bbbk \subset \mathbb{R}$ and let $\mathfrak{O}$ be the ring of integers of $\Bbbk$. Consider a hyperbolic integral lattice M and a hyperbolic sublattice $M' \subset M \otimes \Bbbk$ defined over $\mathfrak{O}$, *i.e.*, such that $\mathfrak{O}M' \subset M'$. Let $\mathcal{H}' = \mathcal{H}(M' \otimes_\mathfrak{O} \mathbb{R})$. Then any group A acting by isometries on M and preserving M' acts on $\mathcal{H}'$. Since M is a hyperbolic integral lattice and $(M')^\perp \subset M$ is elliptic, the induced action is discrete, and the Dirichlet domain with center at a generic $\Bbbk$-rational point of $\mathcal{H}'$ is a $\Bbbk$-rational polyhedral fundamental domain of the action. Any such domain will be called a *rational Dirichlet polyhedron* of A (in $\mathcal{H}'$).

The following theorem treats the classical case where $M = M'$ is an integral lattice and $A = \operatorname{Aut} M$. It is due to C. L. Siegel [Sie], H. Garland, M. S. Raghunathan [GR], and N. J. Wielenberg [Wie].

2.5.1. Theorem. *Let M be a hyperbolic integral lattice. Then the rational Dirichlet polyhedra of the full automorphism group $\operatorname{Aut} M$ in $\mathcal{H}(M)$ are finite. Unless M has rank 2 and represents 0, the polyhedra have finite volume.* □

2.5.2. Corollary. *Let M be a hyperbolic integral lattice. Then the closure in $\mathcal{H}(M) \cup \partial\mathcal{H}(M)$ of any rational Dirichlet polyhedron of $\operatorname{Aut} M$ in $\mathcal{H}(M)$ is the convex hull of a finite collection of rational points.* □

2.6. The fundamental representations. Let $\theta\colon G \to \operatorname{Aut} L$ be a finite group action on a nondegenerate lattice L with $\sigma_+ L = 3$. We will say that θ is *almost geometric* if there is a G-invariant flag $\ell \subset \mathfrak{w}$, where $\mathfrak{w} \subset L \otimes \mathbb{R}$ is a positive definite 3-subspace and ℓ is a 1-subspace with trivial G-action.

2.6.1. Lemma. *Let $\theta\colon G \to \operatorname{Aut} L$ be a finite group action on a lattice L with $d = \sigma_+ L > 0$. Then, for any positive definite G-invariant d-subspace $\mathfrak{w} \subset L \otimes \mathbb{R}$, the induced action $\theta_{\mathfrak{w}}\colon G \to O(\mathfrak{w}) = O(d)$ is determined by θ up to conjugation in $O(d)$. In particular, the augmentation $\kappa\colon G \to O(\mathfrak{w}) \xrightarrow{\det} \{\pm 1\}$ is uniquely determined by θ.*

Proof. Given another subspace $\mathfrak{w}'$ as in the statement, the orthogonal projection $\mathfrak{w}' \to \mathfrak{w}$ is non-degenerate and G-equivariant. Hence, the induced representations $\theta_{\mathfrak{w}}, \theta_{\mathfrak{w}'}\colon G \to O(d)$ are conjugated by an element of $GL(d)$. Since G is finite, they are also conjugated by an element of $O(d)$. Indeed, it is sufficient to treat the case of irreducible representation, where the result follows from the uniqueness of a G-invariant scalar product up to a constant factor. □

Given an almost geometric action $\theta\colon G \to \operatorname{Aut} L$, we will always assume G augmented via κ above, so that an element $c \in G$ does not belong to $G^0 = \operatorname{Ker} \kappa$ if and only if it reverses the orientation of $\mathfrak{w}$. From 2.6.1 it follows that the existence of a 1-subspace ℓ with trivial G-action does not depend on the choice of a G-invariant positive definite 3-subspace $\mathfrak{w}$. Furthermore, the induced action on $\mathfrak{w}_0 = \ell^\perp \subset \mathfrak{w}$ is also independent of $\mathfrak{w}$. Choosing an orientation of $\mathfrak{w}_0$, one obtains a 2-dimensional representation $\rho\colon G^0 \to SO(\mathfrak{w}_0) = S^1$. In what follows, we identify S^1 with the unit circle in $\mathbb{C}$ and often regard representations in S^1 as one-dimensional complex representations. In particular, we consider the spaces (lattices) $L_\rho(\Lambda)$ (see 2.3) associated with θ. Note that $L_\rho(\mathbb{C})$ is the ρ-eigenspace of G^0. Changing the orientation of $\mathfrak{w}_0$ replaces ρ with its conjugate $\bar\rho$. In view of 2.6.1, the unordered pair $(\rho, \bar\rho)$ is determined by θ; we will call ρ and $\bar\rho$ the *fundamental representations* associated with θ. The order of the image $\rho(G^0)$ is called the *order* of θ and is denoted $\operatorname{ord}\theta$.

2.6.2. Lemma. *Let $\xi\colon G^0 \to S^1$ be a non-real representation* (i.e., $\bar\xi \neq \xi$). *Then the map $L_\xi(\mathbb{C}) \to L_\xi(\mathbb{R})$, $\omega \mapsto \frac{1}{2}(\omega + \bar\omega)$, is an isomorphism of $\mathbb{R}$-vector spaces. In* *particular, the space $L_\xi(\mathbb{R})$ inherits a natural complex structure J_ξ (induced from the multiplication by i in $L_\xi(\mathbb{C})$), which is an anti-selfadjoint isometry. One has $J_{\bar\xi} = -J_\xi$.*

Proof is straightforward. The metric properties of J_ξ follow from the fact that $\omega^2 = 0$ for any eigenvector ω (of any isometry) corresponding to an eigenvalue α with $\alpha^2 \ne 1$. □

2.6.3. Lemma. *Let θ be an almost geometric action and ρ an associated fundamental representation. Assume that $\kappa \ne 1$. Then any element $c \in G \smallsetminus G^0$ restricts to an involution $c_\rho\colon L_\rho(\mathbb{R}) \to L_\rho(\mathbb{R})$. If ρ is not real, then c_ρ is J_ρ-anti-linear; in particular, the (±1)-eigenspaces $V_\rho^\pm$ of c_ρ are interchanged by J_ρ.*

Proof. Clearly, c takes ρ-eigenvectors of G^0 to ρ^c-eigenvectors, where ρ^c is the representation $g \mapsto \rho(c^{-1}gc)$. Since, by the definition of fundamental representations, there is a ρ-eigenvector ω taken to a $\bar\rho$-eigenvector, one has $\rho^c = \bar\rho$ and the space $L_\rho(\mathbb{R})$ is c-invariant. Furthermore, the vector $\operatorname{Re}\omega$ is invariant under c_ρ^2. Since $c^2 \in G^0$, one has $c_\rho^2 = \mathrm{id}$.

If ρ is non-real, then c interchanges $L_\rho(\mathbb{C})$ and $L_{\bar\rho}(\mathbb{C})$. Since c commutes with the complex conjugation, the isomorphism $\omega \mapsto \frac{1}{2}(\omega + \bar\omega)$ (see 2.6.2) conjugates c_ρ with the anti-linear involution $\omega \mapsto c(\bar\omega)$ on $L_\rho(\mathbb{C})$. □

2.6.4. Lemma. *Let θ be an almost geometric action, ρ an associated fundamental representation, and $\Bbbk \subset \mathbb{R}$ a field. Then the space $L_\rho(\Bbbk)$ is G-invariant and the induced G-action on $L_\rho(\Bbbk)$ factors through an action of the cyclic group $\mathbb{Z}_n$ (if $\kappa = 1$) or the dihedral group $\mathbb{D}_n$ (if $\kappa \ne 1$), where $n = \operatorname{ord}\theta$. The induced $\mathbb{Z}_n$-action is $\Bbbk$-isotypic; the $\mathbb{D}_n$-action is $\Bbbk$-isotypic unless $n \leqslant 2$.*

Proof. All statements are obvious if $\kappa = 1$. Assume that $\kappa \ne 1$ and pick an element $c \in G \smallsetminus G^0$. The intersection $Q = L_\rho(\Bbbk) \cap c(L_\rho(\Bbbk))$ is defined over $\Bbbk$, and $Q \otimes_\Bbbk \mathbb{R}$ contains $L_\rho(\mathbb{R})$ (see 2.6.3). Hence, $Q \supset L_\rho(\Bbbk)$ and $L_\rho(\Bbbk)$ is G-invariant. Further, the endomorphisms c^2 and $g - c^{-1}gc$ of $L_\rho(\Bbbk) \otimes_\Bbbk \mathbb{R}$ (where $g \in G^0$) are defined over $\Bbbk$ and annihilate $L_\rho(\mathbb{R})$ (see 2.6.3 again); due to the minimality of $L_\rho(\Bbbk)$, they are trivial. □

3. Folding the walls

3.1. Geometric actions. A finite group action $\theta\colon G \to \operatorname{Aut} L$ on an even nondegenerate lattice L with $\sigma_+ L = 3$ is called *geometric* if it is almost geometric and the sublattice $L^\bullet = (L^G + L_\rho(\mathbb{Z}))^\perp$ contains no roots, where ρ is a fundamental representation of θ.

Consider a geometric action θ and fix an associated fundamental representation ρ. If $\kappa \ne 1$, fix an element $c \in G \smallsetminus G^0$ and denote by $V_\rho^\pm$ and $V^\pm$ its (±1)-eigenspaces in $L_\rho(\mathbb{R})$ and $L_\rho(\mathbb{Q})$, respectively (see 2.6.3 and 2.6.4). Let $M^\pm = V^\pm \cap L$ be the (±1)-eigenlattices of c in $L_\rho(\mathbb{Z})$. If $\rho \ne 1$, the spaces $V_\rho^\pm$ and $V^\pm$ are hyperbolic. The following lemma is a straightforward consequence of 2.6.3 and 2.6.4.

3.1.1. Lemma. *The subspaces $V_\rho^\pm$ and $V^\pm$ and the sublattices $M^\pm$ are G-characteristic; they are G-invariant if and only if $\operatorname{ord}\theta \leqslant 2$. If $\rho \ne 1$, there is a well defined action of $\operatorname{Aut}_G L$ on $\mathcal{H}(V_\rho^\pm)$; it is discrete and, up to isomorphism, independent of the choice of an element $c \in G \smallsetminus G^0$.* □

In view of this lemma one can consider corresponding G-actions and introduce the following rational Dirichlet polyhedra.

- $\Gamma_1 \subset \mathcal{H}(L^G \otimes \mathbb{R})$ is a rational Dirichlet polyhedron of $W_G((L^G \oplus L^\bullet)\widehat{\ })$; it is defined whenever $\rho \neq 1$, so that $\sigma_+ L^G = 1$.
- $\Gamma^\pm_\rho \subset \mathcal{H}(V^\pm_\rho)$ are some rational Dirichlet polyhedra of $W_G((M^\pm \oplus L^\bullet)\widehat{\ })$; they are defined whenever ρ is real and $\kappa \neq 1$. (To define Γ^+_ρ, one needs to assume, in addition, that $\rho \neq 1$, so that $\sigma_+ M^+ = 1$.)
- $\Sigma^\pm_\rho \subset \mathcal{H}(V^\pm_\rho)$ are some rational Dirichlet polyhedra of $\operatorname{Aut}^0_G(L_\rho(\mathbb{Z}))$; they are defined whenever ρ is non-real and $\kappa \neq 1$.

Given a vector $v \in L$, put $\mathrm{h}(v) = \{x \in L \otimes \mathbb{R} \mid x \cdot v = 0\}$ and introduce the following notation:

- $\mathrm{h}_1(v) = \mathrm{h}(v) \cap (L^G \otimes \mathbb{R})$;
- if ρ is real and $\kappa \neq 1$, then $\mathrm{h}^\pm_\rho(v) = \mathrm{h}(v) \cap V^\pm_\rho$;
- if ρ is non-real, then $\mathrm{h}_\rho(v) = \{x \in L_\rho(\mathbb{R}) \mid x \cdot v = J_\rho x \cdot v = 0\}$; if, besides, $\kappa \neq 1$, then $\mathrm{h}^\pm_\rho(v) = \mathrm{h}_\rho(v) \cap V^\pm_\rho$.

We use the same notation $\mathrm{h}_1(v)$ and $\mathrm{h}^\pm_\rho(v)$ for the projectivizations of the corresponding spaces in $\mathcal{H}(L^G \otimes \mathbb{R})$ and $\mathcal{H}(V^\pm_\rho)$, respectively (whenever the space is hyperbolic).

The goal of this section is to prove the following two theorems.

3.1.2. Theorem. *Let $\theta\colon G \to \operatorname{Aut} L$ be a geometric action and ρ an associated fundamental representation. If $\rho \neq 1$, then for any root $v \in L_\rho(\mathbb{Z})^\perp$ the intersection $\mathrm{h}_1(v) \cap \operatorname{Int}\Gamma_1$ is empty. If ρ is real and $\kappa \neq 1$, then for any root $v \in (L^G \oplus M^\mp)^\perp$ the intersection $\mathrm{h}^\pm_\rho(v) \cap \operatorname{Int}\Gamma^\pm_\rho$ is empty. (For Γ^+_ρ to be well defined, one needs to assume, in addition, that $\rho \neq 1$.)*

3.1.3. Theorem. *Let $\theta\colon G \to \operatorname{Aut} L$ be a geometric action with non-real associated fundamental representation ρ and $\kappa \neq 1$. Then $\Sigma^\pm_\rho$ intersects finitely many distinct subspaces $\mathrm{h}^\pm_\rho(v)$ defined by roots $v \in (L^G)^\perp$.*

Theorem 3.1.2 is proved at the end of 3.2. Theorem 3.1.3 is proved in 3.6.

3.2. Walls in the invariant sublattice.

3.2.1. Theorem. *Let N be an even lattice and G a finite group acting on N so that $(N^G)^\perp \subset N$ is negative definite. Let $v \in N$ be a root whose projection to $N^G \otimes \mathbb{R}$ has negative square. Then either*

1. *the orthogonal complement $(N^G)^\perp$ contains a root, or*
2. *there is an element of $W_G(N)$ whose restriction to N^G is the reflection against the hyperplane $\mathrm{h}(v) \cap (N^G \otimes \mathbb{R})$.*

3.2.2. Corollary. *In the above notation, assume that N is hyperbolic and $(N^G)^\perp$ contains no roots. Then for any root $v \in N$ the intersection of $\mathrm{h}(v)$ with the interior of a rational Dirichlet polyhedron of $W_G(N)$ in $\mathcal{H}(N^G)$ is empty.* □

To prove Theorem 3.2.1 we need a few facts on automorphisms of root systems. Let R be an even root system and G a finite group acting on R. The action is called *admissible* if the orthogonal complement $(R^G)^\perp$ contains no roots, and it is called *b-transitive* if there is a root whose orbit generates R.

3.2.3. Lemma. *Given a finite group G action on an elliptic root system R, the following statements are equivalent*:

(1) *the action is admissible*;
(2) *the action preserves a camera of R*;
(3) *the action factors through the action of a subgroup of the symmetry group of a camera of R.*

Proof. An action is admissible if and only if R^G does not belong to a wall $\mathrm{h}(v)$ defined by a root $v \in R$. On the other hand, R^G contains an inner point of a camera if and only if this camera is preserved by the action. □

3.2.4. Corollary. *Up to isomorphism, there are two faithful admissible b-transitive actions on irreducible even root systems*: *the trivial action on A_1 and a $\mathbb{Z}_2$-action on A_2 interchanging two roots u, v with $u \cdot v = 1$.*

Proof. The statement follows from Lemma 3.2.3, the classification of irreducible root systems, and the natural bijection between the symmetries of a camera and the symmetries of its Dynkin diagram. □

Proof of Theorem 3.2.1. Pick a vector v as in the statement, and consider the sublattice $R \subset N$ generated by the orbit of v. Under the assumptions, R is an even root system, and the induced G-action on R is b-transitive. Assume that the action on R is admissible (as otherwise $(R^G)^\perp$, and thus $(N^G)^\perp$, would contain a root). Then, in view of 3.2.4, the lattice R splits into orthogonal sum of several copies of either A_1 or A_2, and the vector $\bar{v} = \sum_{g \in G} g(v)$ has the form $\sum m_i a_i$, $m_i \in \mathbb{Z}$, where each a_i is a generator of A_1 or the sum of two generators of A_2 interchanged by the action. Since the a_i's are mutually orthogonal roots, the composition of the reflections $\mathfrak{s}_{a_i}$ is the desired automorphism of N. □

Proof of Theorem 3.1.2. The statement for Γ_1 follows immediately from Theorem 3.2.1 applied to $N = L_\rho(\mathbb{Z})^\perp$. To prove the assertion for $\Gamma_\rho^\pm$, consider the induced G-action $\theta_{\mathfrak{w}} \colon G \to O(\mathfrak{w})$, where $\mathfrak{w}$ is as in the definition of an almost geometric action, see 2.6, and note that, under the hypotheses ($\rho \neq 1$ is real), $\theta_{\mathfrak{w}}$ factors through the abelian subgroup $C \subset O(\mathfrak{w})$ generated by the central symmetry c and a reflection s. Thus, the statement for Γ_ρ^+ (respectively, Γ_ρ^-) follows from 3.2.1 applied to the lattice $N = (L^G \oplus M^-)^\perp$ (respectively, $N = (L^G \oplus M^+)^\perp$) with the twisted action $g \mapsto r(g)\theta(g)$, where $r \colon G \to \{\pm 1\}$ is the composition of $\theta_{\mathfrak{w}}$ and the homomorphism $c \mapsto -1$, $s \mapsto 1$ (respectively, $c \mapsto -1$, $s \mapsto -1$). □

3.3. The group $\mathrm{Aut}_G L$. Let, as before, $\theta \colon G \to \mathrm{Aut}\, L$ be an almost geometric action and ρ a fundamental representation of θ. Recall (see 2.6.4) that the induced G-action on $L_\rho(\mathbb{Z})$ factors through the group $G' = \mathbb{Z}_n$ (if $\kappa = 1$) or $\mathbb{D}_n$ (if $\kappa \neq 1$), where $n = \operatorname{ord} \theta > 2$. Let $\mathbb{K}$ be the cyclotomic field $\mathbb{Q}(\exp(2\pi i/n))$ and let $\Bbbk \subset \mathbb{K}$ be the real part of $\mathbb{K}$, *i.e.*, the extension of $\mathbb{Q}$ obtained by adjoining the real parts of the primitive n-th roots of unity. Both $\mathbb{K}$ and $\Bbbk$ are abelian Galois extensions of $\mathbb{Q}$. Denote by $\mathfrak{O}_{\mathbb{K}}$ and $\mathfrak{O}$ the rings of integers of $\mathbb{K}$ and $\Bbbk$, respectively. Unless specified otherwise, we regard $\Bbbk$ and $\mathbb{K}$ as subfields of $\mathbb{C}$ via their standard embeddings. An isotypic $\Bbbk$-representation of G' corresponding to a pair of conjugate primitive n-th roots of unity will be called *primitive.*

3.3.1. Lemma. *For any primitive irreducible $\Bbbk$-representation ξ of G', the restriction homomorphism $\mathrm{Aut}_G L \to \mathrm{Aut}_G L_\xi(\mathfrak{O})$ is well defined and its image has finite* *index. If $L = L_\xi(\mathbb{Z})$, the restriction is a monomorphism.*

Proof. In view of 2.4.2 and 2.6.4, it suffices to consider the case when $L = L_\xi(\mathbb{Z})$ and $G = G'$. The restriction homomorphism is well defined as any G-equivariant isometry of $L_\xi(\mathbb{Z})$, after extension to $L_\xi(\mathbb{Z}) \otimes \Bbbk$, must preserve the $\Bbbk$-isotypic subspaces. It is a monomorphism, since $L_\xi(\mathbb{Q})$ is the minimal $\mathbb{Q}$-vector space such that $L_\xi(\mathbb{Q}) \otimes \Bbbk$ contains $L_\xi(\Bbbk)$. (If an element $g \in \operatorname{Aut}_G L_\xi(\mathbb{Z})$ restricts to the identity of $L_\xi(\mathfrak{O})$, then $\operatorname{Ker}(g - \mathrm{id})$ is a $\mathbb{Q}$-vector space with the above property; hence, it must contain $L_\xi(\mathbb{Q})$.)

It remains to prove that, up to finite index, any G-equivariant $\mathfrak{O}$-automorphism g of $L_\xi(\mathfrak{O})$ extends to a G-equivariant automorphism of $L_\xi(\mathbb{Z}) \otimes \mathfrak{O}$ defined over $\mathbb{Z}$. Up to finite index, one has an orthogonal decomposition $L_\xi(\mathbb{Z}) \otimes \mathfrak{O} \supset \bigoplus L_{\xi_i}(\mathfrak{O})$, the summation over all primitive irreducible representations ξ_i of G. For each such representation ξ_i there is a unique element $\mathfrak{g}_i \in \operatorname{Gal}(\Bbbk/\mathbb{Q})$ such that $\xi_i = \mathfrak{g}_i\xi$, and the automorphism $\bigoplus \mathfrak{g}_i g \mathfrak{g}_i^{-1}$ of $\bigoplus L_{\xi_i}(\mathfrak{O})$ is Galois invariant, *i.e.*, defined over $\mathbb{Z}$. ☐

Let now $\kappa \neq 1$, *i.e.*, $G' = \mathbb{D}_n$. Put $M_\xi^\pm = V_\xi^\pm \cap (L \otimes \mathfrak{O})$ and denote by $\operatorname{Aut} M_\xi^\pm$ the group of isometries of $M_\xi^\pm$ defined over $\mathfrak{O}$. (Note that $V_\rho^\pm$ are defined over $\Bbbk$ and thus can be regarded as subspaces of $L_\rho(\Bbbk)$.)

3.3.2. Lemma. *For any primitive irreducible $\Bbbk$-representation ξ of $G' = \mathbb{D}_n$, the restriction homomorphism $\operatorname{Aut}_G L_\xi(\mathfrak{O}) \to \operatorname{Aut} M_\xi^\pm$ is a well defined monomorphism, and its image has finite index.*

Proof. Again, it suffices to consider the case $G = G'$. Obviously, any G-equivariant automorphism of $L_\xi(\mathfrak{O})$ preserves $M_\xi^\pm$. To prove the converse (say for M_ξ^+), note that, up to a factor, the map J_ξ is defined over $\Bbbk$ (as this is obviously true for an irreducible representation, where $\dim_\Bbbk V_\xi^+ = \dim_\Bbbk V_\xi^- = 1$), *i.e.*, there is a dilation $J = kJ_\xi$ of $L_\xi(\Bbbk)$ interchanging V_ξ^+ and V_ξ^-. Furthermore, the factor can be chosen so that $J(M_\xi^-) \subset M_\xi^+$. Since any extension of an isometry $a \in \operatorname{Aut} M_\xi^+$ to $L_\xi(\mathfrak{O})$ must commute with J, on $M_\xi^+ \oplus M_\xi^-$ it must be given by $a \oplus J^{-1}aJ'$. On the other hand, due to 2.4.4, the latter expression does define an extension for all a in a finite index subgroup of $\operatorname{Aut} M_\xi^+$. ☐

3.3.3. Corollary. *The polyhedron $\Sigma_\rho^\pm$ is the union of finitely many copies of a rational Dirichlet polyhedron of $\operatorname{Aut} M_\rho^\pm$ in $\mathcal{H}_\rho^\pm$.* ☐

3.4. Dirichlet polyhedra: the case $\varphi(\operatorname{ord}\rho) = 2$. Recall that φ is the Euler function, *i.e.*, $\varphi(n)$ is the number of positive integers $< n$ prime to n. Alternatively, $\varphi(n)$ is the degree of the cyclotomic extension of $\mathbb{Q}$ of order n. Consider a hyperbolic sublattice $M \subset L$ and denote by $\mathcal{H} = \mathcal{H}(M \otimes \mathbb{R})$ the corresponding hyperbolic space. Given a vector $v \in M$, let $\mathrm{h}_M(v) = (\mathrm{h}(v) \cap \mathcal{C}(M \otimes \mathbb{R}))/\mathbb{R}^* \subset \mathcal{H}$.

3.4.1. Lemma. *Let $\ell \subset \mathcal{H}$ be a line whose closure intersects the absolute $\partial\mathcal{H}$ at rational points. Then for any integer a there are at most finitely many vectors $v \in M$ such that $v^2 = a$ and the hyperplane $\mathrm{h}_M(v)$ intersects ℓ.*

Proof. Let $u_1, u_2 \in M$ be some vectors corresponding to the intersection points $\ell \cap \partial\mathcal{H}$. Then u_1, u_2 span a (scaled) hyperbolic plane $U \subset M$ and the orthogonal complement $U^\perp \subset M$ is elliptic. Therefore, $U \oplus U^\perp$ is of finite index d in M.

Let v be a vector as in the statement. Since $\mathrm{h}_M(v)$ intersects ℓ, one has $v = \lambda b u_1 + (\lambda - 1) b u_2 + v'$ for some $v' \in \frac{1}{d} U^\perp$ and $\lambda \in (0, 1)$. Thus, the equation $v^2 = a$

turns into $-b^2\lambda(1-\lambda)+(v')^2=a$. Since dv' belongs to a negative definite lattice, $\lambda(1-\lambda)>0$, and both λbd and $(1-\lambda)bd$ are integers, this equation has finitely many solutions. $\square$

3.4.2. Corollary. *Let $Q\subset\mathcal{H}$ be a polyhedron whose closure in $\mathcal{H}\cup\partial\mathcal{H}$ is a convex hull of finitely many rational points. Then for any integer a there are at most finitely many vectors $v\in M$ such that $v^2=a$ and the hyperplane $\mathrm{h}_M(v)$ intersects Q.*

Proof. Each edge of Q either is a compact subset of $\mathcal{H}$ or has a rational endpoint on the absolute. In the former case, the edge intersects finitely many hyperplanes $\mathrm{h}_M(v)$, as they form a discrete set. In the latter case, both the intersection points of the absolute and the line containing the edge are rational, and the edge intersects finitely many hyperplanes $\mathrm{h}_M(v)$ due to 3.4.1. Finally, if a hyperplane does not intersect any edge of Q, it contains at least $\dim\mathcal{H}$ vertices of Q at the absolute and is determined by those vertices. Since Q has finitely many vertices, the number of such hyperplanes is also finite. $\square$

3.4.3. Corollary (of 3.4.2 and 2.5.2). *Assume that $\kappa\ne1$ and $\varphi(\operatorname{ord}\theta)=2$ (so that $M^\pm_\rho$ are defined over $\mathbb{Z}$) and let $\Pi^\pm_\rho$ be some rational Dirichlet polyhedra of $\operatorname{Aut}M^\pm_\rho$ in $\mathcal{H}^\pm_\rho$. Then for any integer a there are at most finitely many vectors $v\in M^\pm_\rho$ such that $v^2=a$ and the subspace $\mathrm{h}^\pm_\rho(v)$ intersects $\Pi^\pm_\rho$ or $J_\rho(\Pi^\mp_\rho)$.* $\square$

3.5. Dirichlet polyhedra: the case $\varphi(\operatorname{ord}\theta)\geqslant4$. Recall that an algebraic number field F has exactly $\deg(F/\mathbb{Q})$ distinct embeddings to $\mathbb{C}$. Denote by $r(F)$ the number of real embeddings (*i.e.*, those whose image is contained in $\mathbb{R}$), and by $c(F)$, the number of pairs of conjugate non-real ones. Clearly, $r(F)+2c(F)=\deg F$. The following theorem is due to Dirichlet (see, *e.g.*, [BSh]).

3.5.1. Theorem. *The rank of the group of units* (i.e., *invertible elements of the ring of integers*) *of an algebraic number field F is $r(F)+c(F)-1$.* $\square$

Let $n=\operatorname{ord}\theta$ and assume that $\varphi(n)\geqslant4$. Let $\Bbbk$, $\mathfrak{O}$, and $M^\pm_\rho$ be as in 3.3. Note that $r(\Bbbk)=\deg\Bbbk=\frac12\varphi(n)\geqslant2$ and $c(\Bbbk)=0$.

3.5.2. Lemma. *If $\kappa\ne1$, $\varphi(n)\geqslant4$, and $\dim_\Bbbk V^\pm_\rho=2$, then the rational Dirichlet polyhedra of $\operatorname{Aut}M^\pm_\rho$ in $\mathcal{H}^\pm_\rho$ are compact.*

Proof. Since $\mathcal{H}^\pm_\rho$ are hyperbolic lines, it suffices to show that the groups $\operatorname{Aut}M^\pm_\rho$ are infinite. Consider one of them, say, $\operatorname{Aut}M^+_\rho$. The lattice M^+_ρ contains a finite index sublattice M' whose Gramm matrix (after, possibly, dividing the form by an element of $\mathfrak{O}$) is of the form

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\qquad\text{or}\qquad\begin{bmatrix}1&0\\0&-d\end{bmatrix}\quad\text{with } d>0 \text{ and } \sqrt{d}\notin\Bbbk.$$

In the former case (which occurs if the form represents 0 over $\Bbbk$), the automorphisms of M' are of the form

$$A_\lambda=\begin{bmatrix}\pm\lambda&0\\0&\pm1/\lambda\end{bmatrix},$$

where $\lambda\in\mathfrak{O}^*$ is a unit of $\Bbbk$. Thus, in this case $\operatorname{Aut}M^+_\rho$ contains a free abelian group of rank $r(\Bbbk)-1>0$.

In the latter case, the automorphisms of M' are of the form

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \qquad \text{or} \qquad B_\lambda = \begin{bmatrix} \alpha & d\beta \\ \beta & \alpha \end{bmatrix},$$

where $\alpha, \beta \in \mathfrak{O}$ and $\lambda = \alpha + \beta\sqrt{d}$ is a unit of $F = \Bbbk(\sqrt{d})$ such that $\alpha^2 - \beta^2 d = 1$. We will show that the group of such units is at least $\mathbb{Z}$.

The map $\mu\colon \alpha + \beta\sqrt{d} \mapsto \alpha^2 - \beta^2 d$ is a homomorphism from the group of units of F to the group of units of $\Bbbk$, and its cokernel is finite. As $d > 0$, the quadratic extension F of $\Bbbk$ has at least two real embeddings to $\mathbb{C}$, *i.e.*, $r(F) \geqslant 2$. Since $r(F) + 2c(F) = 2\deg \Bbbk = 2r(\Bbbk)$, one has[2] $\operatorname{rk}\operatorname{Ker}\mu = \frac{1}{2}r(F) \geqslant 1$.

The coefficients α, β of all integers $\alpha + \beta\sqrt{d}$ of F have 'bounded denominators', *i.e.*, $\alpha, \beta \in \frac{1}{N}\mathfrak{O}$ for some $N \in \mathbb{N}$ (since the abelian group generated by α's and β's has finite rank and $\mathfrak{O}$ has maximal rank). Hence, for any $\lambda \in \operatorname{Ker}\mu$, the map B_λ defines an isometry of V^+ taking $N \cdot M'$ into M', and Lemma 2.4.3 applies. □

Remark. Note that, if $\varphi(n) > 4$, the form cannot represent 0 over $\Bbbk$. Indeed, otherwise $\operatorname{Aut} M^\pm_\rho$ would contain a free abelian group of rank $\geqslant 2$, which would contradict to the discreteness of the action.

Next theorem (as well as Lemma 3.5.2) can probably be deduced from the Godement criterion. We chose to give here an alternative self-contained proof.

3.5.3. Theorem. *If $\kappa \neq 1$ and $\varphi(n) \geqslant 4$, then the rational Dirichlet polyhedra of* $\operatorname{Aut} M^\pm_\rho$ *in $\mathcal{H}^\pm_\rho$ are compact.*

Proof. Let $m = \dim_\Bbbk V^\pm_\rho$. The assertion is obvious if $m = 1$, and it is the statement of 3.5.2 if $m = 2$. If $m > 2$ and a rational Dirichlet polyhedron $\Pi \subset \mathcal{H}^+_\rho$ is not compact, one can find a line $\mathcal{H}' = \mathcal{H}(V' \otimes_\Bbbk \mathbb{R})$, $V' \subset V^+_\xi$, such that $\Pi \cap \mathcal{H}'$ is not compact. (If $\Pi = \mathcal{H}^+_\rho$, one can take for V' any hyperbolic 2-subspace. Otherwise, one can replace Π with one of its non-compact facets and proceed by induction.) Applying 3.5.2 to $M' = V' \cap L_\rho(\mathfrak{O})$, one concludes that the polyhedron $\Pi' \subset \mathcal{H}'$ of $\operatorname{Aut} M'$ is compact. On the other hand, in view of 2.4.2, $\Pi \cap \mathcal{H}'$ must be a finite union of copies of Π'. □

3.5.4. Corollary. *Assume that $\kappa \neq 1$ and $\varphi(\operatorname{ord}\theta) \geqslant 4$, and let $\Pi^\pm_\rho$ be some rational Dirichlet polyhedra of* $\operatorname{Aut} M^\pm_\rho$ *in $\mathcal{H}^\pm_\rho$. Then for any integer a there are at most finitely many vectors $v \in M^\pm$ such that $v^2 = a$ and the subspace $\mathrm{h}^\pm_\rho(v)$ intersects $\Pi^\pm_\rho$ or $J_\rho(\Pi^\mp_\rho)$.* □

3.6. Proof of Theorem 3.1.3. In view of 3.3.3, one can replace $\Sigma^\pm_\rho$ in the statement with the rational Dirichlet polyhedra $\Pi^\pm_\rho$ of $\operatorname{Aut} M^\pm$ in $\mathcal{H}^\pm_\rho$.

For a root $v \in (L^G)^\perp$ denote by $v^\pm$ its projections to $V^\pm$ (the (±1)-eigenspaces of c on $L_\rho(\mathbb{Q}) \otimes \mathbb{R}$) and by $v^\pm_\rho$, its projections to $V^\pm_\rho$. The projections $v^\pm$ are rational vectors with uniformly bounded denominators, *i.e.*, there is an integer N, depending only on θ, such that $Nv^\pm \in M^\pm$. Under the assumption (ρ is non-real and $\kappa \neq 1$), the set $\mathrm{h}^+_\rho(v)$ is not empty if and only if each of $v^\pm_\rho$ either is trivial

[2] In fact, under the assumption on the signature of the form, F has exactly two real embeddings to $\mathbb{C}$, namely, $\Bbbk(\sqrt{d})$ and $\Bbbk(-\sqrt{d})$. In particular, modulo torsion one has $\operatorname{Ker}\mu = \mathbb{Z}$. Indeed, the other embeddings are $\Bbbk(\pm\sqrt{\mathfrak{g}(d)})$, $\mathfrak{g} \in \operatorname{Gal}(\Bbbk/\mathbb{Q})$, $\mathfrak{g} \neq 1$, and since all spaces $L_{\mathfrak{g}\rho}(\Bbbk)$ are negative definite, one has $\mathfrak{g}(d) < 0$.

or has negative square. In any case, $(v^{\pm}_{\rho})^2 \leqslant 0$ and, hence, $(v^{\pm})^2 \leqslant 0$. Thus, the squares $(Nv^{\pm})^2$ take finitely many distinct integral values, and the statement of the theorem follows from 3.4.3 and 3.5.4. □

4. The proof

4.1. Period spaces related to $K3$-surfaces. Let $L = 2E_8 \oplus 3U$. Consider the variety Per of positive definite 3-subspaces in $L \otimes \mathbb{R}$. It is a homogeneous symmetric space (of noncompact type):

$$\operatorname{Per} = SO^+(3, 19)/SO(3) \times SO(19).$$

The orthogonal projection of a positive definite 3-subspace to another one is nondegenerate. Hence, one can orient all the subspaces in a coherent way; this gives an orientation of the canonical 3-dimensional vector bundle over Per. In what follows we assume such an orientation fixed; the corresponding orientation of a space $\mathfrak{w} \in \operatorname{Per}$ is referred to as its *prescribed orientation.*

Given a vector $v \in L$ with $v^2 = -2$, let $\mathfrak{h}_v \subset \operatorname{Per}$ be the set of the 3-subspaces orthogonal to v. Put

$$\operatorname{Per}_0 = \operatorname{Per} \smallsetminus \textstyle\bigcup_{v \in L,\, v^2=-2} \mathfrak{h}_v.$$

The space Per_0 is called the *period space of marked Einstein $K3$-surfaces.*

There is a natural S^2-bundle $K\Omega \to \operatorname{Per}$, where

$$K\Omega = \{(\mathfrak{w}, \gamma) \mid \mathfrak{w} \in \operatorname{Per},\ \gamma \in \mathfrak{w},\ \gamma^2 = 1\}.$$

The pull-back $K\Omega_0$ of Per_0 is called the *period space of marked Kähler $K3$-surfaces.* Finally, let Ω be the variety of oriented positive definite 2-subspaces of $L \otimes \mathbb{R}$; it is called the *period space of marked $K3$-surfaces.* One can identify Ω with the projectivization

$$\{\omega \in L \otimes \mathbb{C} \mid \omega^2 = 0,\ \omega \cdot \bar{\omega} > 0\}/\mathbb{C}^*, \tag{4.1.1}$$

associating to a complex line generated by ω the plane $\{\operatorname{Re}(\lambda\omega) \mid \lambda \in \mathbb{C}\}$ with the orientation given by a basis $\operatorname{Re}\omega$, $\operatorname{Re} i\omega$. Thus, Ω is a 20-dimensional complex variety, which is an open subset of the quadric defined in the projectivization of $L \otimes \mathbb{C}$ by $\omega^2 = 0$. The spaces $K\Omega_0$ and Per_0 are (noncompact) real analytic varieties of dimensions 59 and 57, respectively.

4.2. Period maps. A *marking* of a $K3$-surface X is an isometry $\varphi\colon H^2(X) \to L$. It is called *admissible* if the orientation of the space $\mathfrak{w} = \langle \operatorname{Re}\varphi(\omega), \operatorname{Im}\varphi(\omega), \varphi(\gamma)\rangle$, where $\omega \in H^{2,0}(X)$ and γ is the fundamental class of a Kähler structure on X, coincides with its prescribed orientation. A *marked $K3$-surface* is a $K3$-surface X equipped with an admissible marking. Two marked $K3$-surfaces (X, φ) and (Y, ψ) are *isomorphic* if there exists a biholomorphism $f\colon X \to Y$ such that $\psi = \varphi \circ f^*$. Denote by $\mathcal{T}$ the set of isomorphism classes of marked $K3$-surfaces.

The *period map* $\operatorname{per}\colon \mathcal{T} \to \Omega$ sends a marked $K3$-surface (X, φ) to the 2-subspace $\{\operatorname{Re}\varphi(\omega) \mid \omega \in H^{2,0}(X)\}$, the orientation given by $(\operatorname{Re}\varphi(\omega), \operatorname{Re}\varphi(i\omega))$. (We will always use the same notation φ for various extensions of the marking to other coefficient groups.) Alternatively, $\operatorname{per}(X, \varphi)$ is the line $\varphi(H^{2,0}(X))$ in the complex model (4.1.1) of Ω.

A *marked polarized $K3$-surface* (see a remark in 1.4) is a $K3$-surface X equipped with the fundamental class γ_X of a Kähler structure and an admissible marking $\varphi\colon H^2(X) \to L$. Two marked polarized $K3$-surfaces (X, φ, γ_X) and (Y, ψ, γ_Y) are *isomorphic* if there exists a biholomorphism $f\colon X \to Y$ such that $\psi = \varphi \circ f^*$ and $f^*(\gamma_Y) = \gamma_X$. Denote by $K\mathcal{T}$ the set of isomorphism classes of marked polarized $K3$-surfaces.

The *period map* $\operatorname{per}^K\colon K\mathcal{T} \to K\Omega$ sends a triple $(X, \varphi, \gamma_X) \in K\mathcal{T}$ to the point $(\mathfrak{w}, \varphi(\gamma_X)) \in K\Omega$, where $\mathfrak{w} = \operatorname{per}(X, \varphi) \oplus \varphi(\gamma_X) \in \operatorname{Per}$ is as above. When this does not lead to a confusion, we abbreviate $\operatorname{per}^K(X, \varphi, \gamma_X)$ to $\operatorname{per}^K(X)$.

As is known (see [PSS] and [K], or [Siu]), the period map per^K is a bijection to $K\Omega_0$, and the image of per is Ω_0. Moreover, $K\Omega_0$ is a fine period space of marked polarized $K3$-surfaces, *i.e.*, the following statement holds (see [Bea]).

4.2.1. Theorem. *The space $K\Omega_0$ is the base of a universal smooth family of marked polarized $K3$-surfaces,* i.e., *a family $p\colon \Phi \to K\Omega_0$ such that any other smooth family $p'\colon X \to S$ of marked polarized $K3$-surfaces is induced from p by a unique smooth map $S \to K\Omega_0$. The latter is given by $s \mapsto \operatorname{per}^K(X_s)$, where X_s is the fiber over $s \in S$.*

Since the only automorphism of a $K3$-surface identical on the homology is the identity (see [PSS]), Theorem 4.2.1 can be rewritten in a slightly stronger form.

4.2.2. Theorem. *For any smooth family $p'\colon X \to S$ of marked polarized $K3$-surfaces there is a unique smooth fiberwise map $X \to \Phi$ (see 4.2.1) that covers the map $S \to K\Omega_0$, $s \mapsto \operatorname{per}^K(X_s)$ of the bases and is an isomorphism of marked polarized $K3$-surfaces in each fiber.*

4.2.3. Corollary. *Let (X, γ_X) and (Y, γ_Y) be two polarized $K3$-surfaces and let $g\colon H^2(Y) \to H^2(X)$ be an isometry such that $g(\gamma_Y) = \gamma_X$. Then:*

(1) *if $g(H^{2,0}(Y)) = H^{2,0}(X)$, then g is induced by a unique holomorphic map $X \to Y$, which is a biholomorphism;*
(2) *if $g(H^{2,0}(Y)) = H^{0,2}(X)$, then $-g$ is induced by a unique anti-holomorphic map $X \to Y$, which is an anti-biholomorphism.* □

4.3. Equivariant period spaces. In this section we construct the period space of marked polarized $K3$-surfaces with a G-action of a given homological type. Recall that we define the homological type as the class of the twisted induced action $\theta_X\colon G \to \operatorname{Aut} H^2(X)$ modulo conjugation by elements of $\operatorname{Aut} H^2(X)$. A marking takes θ_X to an action $\theta\colon G \to \operatorname{Aut} L$. Note in this respect that, since we work with admissible markings only, it would be more natural to consider θ_X up to conjugation by elements of the subgroup $\operatorname{Aut} L \cap O^+(L \otimes \mathbb{R})$. However, this stricter definition would be equivalent to the original one, as the central element $-\operatorname{id} \in \operatorname{Aut} L$ belongs to $O^-(L \otimes \mathbb{R})$.

4.3.1. Proposition. *Let X be a $K3$-surface supplied with a Klein action of a finite group G. Then the twisted induced action $\theta_X\colon G \to \operatorname{Aut} H^2(X)$ is geometric, and the augmentation $\kappa\colon G \to \{\pm 1\}$ and the pair $\rho, \bar\rho\colon G^0 \to S^1$ of complex conjugated fundamental representations introduced in 1.7 coincide with those determined by θ_X (see 2.6).*

Proof. Since G is finite, X admits a Kähler metric preserved by the holomorphic elements of G and conjugated by the anti-holomorphic elements. Take for γ_X the fundamental class of such a metric. Pick also a holomorphic form on X and denote by ω its cohomology class. Let $\mathfrak{w}$ be the space spanned by γ_X, $\operatorname{Re}\omega$, and $\operatorname{Im}\omega$, and let $\ell \subset \mathfrak{w}$ be the subspace generated by γ_X. Then the flag $\ell \subset \mathfrak{w}$ attests the fact that θ_X is almost geometric, and this flag can be used to define κ and ρ. As γ_X and ω cannot be simultaneously orthogonal to an integral vector $v \in H^2(X)$ of square (-2), the action is geometric. □

Let $\theta\colon G \to \operatorname{Aut} L$ be an almost geometric action on L. The assignment $g\colon \mathfrak{w} \mapsto \kappa(g)g(\mathfrak{w})$, where $g \in G$ and $-\mathfrak{w}$ stands for $\mathfrak{w}$ with the opposite orientation, defines a G-action on the space Per. Denote by Per^G the subspace of the G-fixed points and let $\operatorname{Per}^G_0 = \operatorname{Per}^G \cap \operatorname{Per}_0$. There is a natural map $K\Omega^G \to \operatorname{Per}^G$, where

$$K\Omega^G = \{(\mathfrak{w}, \gamma) \mid \mathfrak{w} \in \operatorname{Per}^G,\ \gamma \in \mathfrak{w}^G,\ \gamma^2 = 1\},$$

$\mathfrak{w}^G$ standing for the G-invariant part of $\mathfrak{w}$. Put $K\Omega^G_0 = \{(\mathfrak{w}, \gamma) \in K\Omega^G \mid \mathfrak{w} \in \operatorname{Per}^G_0\}$ and denote by Ω^G (respectively, Ω^G_0) the image of $K\Omega^G$ (respectively, $K\Omega^G_0$) under the projection $K\Omega \to \Omega$. The following statement is a paraphrase of the definitions.

4.3.2. Proposition. *An almost geometric action $\theta : G \to \operatorname{Aut} L$ is geometric if and only if the space $K\Omega^G_0$ (as well as Per^G_0 and Ω^G_0) is non-empty.* □

Let (X, φ) be a marked $K3$-surface. We will say that a Klein G-action on X and an action $\theta : G \to \operatorname{Aut} L$ are *compatible* if for any $g \in G$ one has $\theta_X g = \varphi^{-1} \circ \theta g \circ \varphi$, where $\theta_X : G \to \operatorname{Aut} H^2(X)$ is the twisted induced action. If a marking is not fixed, we say that a Klein G-action on X is compatible with θ if X admits a compatible admissible marking, *i.e.*, if θ_X is isomorphic to θ.

4.3.3. Proposition. *An action $\theta\colon G \to L$ is compatible with a Klein G-action on a marked $K3$-surface if and only if θ is geometric. Furthermore, $K\Omega^G_0$ is a fine period space of marked polarized $K3$-surfaces with a Klein G-action compatible with θ,* i.e., *it is the base of a universal smooth family of marked polarized $K3$-surfaces with a Klein G-action compatible with θ.*

Proof. The 'only if' part follows from 4.3.1, and the 'if', from 4.2.3 and 4.3.2. The fact that $K\Omega^G_0$ is a fine period space is an immediate consequence of 4.2.2. □

4.3.4. Proposition. *Let $\kappa\colon G \to \{\pm 1\}$ be the augmentation and $\rho\colon G^0 \to S^1$ a fundamental representation associated with θ. If $\rho = 1$, then the spaces $K\Omega^G$ and Ω^G are connected. If $\rho \neq 1$, then the space $K\Omega^G$ (respectively, Ω^G) consists of two components, which are transposed by the involution $(\mathfrak{w}, \gamma) \mapsto (\mathfrak{w}, -\gamma)$ (respectively, the involution reversing the orientation of 2-subspaces). If, besides, $\rho \neq \bar\rho$, the two components of $K\Omega^G$ (or Ω^G) are in a one-to-one correspondence with the two fundamental representations ρ, $\bar\rho$.*

Proof. Since Per is a hyperbolic space and G acts on Per by isometries, the space Per^G is contractible. The projections $K\Omega^G \to \operatorname{Per}^G$ and $K\Omega^G_0 \to \operatorname{Per}^G_0$ are (trivial) S^p-bundles, where $p = 0$ if $\rho \neq 1$, $p = 1$ if $\rho = 1$ and $\kappa \neq 1$, and $p = 2$ if $\rho = 1$ and $\kappa = 1$. Finally, since each space $\mathfrak{w} \in \operatorname{Per}$ has its prescribed orientation, a choice of a G-invariant vector $\gamma \in \mathfrak{w}$ determines an orientation of $\gamma^\perp \subset \mathfrak{w}$ and, hence, a fundamental representation. □

4.4. The moduli spaces. Fix a geometric action $\theta : G \to \operatorname{Aut} L$ and consider the space $K\mathfrak{M}^G = K\Omega_0^G/\operatorname{Aut}_G L$. In view of 4.3.3, it is the 'moduli space' of polarized $K3$-surfaces with Klein G-actions compatible with θ. Given such a surface (X, γ_X), pick a marking $\varphi\colon H^2(X) \to L$ compatible with θ and denote by $\mathfrak{m}^K(X, \gamma_X) = \mathfrak{m}^K(X)$ the image of $\operatorname{per}^K(X, \varphi, \gamma_X)$ in $K\mathfrak{M}^G$. Since any two compatible markings differ by an element of $\operatorname{Aut}_G L$, the point $\mathfrak{m}^K(X, \gamma_X)$ is well defined. The following statement is an immediate consequence of 4.3.3 and the local connectedness of $K\Omega_0^G$.

4.4.1. Proposition. *Let (X, γ_X) and (Y, γ_Y) be two polarized $K3$-surfaces with Klein G-actions compatible with θ. Then X and Y are G-equivariantly deformation equivalent if and only if $\mathfrak{m}^K(X)$ and $\mathfrak{m}^K(Y)$ belong to the same connected component of $K\mathfrak{M}^G$.* □

In 4.4.2–4.4.7 below we give a more detailed description of period and moduli spaces. We use the notations of 3.1.

4.4.2. The case $\rho = 1$, $\kappa = 1$. *If $\rho = 1$ and $\kappa = 1$, then $K\Omega_0^G \cong (\mathcal{H}(L^G) \smallsetminus \Delta) \times S^2$, where $\operatorname{codim} \Delta \geqslant 3$. In particular, $K\Omega_0^G$ and, hence, $K\mathfrak{M}^G$ are connected.*

4.4.3. The case $\rho = 1$, $\kappa \neq 1$. *If $\rho = 1$ and $\kappa \neq 1$, then $K\mathfrak{M}^G$ is a quotient of the connected space $((\mathcal{H}(L^G) \times \operatorname{Int} \Gamma_\rho^-) \smallsetminus \Delta) \times S^1$, where $\operatorname{codim} \Delta \geqslant 2$. In particular, $K\mathfrak{M}^G$ is connected.*

4.4.4. The case $\rho \neq 1$ real, $\kappa = 1$. *If $\rho \neq 1$ is real and $\kappa = 1$, then $K\mathfrak{M}^G$ is a quotient of the two-component space $((\operatorname{Int} \Gamma_1 \times \mathcal{H}(L_\rho(\mathbb{R}))) \smallsetminus \Delta) \times S^0$, where $\operatorname{codim} \Delta \geqslant 2$. In particular, $K\mathfrak{M}^G$ has at most two connected components, which are interchanged by the complex conjugation $X \mapsto \bar{X}$.*

4.4.5. The case $\rho \neq 1$ real, $\kappa \neq 1$. *If $\rho \neq 1$ is real and $\kappa \neq 1$, then $K\mathfrak{M}^G$ is a quotient of the two-component space $((\operatorname{Int} \Gamma_1 \times \operatorname{Int} \Gamma_\rho^+ \times \operatorname{Int} \Gamma_\rho^-) \smallsetminus \Delta) \times S^0$, where $\operatorname{codim} \Delta \geqslant 2$. In particular, $K\mathfrak{M}^G$ has at most two connected components, which are interchanged by the complex conjugation $X \mapsto \bar{X}$.*

4.4.6. The case ρ non-real, $\kappa = 1$. *If ρ is non-real and $\kappa = 1$, then $K\mathfrak{M}^G$ is a quotient of the two-component space $((\operatorname{Int} \Gamma_1 \times \mathbb{P}_J \mathcal{C}_\rho) \smallsetminus \Delta) \times S^0$, where $\mathbb{P}_J \mathcal{C}_\rho$ is the space of positive definite (over $\mathbb{R}$) J_ρ-complex lines in $L_\rho(\mathbb{R})$ and $\operatorname{codim} \Delta \geqslant 2$. In particular, $K\mathfrak{M}^G$ has at most two connected components, which are interchanged by the complex conjugation $X \mapsto \bar{X}$.*

4.4.7. The case ρ non-real, $\kappa \neq 1$. *If ρ is non-real and $\kappa \neq 1$, then $K\mathfrak{M}^G$ is a quotient of the space $((\operatorname{Int} \Gamma_1 \times \Sigma_\rho^+) \smallsetminus \Delta) \times S^0$, where Δ is the union of a subset of codimension $\geqslant 2$ and finitely many hyperplanes of the form $\operatorname{Int} \Gamma_1 \times (\mathrm{h}_\rho^\pm(v) \cap \Sigma_\rho^+)$ defined by roots $v \in (L^G)^\perp$. This space has finitely many connected components; hence, so does $K\mathfrak{M}^G$.*

Proof of 4.4.2–4.4.5. One has

- $\operatorname{Per}^G = \mathcal{H}(L^G \otimes \mathbb{R})$ in case 4.4.2,
- $\operatorname{Per}^G = \mathcal{H}(L^G \otimes \mathbb{R}) \times \mathcal{H}(V_\rho^-)$ in case 4.4.3,
- $\operatorname{Per}^G = \mathcal{H}(L^G \otimes \mathbb{R}) \times \mathcal{H}(L_\rho(\mathbb{R}))$ in case 4.4.4, and
- $\operatorname{Per}^G = \mathcal{H}(L^G \otimes \mathbb{R}) \times \mathcal{H}(V_\rho^+) \times \mathcal{H}(V_\rho^-)$ in case 4.4.5.

Thus, in each case, Per^G is a product $\prod \mathcal{H}(L_i \otimes \mathbb{R})$ of the hyperbolic spaces of orthogonal indefinite sublattices $L_i \subset L$ such that $\bigoplus_i L_i \oplus L^\bullet$ is a finite index sublattice in L. Consider the quotient $\mathcal{Q}_0 = \operatorname{Per}_0^G/W$, where $W = \prod W_i$ (the product in $W_G(L)$) and $W_i = 1$ if $\sigma_+ L_i > 1$ or $W_i = W_G((L_i \oplus L^\bullet)\widehat{\ })$ if $\sigma_+ L_i = 1$. The quotient $\mathcal{Q}_0$ can be identified with a subspace of $\mathcal{Q} = \prod \operatorname{Int}\Gamma_i$, where Γ_i is a fundamental Dirichlet polyhedron of W_i in $\mathcal{H}(L_i \otimes \mathbb{R})$. (Note that $\Gamma_i = \mathcal{H}(L_i \otimes \mathbb{R})$ unless $\sigma_+ L_1 = 1$.) Put $\Delta = \mathcal{Q} \smallsetminus \mathcal{Q}_0$; it is the union of the walls $\mathfrak{h}_v \cap \mathcal{Q}$ over all roots $v \in L$.

For a root $v \in L$ one has $\operatorname{codim}(\mathfrak{h}_v \cap \mathcal{Q}) \geqslant \sum \sigma_+ L_i$, the summation over all i such that the projection of v to L_i is nontrivial. Thus, a wall $\mathfrak{h}_v \cap \mathcal{Q}$ may have codimension 1 only if $v \in (L_i \oplus L^\bullet)\widehat{\ }$ and $\sigma_+ L_i = 1$. However, in this case $\mathfrak{h}_v \cap \mathcal{Q} = \varnothing$ due to 3.1.2. Hence, $\operatorname{codim}\Delta \geqslant 2$ and the space $\mathcal{Q}_0$ is connected. □

Proof of 4.4.6. In this case, $\operatorname{Per}_0^G/W_G((L^G \oplus L^\bullet)\widehat{\ })$ can be identified with a subset of $\operatorname{Int}\Gamma_1 \times \mathbb{P}_J \mathcal{C}_\rho$, and the proof follows the lines of the previous one. □

Proof of 4.4.7. One has $\operatorname{Per}^G = \mathcal{H}(L^G \otimes \mathbb{R}) \times \mathcal{H}(V_\rho^+)$, and the quotient space $\mathcal{Q}_0 = \operatorname{Per}_0^G/(W_G(L_\rho(\mathbb{Z})^\perp) \cdot \operatorname{Aut}_G^0(L_\rho(\mathbb{Z}))$ can be identified with a subset of $\operatorname{Int}\Gamma_1 \times \Sigma_\rho^+$. Now, the statement follows from 3.1.2 and 3.1.3. □

4.5. Proof of Theorems 1.7.1 and 1.7.2. Theorem1.7.2 follows from 4.4.2–4.4.6. Theorem 1.7.1 consists, in fact, of two statements: finiteness of the number of equivariant deformation classes within a given homological type of G-actions (of a given group G), and finiteness of the number of homological types of faithful actions. The former is a direct consequence of 4.4.2–4.4.7. The latter is a special case of the finiteness of the number of conjugacy classes of finite subgroups in an arithmetic group, see [BH] and [B]. □

5. Degenerations

5.1. Passing through the walls. Let $L = 2E_8 \oplus 3U$. Consider a geometric G-action $\theta\colon G \to \operatorname{Aut} L$. Pick a G-invariant elliptic root system $R \subset L$. Denote by $\bar{R}$ the sublattice of L generated by all roots in $(R + L^\bullet)\widehat{\ }$. Clearly, $\bar{R}$ is a G-invariant root system; it is called the *θ-saturation* of R. We say that R is *θ-saturated* if $R = \bar{R}$. Any θ-saturated root system R is *saturated*, *i.e.*, R contains all roots in $R\widehat{\ }$.

Fix a camera C of $\bar{R}$ and denote by S_C its group of symmetries. Then, for any $g \in G$, the restriction of θg to $\bar{R}$ admits a unique decomposition $s_g w_g$, $s_g \in S_C$, $w_g \in W(\bar{R})$. Let $\theta_R(g) = (\theta g)w_g^{-1} \in \operatorname{Aut} L$, w_g being extended to L identically on $\bar{R}^\perp$. We will call the map $\theta_R\colon G \to \operatorname{Aut} L$ the *degeneration* of θ at R.

5.1.1. Proposition. *The map θ_R is a geometric G-action. Up to conjugation by an element of $W(\bar{R})$, it does not depend on the choice of a camera C of $\bar{R}$ and is the only action with the following properties*:

1. *the action induced by θ_R on $\bar{R}$ is admissible*;
2. *θ and θ_R induce the same action on each of the following sets*: $\bar{R}^\perp$, $\operatorname{discr}\bar{R}$, *the set of irreducible components of $\bar{R}$.*

Conversely, if $\bar{R} \subset L$ is a saturated root system and $\theta_R\colon G \to L$ is an action satisfying (1)–(2) *above, then $\bar{R}$ is θ-saturated and θ_R is a degeneration of θ at $\bar{R}$.*

Proof. Clearly, both θ and θ_R factor through a subgroup of $\operatorname{Aut}\bar{R} \times \operatorname{Aut}\bar{R}^\perp$. The composition of θ_R with the projection to $\operatorname{Aut}\bar{R}^\perp$ coincides with that of θ; the composition of θ_R with the projection to $\operatorname{Aut}\bar{R}$ is the composition of θ, the projection to $\operatorname{Aut}\bar{R}$, and the quotient homomorphism $\operatorname{Aut}\bar{R} \to S_C \subset \operatorname{Aut}\bar{R}$. Hence, θ_R is a homomorphism. Furthermore, another choice of a camera C' of $\bar{R}$ leads to another representation $\operatorname{Aut}\bar{R} \to S_{C'} \subset \operatorname{Aut}\bar{R}$, which is conjugated to the original one by a unique element $w_0 \in W(\bar{R})$; the latter can be regarded as an automorphism of L.

All other statements follow directly from the construction. For the uniqueness, it suffices to notice that, for any irreducible root system R' and a camera C' of R', the natural homomorphism $S_{C'} \to \operatorname{Aut}\operatorname{discr} R'$ is a monomorphism. □

5.1.2. Proposition. *Let R be a θ-saturated root system and $R' \subset R$ the sublattice generated by all roots in $R \cap (L^G)^\perp$. Then, up to conjugation by an element of $W(R)$, the degenerations θ_R and $\theta_{R'}$ coincide. In particular, θ_R can be chosen to coincide with θ on $(R')^\perp$.*

Proof. Take for C a camera adjacent to the intersection of the mirrors defined by the roots of R'. Then C has an invariant face (possibly, empty), and the decomposition $\theta g|_R = s_g w_g$ has $w_g \in W(R')$ for any $g \in G$. □

If the action is properly Klein, one can take for R the θ-saturated root system generated by all roots in $(L^G)^\perp$ orthogonal to a given wall $\mathrm{h}_\rho^+(v)$. The resulting degeneration is called the *degeneration* at the wall $\mathrm{h}_\rho^+(v)$.

Remark. The degeneration construction gives rise to a partial order on the set of homological types of geometric actions of a given finite group G.

5.2. Degenerations of $K3$-surfaces. Let (G, κ) be an augmented group. Denote by D_ε the disk $\{s \in \mathbb{C} \mid |s| < \varepsilon\}$. The composition of κ and the $\{\pm 1\}$-action via the complex conjugation $s \mapsto \bar{s}$ is a Klein G-action on D_ε. A *G-equivariant degeneration of $K3$-surfaces* is a nonsingular complex 3-manifold X supplied with a Klein G-action and a G-equivariant (with respect to the above G-action on D_ε) proper analytic map $p\colon X \to D_\varepsilon$ so that the following holds:

- the projection p has no critical values except $s = 0$;
- the fibers $X_s = p^{-1}(s)$ of p are normal $K3$-surfaces, nonsingular unless $s = 0$.

(By a singular $K3$-surface we mean a surface whose desingularization is $K3$.) Given a degeneration X, denote by $\pi_s\colon \tilde{X}_s \to X_s$, $s \in D_\varepsilon$, the minimal resolution of singularities of X_s, see, *e.g.*, [L]. (Note that $\tilde{X}_s = X_s$ unless $s = 0$.) From the uniqueness of the minimal resolution it follows that any Klein action lifts from X_s to $\tilde{X}_s$. Thus, if either $\kappa = 1$ or s is real, $\tilde{X}_s$ inherits a natural Klein action of G.

5.2.1. Theorem. *Let $p\colon X \to D_\varepsilon$ be a G-equivariant degeneration of $K3$-surfaces. Pick a regular value $t \in D_\varepsilon$, real, if $\kappa \neq 1$. Denote by $R \subset H^2(X_t)$ the subgroup Poincaré dual to the kernel of the inclusion homomorphism $H_2(X_t) \to H_2(X) = H_2(X_0)$. Then R is a saturated elliptic root system and the twisted induced G-action on $H^2(\tilde{X}_0)$ is isomorphic to the degeneration at R of the twisted induced G-action on $H^2(X_t)$.*

Remark. A statement analogous to Theorem 5.2.1 holds in a more general situation, for a family of complex surfaces whose singular fiber at $s = 0$ has at worst simple singularities, *i.e.*, those of type A_n, D_n, E_6, E_7, or E_8. The only difference is the fact that one can no longer claim that the root system R is saturated, and one should consider the degeneration at R **without** passing to its saturation first. (In particular, the algebraic definition of degeneration should be changed. Our choice of the definition, incorporating the saturation operation, was motivated by our desire to assure that the result should be a geometric action.) The proof given below applies to the general case with obvious minor modifications.

Proof. It is more convenient to switch to the twisted induced actions θ_s in the homology groups $H_2(X_s)$, $s \in D_\varepsilon$; they are Poincaré dual to the twisted induced actions in the cohomology.

Let $\iota_s\colon H_2(\tilde X_s) \to H_2(X)$, $s \in D_\varepsilon$, be the composition of $(\pi_s)_*$ and the inclusion homomorphism. Put $R_s = \operatorname{Ker} \iota_s$. Consider sufficiently small G-invariant open balls $B_i \subset X$ about the singular points of X_0 and let $B = \bigcup B_i$. One can assume that t is real and sufficiently small, so that $M_i = X_t \cap B_i$ are Milnor fibers of the singular points. Then there is a G-equivariant diffeomorphism $d'\colon X_t \smallsetminus B \to X_0 \smallsetminus B$.

Recall that all singular points of the $K3$-surface X_0 are simple and R_0 is a saturated elliptic root system (see Lemma 5.2.2 below). In particular, d' extends to a diffeomorphism $d\colon X_t \to \tilde X_0$. Note that neither d nor the induced isomorphism $d_*\colon H_2(X_t) \to H_2(\tilde X_0)$ is canonical and d_* does not need to be G-equivariant. However, d_* does preserve the G-action on the sets of irreducible components of the root systems R_t, R_0 (as it is just the G-action on the set of singular points of X_0), and, in view of natural identifications $R_s^\perp = H_2(X_s \smallsetminus B)/\operatorname{Tors}$ and $\operatorname{discr} R_s = H_1(\partial(X_s \smallsetminus B))$, $s = t, 0$, and the fact that d' commutes with G, the restrictions of d_* to $R_t^\perp$ and $\operatorname{discr} R_t$ are G-equivariant. Finally, the action induced by θ_0 on R_0 is admissible: it preserves the camera defined by the exceptional divisors in $\tilde X_0$ (see 3.2.3). Thus, after identifying $H_2(X_t)$ and $H_2(\tilde X_0)$ via d_*, the actions $\theta = \theta_t$ and $\theta_R = \theta_0$ satisfy 5.1.1(1)–(2), and 5.1.1 implies that θ_0 is the degeneration of θ_t at R_t. $\square$

For completeness, we outline the proof of the following lemma, which refines the well known fact that a $K3$-surface can have at worst simple singular points.

5.2.2. Lemma. *Let X be a $K3$-surface. Then any negative definite sublattice $R \subset H^2(X)$ generated by classes of irreducible curves is a saturated root system.*

Proof. As it follows from the adjunction formula, any irreducible curve $C \subset X$ of negative self-intersection is a (-2)-curve, *i.e.*, a non-singular rational curve of self-intersection (-2). Thus, any sublattice R as in the statement is an elliptic root system generated by classes of irreducible (-2)-curves.

From the Riemann-Roch theorem it follows that, given a root $r \in \operatorname{Pic} X$, there is a unique (-2)-curve $C \subset X$ whose cohomology class is $\pm r$. Thus, the set of all roots in $\operatorname{Pic} X$ splits into disjoint union $\Delta_+ \cup \Delta_-$, where Δ_+ is the set of *effective* roots (those realized by curves) and $\Delta_- = -\Delta_+$. Furthermore, the set Δ_+ is closed with respect to positive linear combinations and the function $\#\colon \Delta_+ \to \mathbb{N}$ counting the number of components of the curve representing a root $r \in \Delta^+$ is a well defined homomorphism, in the sense that, whenever a root r is decomposed into $\sum a_i r_i$ for some $r_i \in \Delta_+$ and $a_i \in \mathbb{N}$, one has $r \in \Delta_+$ and $\#r = \sum a_i \,\#r_i$. (Note that, if X is algebraic, the roots $r \in \Delta_+$ with $\#r = 1$ define the walls of the rational Dirichlet polyhedron of $\operatorname{Aut}\operatorname{Pic} X$ in $\mathcal{H}(\operatorname{Pic} X \otimes \mathbb{R})$ containing the fundamental class of a Kähler structure, see, *e.g.*, [PSS] or [DIK]. If X is non-algebraic, they define the walls of a distinguished camera of $\operatorname{Pic} X$.)

Let now $R \in \operatorname{Pic} X$ be a root system as in the statement and $\bar R \supset R$ its saturation in $\operatorname{Pic} X$. Consider the subsets $\bar\Delta_\pm = \bar R \cap \Delta_\pm$. They form a partition of the set of roots of $\bar R$, one has $\bar\Delta_- = -\bar\Delta_+$, and $\bar\Delta_+$ is closed with respect to positive linear combinations. Hence, there is a unique camera C of $\bar R$ such that $\bar\Delta_+$ is the set of roots positive with respect to C (see, *e.g.*, [Bou]); this means that the roots $r_1, \dots, r_k \in \bar\Delta_+$ defining the walls of C form a basis of $\bar R$ and each root $r \in \bar\Delta_+$ is a positive linear combination of the r_i's. Hence, any root $r \in \bar\Delta_+$ with $\#r = 1$ must be one of r_i's. Since R is generated by such roots, one has $R = \bar R$. □

6. Are $K3$-surfaces quasi-simple?

6.1. $K\mathfrak{M}^G$ with walls. Here, we construct an example of a geometric action of the group $G = \mathbb{D}_3$ (with ρ non-real and $\kappa \neq 1$) whose associated space $K\mathfrak{M}^G$ has more than two components, *i.e.*, the action of $\operatorname{Aut}_G L$ on the set of connected components of Per_0^G is not transitive. This shows that the assumptions on the action in Theorem 1.7.2 cannot be removed. However, the resulting Klein actions on $K3$-surfaces are not diffeomorphic (see 6.2.1), *i.e.*, they do not constitute a counter-example to quasi-simplicity of $K3$-surfaces.

6.1.1. Proposition. *There is a homological type of $\mathbb{D}_3$-action on $L \cong 3U \oplus 2E_8$ realizable by six $\mathbb{D}_3$-equivariant deformation classes of $K3$-surfaces. More precisely, there is a geometric action of $G = \mathbb{D}_3$ on L such that the corresponding moduli space $K\mathfrak{M}^G$ consists of three pairs of complex conjugate connected components.*

Proof. Fix a decomposition $L = P \oplus Q$, where $P \cong 2U$ and $Q \cong U \oplus 2E_8$. Define a $\mathbb{D}_3$-action on L as follows. On Q, the $\mathbb{Z}_3$ part of $\mathbb{D}_3$ acts trivially, and each nontrivial involution of $\mathbb{D}_3$ acts via multiplication by -1. On P, fix a basis u_1, v_1, u_2 and v_2 so that $u_i^2 = v_i^2 = 0$, $u_i \cdot v_i = 1$, and $u_i \cdot u_j = v_i \cdot v_j = u_i \cdot v_j = 0$ for $i \neq j$. Choose an order 3 element t and an order 2 element s in $\mathbb{D}_3$, and define their action on P by the matrices

$$T = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix},$$

respectively. Note that $L^\bullet$ is trivial; hence, according to 4.3.3, the constructed $\mathbb{D}_3$-action on L is realizable by a Klein $\mathbb{D}_3$-action on a $K3$-surface.

The associated fundamental representation of the constructed action is non-real. Hence, $K\mathfrak{M}^G \cong (\operatorname{Per}_0^G/\operatorname{Aut}_G L) \times S^0$, and it suffices to show that $\operatorname{Per}_0^G/\operatorname{Aut}_G L$ has three connected components.

One has $L^G = Q$ and $L_\rho(\mathbb{Z}) = P$. The lattice M^+ (the $(+1)$-eigenlattice of s) is generated by $w_1 = u_1 + v_1 + u_2$ and $w_2 = u_1 + u_2 - v_2$, and one has $w_1^2 = 2$, $w_2^2 = -2$, and $w_1 \cdot w_2 = 0$. We assert that the only nontrivial automorphism of M^+ that extends to an equivariant automorphism of P is the multiplication by -1; thus, $\operatorname{Aut}_G P = \{\pm 1\}$. Indeed, $\operatorname{Aut} M^+$ consists of the four automorphisms $w_1 \mapsto \varepsilon_1 w_1$, $w_2 \mapsto \varepsilon_2 w_2$, where $\varepsilon_1, \varepsilon_2 = \pm 1$, and the equivariant extension to $P \otimes \mathbb{Q}$ is uniquely given by the additional conditions $t(w_i) \mapsto \varepsilon_i t(w_i)$. If $\varepsilon_1 \neq \varepsilon_2$, the extension is not integral.

Thus, the action of $\operatorname{Aut}_G L$ on $\mathcal{H}^+$ is trivial, the fundamental domain Σ_ρ^+ coincides with $\mathcal{H}^+$, and, in view of 4.4.7, one has $\operatorname{Per}_0^G/\operatorname{Aut}_G L = (\tilde\Gamma_1 \times \mathcal{H}^+) \smallsetminus \Delta$, where $\tilde\Gamma_1 = \operatorname{Int}\Gamma_1/\operatorname{Aut} Q$ and Δ is the union of a subset of codimension $\geqslant 2$ and the hyperplanes $\tilde\Gamma_1 \times \mathrm{h}_\rho^\pm(v)$ defined by roots $v \in P$. (Since $\dim \mathcal{H}^+ = 1$, each nonempty set $\mathrm{h}_\rho^\pm(v)$ is a hyperplane.) Let $v \in P$ be a root and $v^\pm$ its projections to $V^\pm$. Since

$2v^{\pm} \in M_\rho^{\pm}$ and M_ρ^+ has no vectors of square -4, the condition $\mathrm{h}_\rho^+(v) \neq \varnothing$ implies that either $v^+ = 0$ (and then $(v^-)^2 = -2$), or $(v^+)^2 = -2$ (and then $v^- = 0$), or $(2v^+)^2 = -8 - (2v^-)^2 = -2$ or -6. Each $M_\rho^{\pm}$ contains, up to sign, one vector of square (-2) and two vectors of square (-6). Comparing their images under J_ρ, one concludes that the space $\mathcal{H}^+$ is divided into three components by the two walls $\mathrm{h}_\rho^+(w_2)$ and $\mathrm{h}_\rho^+(2w_2 - w_1)$. □

6.1.2. Before discussing this example in more details, introduce another geometric $\mathbb{D}_3$-action on L with the same sublattice $L^G = Q = U \oplus 2E_8$. In the above notation, replace S with the matrix

$$S' = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix},$$

and keep the rest unchanged. For the new action, one has $M_\rho^{\pm} \cong U(2)$. The only possible wall in $\mathcal{H}^+$ is $\mathrm{h}_\rho^+(w^+)$, where $w^+ \in M_\rho^+$ is the only vector of square -4. However, $J_\rho w^+$ is not proportional to the vector $w^- \in M_\rho^-$ of square -4; hence, the action is realized by a single $\mathbb{D}_3$-equivariant deformation class of $K3$-surfaces.

In view of the following lemma, there are exactly two (up to isomorphism) geometric $\mathbb{D}_3$-actions on L with $L^G \cong U \oplus 2E_8$.

6.1.3. Lemma. *Up to automorphism, there are three non-trivial $\mathbb{Z}_3$-actions on the lattice $P \cong 2U$; their invariant sublattices are isomorphic to either A_2, or $A_2(-1)$, or 0. The last action admits two, up to isomorphism, extension to a $\mathbb{D}_3$-action.*

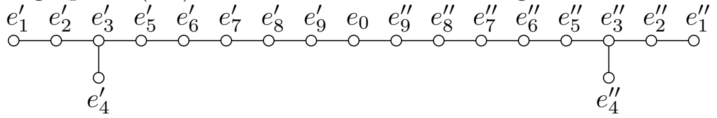
Proof. Let $t \in \mathbb{Z}_3$ be a generator. Pick a primitive vector u_1 of square 0 and let $u_2 = t(u_1)$. If $t(u_1) = u_1$ for any such u_1, the action is trivial. If $u_1 \cdot u_2 = a \neq 0$, then u_1, u_2, and $t^2(u_1)$ span a sublattice P' of rank three. In this case $a = \pm 1$, and the action is uniquely recovered using the fact that its restriction to $(P')^\perp$ (a sublattice of rank one) is trivial. Finally, if $u_1 \cdot u_2 = 0$ and u_1, u_2 are linearly independent, then one must have $t(u_2) = -u_1 - u_2$. Completing u_1, u_2 to a basis u_1, v_1, u_2, v_2 as in the proof of 6.1.1, one can see that the system $T^3 = \mathrm{id}$, $\mathrm{Gr} = T^* \mathrm{Gr}\, T$ (where T is the matrix of t and Gr is the Gramm matrix) has a unique solution (the one indicated in the proof of 6.1.1).

Consider the last action and an involution $s\colon P \to P$, $ts = st^{-1}$. The invariant space M^+ of s is either U, or $U(2)$, or $\langle 2 \rangle \oplus \langle -2 \rangle$. The consideration above shows that the $\mathbb{Z}_3$-orbit of any primitive vector u_1 of square 0 is standard and spans a sublattice of rank 2. Start from $u_1 \in M^+$ and complete it to a basis u_1, v_1, u_2, v_2 as above. The set of solutions to the system $TS = ST^{-1}$, $S^2 = \mathrm{id}$, $\mathrm{Gr} = S^* \mathrm{Gr}\, S$ for the matrix S of s depends on one parameter a, $s(v_2) = au_1 - v_2$, and a change of variables shows that only the values $a = 0$ or 1 produce essentially different actions (with $M^+ \cong U(2)$ or $\langle 2 \rangle \oplus \langle -2 \rangle$, respectively). □

6.2. Geometric models. In this section, we give a geometric description (via elliptic pencils) of the six families constructed in 6.1.1. At a result, at the end of the section we prove the following statement.

6.2.1. Proposition. *All three pairs of complex conjugate deformation families constructed in* 6.1.1 *differ by the topological type of the $\mathbb{D}_3$-action.*

Fix a decomposition $Q = \operatorname{Pic} X \cong 2E_8 \oplus U$. Let $e'_1, \dots, e'_8$, $e''_1, \dots, e''_8$ be some standard bases for the E_8-components and u, v a basis for the U component, so that $u^2 = v^2 = 0$ and $u \cdot v = 1$. Under an appropriate choice of γ (a small perturbation of $u + v$) the graph of (-2)-curves on X is the following:



Here $e_0 = u - v$, $e'_9 = v - 2e'_1 - 4e'_2 - 6e'_3 - 3e'_4 - 5e'_5 - 4e'_6 - 3e'_7 - 2e'_8$, and $e''_9 = v - 2e''_1 - 4e''_2 - 6e''_3 - 3e''_4 - 5e''_5 - 4e''_6 - 3e''_7 - 2e''_8$.

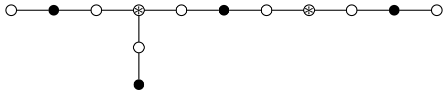
Consider the equivariant elliptic pencil $\pi\colon X \to \mathbb{P}^1$ defined by the effective class v. From the diagram above it is clear that the pencil has a section e_0 and two singular fibers of type $\tilde{E}_8$, whose components are $e'_1, \dots, e'_9$ and $e''_1, \dots, e''_9$, respectively, and has no other reducible singular fibers. (We use the same notation for a (-2)-curve and for its class in L.) Counting the Euler characteristic shows that the remaining singular fibers are either $4\tilde{A}_0^*$, or $2\tilde{A}_0^* + \tilde{A}_0^{**}$, or $2\tilde{A}_0^{**}$. (Here $\tilde{A}_0^*$ and $\tilde{A}_0^{**}$ stand for a rational curve with a node or a cusp, respectively.) In any case, at least one of these singular fibers must also remain fixed under the $\mathbb{Z}_3$-action; hence, the $\mathbb{Z}_3$-action on the base of the pencil has three fixed points and thus is trivial. This implies, in particular, that the pencil has no fibers of type $\tilde{A}_0^*$: the normalization of such a fiber would have three fixed points (the two branches at the node and the point of intersection with e_0) and the $\mathbb{Z}_3$-action on it and, hence, on the whole surface would have to be trivial. Thus, the types of the singular fibers of the pencil are $2\tilde{A}_0^{**} + 2\tilde{E}_8$.

Let us study the action of $\mathbb{Z}_3$ on the fibers of the pencil. Each fiber has at least one fixed point: the point of intersection with e_0. For nonsingular fibers this implies that

(1) they all have j-invariant $j = 0$ (as there is only one elliptic curve admitting a $\mathbb{Z}_3$-action with a fixed point), and
(2) each nonsingular fiber has two fixed points more.

Denote the closure of the union of these additional fixed points by C. This is a curve fixed under the $\mathbb{Z}_3$-action. In particular, it must intersect the cuspidal fibers at the cusps. The action on the $\tilde{E}_8$ singular fibers can easily be recovered starting from the points of intersection with e_0 and using the following simple observation: in appropriate coordinates (x, y) a generator $g \in \mathbb{Z}_3$ acts via $(x, y) \mapsto (x, \varepsilon y)$ in a neighborhood of a point of a fixed curve $y = 0$, and via $(x, y) \mapsto (\varepsilon^2 x, \varepsilon^2 y)$ in a neighborhood of an isolated fixed point $(0, 0)$. (Here ε is the eigenvalue of ω: $g(\omega) = \varepsilon\omega$.) One concludes that the components e'_3, e'_7, e''_3, and e''_7 are fixed, the intersection points of pairs of other components are isolated fixed points, and C intersects the $\tilde{E}_8$ fibers at some points of e'_1 and e''_1. In particular, the restriction $\pi\colon C \to \mathbb{P}^1$ is a double covering with four branch points; hence, C is a nonsingular elliptic curve.

Let $\tilde{X}$ be X with isolated fixed points blown up and $\tilde{Y} = \tilde{X}/\mathbb{Z}_3$. This is a rational ruled surface with two singular fibers $\tilde{F}'$, $\tilde{F}''$ (the images of the $\tilde{E}_8$ fibers of X), whose adjacency graphs are as follows:



(Here $\circ$, $\bullet$, and $\circledast$ stand for a nonsingular rational curve of self-intersection -1, -3, and -6, respectively; an edge corresponds to a simple intersection point of the curves.) The image $\tilde{R}$ of the section e_0 has self-intersection (-6) and intersects the rightmost curve in the graph; the image $\tilde{D}$ of the section C has self-intersection 0 and intersects the leftmost curve in the graph. The branch divisor of the covering $\tilde{X} \to \tilde{Y}$ is $\tilde{R} + \tilde{D} +$ (the (-6)-components) $-$ (the (-3)-components).

Contract the singular fibers of $\tilde{Y}$ to obtain a geometrically ruled surface Y. Denote by R, D, F', and F'' the images of $\tilde{R}$, $\tilde{D}$, $\tilde{F}'$, and $\tilde{F}''$, respectively. The contraction can be chosen so that $R^2 = 0$, *i.e.*, $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then $D^2 = 8$ and D is a curve of bi-degree $(2, 2)$. It is tangent to F', F'', and R passes through the tangency points.

The above construction respects the $\mathbb{D}_3$-action on X, and Y inherits a canonical real structure in respect to which D, R, F', and F'', as well as the base of the pencil, are real; one has $Y_{\mathbb{R}} = S^1 \times S^1$.

Recall that, up to rigid isotopy, the embedding $D_{\mathbb{R}} \subset Y_{\mathbb{R}}$ is one of the following:

(1) $D_{\mathbb{R}}$ is empty;
(2) $D_{\mathbb{R}}$ consists of one oval (a component contractible in $Y_{\mathbb{R}}$);
(3) $D_{\mathbb{R}}$ consists of two ovals;
(4) $D_{\mathbb{R}}$ consists of two components, each realizing the class $(0, 1)$ in $H_1(Y_{\mathbb{R}})$;
(5) $D_{\mathbb{R}}$ consists of two components, each realizing the class $(1, 0)$ in $H_1(Y_{\mathbb{R}})$;
(6) $D_{\mathbb{R}}$ consists of two components, each realizing the class $(1, 1)$ in $H_1(Y_{\mathbb{R}})$.

(The basis in $H_1(Y_{\mathbb{R}})$ is chosen so that $R_{\mathbb{R}}$ realizes $(1, 0)$ and $F'_{\mathbb{R}}$ realizes $(0, 1)$.) Now, one can easily indicate four topologically distinct types of the action. Since p' and p'' are on the same generatrix R, the embedding $D_{\mathbb{R}} \subset Y_{\mathbb{R}}$ is either

(a) as in (2), or
(b) as in (3) (the points p', p'' are in the same component of $D_{\mathbb{R}}$), or
(c) as in (4) (the points p', p'' are in the different components of $D_{\mathbb{R}}$).

In the latter case, there are two possibilities:

(c$'$) $F'_{\mathbb{R}}$ and $F''_{\mathbb{R}}$ belong to (the closure of) the same component of $Y_{\mathbb{R}} \smallsetminus D_{\mathbb{R}}$,
(c$''$) $F'_{\mathbb{R}}$ and $F''_{\mathbb{R}}$ belong to (the closure of) distinct components of $Y_{\mathbb{R}} \smallsetminus D_{\mathbb{R}}$.

Note that, according to Lemma 3.2.4, any model constructed does necessarily realize either the action of 6.1.1, or the action of 6.1.2.

The models of types (a) and (b) (resp., (a) and (c$'$)) can be joined through a singular elliptic $K3$-surface whose desingularization has a fiber of type $\tilde{A}_2$. In view of Proposition 5.2.1, these types realize the action of 6.1.1. Hence, the remaining type (c$''$) realizes the action of 6.1.2.

Proof of 6.2.1. The surfaces in question are represented by the above models of types (a), (b) and (c$'$), which differ topologically: by the number of components of $C_{\mathbb{R}} \cong D_{\mathbb{R}}$ and by whether $C_{\mathbb{R}}$ has a component bounding a disk in $X_{\mathbb{R}}$. $\square$

6.3. The four families in their Weierstraß form. Since the four families constructed above are Jacobian fibrations (*i.e.*, have sections), are isotrivial, and have singular fibers of type $2\tilde{A}_0^{**} + 2\tilde{E}_8$, their Weierstraß equations are of the form

$$y^2 z = x^3 + (u^2 - v^2)^5 p_2(u, v) z^3,$$

where $(u : v)$ are homogeneous real coordinates in $\mathbb{P}^1$, p_2 is a degree 2 homogeneous real polynomial with simple roots other than $u = \pm v$, and (x, y, z) are regarded as coordinate in the bundle $\mathbb{P}(\mathcal{O}(6) \oplus \mathcal{O}(4) \oplus \mathcal{O})$ over $\mathbb{P}^1(u : v)$. Isomorphisms between such elliptic fibrations are given by projective transformations in $\mathbb{P}^1(u : v)$ and coordinates changes of the form $x \mapsto k^4 x$, $y \mapsto k^6 y$, $z \mapsto z$, $u \mapsto ku$, $v \mapsto kv$, $k \in \mathbb{R}^*$. By means of such isomorphisms the equation can be reduced to one of the following four families:

$$y^2 z = x^3 + (u^2 - v^2)^5 (u - cv)(u - \bar{c}v) z^3, \quad c \neq \bar{c},$$
$$y^2 z = x^3 \pm (u^2 - v^2)^5 (u - av)(u - bv) z^3, \quad -1 < a < b < 1, \quad \text{and}$$
$$y^2 z = x^3 + (u^2 - v^2)^5 (u - av)(u - bv) z^3, \quad -1 < a < 1 < b.$$

The $\tilde{E}_8$ singular fibers are those with $u^2 = v^2$. Each of the surfaces can be equipped with any of the two $\mathbb{D}_3$-actions generated by the complex conjugation and the multiplication of x by either $\exp(2\pi i/3)$ or $\exp(-2\pi i/3)$.

The exceptional family, *i.e.*, that with the action of 6.1.2, is the one with the last equation. To see this, one can explicitly construct two cycles in M_- with square 0 and intersection 2. For one of them, we pick a skew-invariant under the complex conjugation circle ξ in an elliptic fiber between $u = av$ and $u = v$ and drag it along a loop in $\mathbb{P}^1(u : v)$ around $u = -v$ and $u = av$. The other (singular) cycle is constructed from a circle η in the same fiber with $T\eta = \bar{\eta}$, where T is the monodromy operator about the fiber $u = v$. We drag it along a loop around $u = v$ and pull its ends together into the cusp of the fiber $u = av$.

Note that the real part of the double section of the surfaces in the first family has only one connected component, so it correspond to the series (a). One component of the double section of the surfaces given by the second equation with the sign $-$ bounds a disc in the real part of the surface, so it corresponds to series (b). The same equation with the sign $+$ gives series (c$'$).

Thus, one obtains another description of the six disjoint families constructed in 6.1.1. The bijection between the set of isomorphism classes of $K3$-surfaces with a $\mathbb{D}_3$-action such that $L^G = U \oplus 2E_8$ and the set of surfaces given by the above four equations (considered up to projective transformations of the base and rescalings) can be used for an alternative proof of 6.1.1.

6.4. Distinct conjugate components with the same real ρ. In this section, we construct an example of a geometric action θ of a certain group $G = \tilde{T}_{192}$ (with $\rho \neq 1$ real and $\kappa = 1$) whose moduli space has two distinct components interchanged by the conjugation $X \mapsto \bar{X}$. Note that, since ρ is real, the components are **not** distinguished by the associated fundamental representations.

Recall that the group T_{192} can be described as follows. Consider the form $\Phi(u, v) = u^4 + v^4 - 2\sqrt{-3}\, u^2 v^2$. Its group of unitary isometries is the so called *binary tetrahedral group* $T_{24} \subset U(2)$; it can be regarded as a $\mathbb{Z}_3$-extension of the Klein group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$. (Note that the double projective line ramified at the roots of Φ is a hexagonal elliptic curve. An order three element of T_{24} can be given, *e.g.*, by the matrix

$$q = \frac{1}{-1 + i\sqrt{3}} \begin{bmatrix} -1 - i & 1 - i \\ -1 - i & -1 + i \end{bmatrix},$$

whose determinant is $(-1 + i\sqrt{3})/2$.) The center of T_{24} is $\{\pm 1\} \subset Q_8$. Identify two copies of $T_{24}/Q_8 \cong \mathbb{Z}_3$ via $[q] \mapsto [q]^{-1}$ and let T' be the fibered central product

$(T_{24} \times_{\mathbb{Z}_3} T_{24})/\{c_1 = c_2\}$, where c_1 and c_2 are the central elements in the two factors. Then T_{192} is the semi-direct product $T' \rtimes \mathbb{Z}_2$, the generator t of $\mathbb{Z}_2$ acting via transposing the factors.

Denote by $\tilde{T}_{192}$ the extension of T_{192} by an element c subject to the relations $c^2 = c_1 = c_2$, $c^{-1}tc = c_1 t$, and $ac = ca$ for any a in either of the two copies of $T_{24} \subset T_{192}$. Augment this group via $\kappa\colon \tilde{T}_{192} \to \tilde{T}_{192}/T_{192} = \mathbb{Z}_2$.

6.4.1. Proposition. *There is a geometric action of $G = \tilde{T}_{192}$ on $L = 3U \oplus 2E_8$ such that the associated fundamental representation ρ is real and the corresponding moduli space $K\mathfrak{M}^G$ consists of a pair of conjugate points X, $\bar{X}$.*

Proof. Consider the quartic $X \subset \mathbb{P}^3$ given by the polynomial $\Phi(x_0, x_1) + \Phi(x_2, x_3)$. According to Mukai [Mu], there is a T_{192}-action on X with $\rho = 1$. It can be described as follows. The central product $(T_{24} \times T_{24})/\{c_1 = c_2\}$ acts via block diagonal linear automorphisms of $\Phi \oplus \Phi$, the two factors acting separately in (x_0, x_1) and (x_2, x_3). The fundamental representation of the induced action on X has order 3, and its kernel extends to a symplectic T_{192}-action via the involution $(x_0, x_1) \leftrightarrow (x_2, x_3)$.

The described T_{192}-action on X extends to a $\tilde{T}_{192}$-action, the element $c \in \tilde{T}_{192}$ acting via $(x_0 : x_1 : x_2 : x_3) \mapsto (ix_0 : ix_1 : x_2 : x_3)$, so that $\rho(c) = -1$. Choosing an isometry $H^2(X) \to L$, one obtains a geometric $\tilde{T}_{192}$-action on L.

Fix a marking $H^2(X) = L$ and consider the twisted induced action on L. We assert that the corresponding period space consists of two points X and $\bar{X}$, both admitting a unique embedding into $\mathbb{P}^3$ compatible with a projective representation of $\tilde{T}_{192}$. Indeed, as it follows, *e.g.*, from the results of Xiao [X], for any action of the group $G' = T_{192}$ with $\rho = 1$ one has $\operatorname{rk} L^\bullet = 19$; hence, $\operatorname{Per}^{G'}$ is a single point $\mathfrak{w} \subset L \otimes \mathbb{R}$ and $K\Omega^{G'} = S^2$. Passing to $G = \tilde{T}_{192}$ decomposes $\mathfrak{w}$ into $\ell = \mathfrak{w}^G$ and $\ell^\perp$ and reduces $K\Omega^G$ to a pair of points. Since the action is induced from $\mathbb{P}^3$, the line ℓ is generated by an integral vector of square 4, and this is the only (primitive) polarization of the surface compatible with the action.

It remains to show that X does not admit an anti-holomorphic automorphism commuting with $\tilde{T}_{192}$. Any such automorphism would preserve ℓ and, hence, would be induced from an anti-holomorphic automorphism a of $\mathbb{P}^3$. Since a commutes with $\tilde{T}_{192}$, it must fix the four intersection points of X with the line C given by $\{x_0 = x_1 = 0\}$. In particular, a must preserve C. On the other hand, the roots of Φ do not lie on a circle and, thus, cannot be fixed by an anti-homography. □

Appendix A. Finiteness and quasi-simplicity for 2-tori

A.1. Klein actions on 2-tori. In this section we prove analogs of Theorems 1.7.1 and 1.7.2 for complex 2-tori (or just 2-tori, for brevity). The *homological type* of a finite group G Klein action on a 2-torus X is the twisted induced action $\theta_X\colon G \to \operatorname{Aut} H^2(X)$ on the lattice $H^2(X) \cong 3U$, considered this time up to conjugation by **orientation preserving** lattice automorphisms. As in the case of $K3$-surfaces, one has $H^{2,0}(X) \cong \mathbb{C}$, and the action of G^0 on $H^{2,0}(X)$ gives rise to a natural representation $\rho\colon G^0 \to \mathbb{C}^*$, called the *associated fundamental representation.* Both θ_X and ρ are deformation invariants of the action; θ_X is also a topological invariant.

Our principal results for 2-tori are the following two theorems.

A.1.1. Finiteness Theorem. *The number of equivariant deformation classes of complex* 2*-tori with faithful Klein actions of finite groups of uniformly bounded order* (*for any given bound*) *is finite.*

Remark. Note that the order of groups acting on 2-tori and not containing pure translations is bounded (*cf.* A.1.4 below). In particular, there are finitely many deformation classes of such actions.

A.1.2. Quasi-simplicity Theorem. *Let X and Y be two complex* 2*-tori with diffeomorphic finite group G Klein actions. Then either X or $\bar X$ is G-equivariantly deformation equivalent to Y. If the associate fundamental representation is trivial, then X and $\bar X$ are G-equivariantly deformation equivalent.*

A.1.3. Corollary. *The number of equivariant deformation classes of hyperelliptic surfaces with faithful Klein actions of finite groups is finite. If X and Y are two hyperelliptic surfaces with diffeomorphic finite group G Klein actions, then either X or $\bar X$ is G-equivariantly deformation equivalent to Y.* $\square$

Recall that, after fixing a point 0 on a 2-torus X, one can identify X with the quotient space $T_0(X)/H_1(X;\mathbb{Z})$ and thus regard it as a group. Then with each (anti-)automorphism t of X one can associate its linearization dt preserving 0, and, hence, any Klein action θ on X gives rise to its linearization $d\theta$ consisting of (anti-)holomorphic autohomomorphisms of X. As is known (see, *i.e.*, [VS] or [Ch]), the original action θ is uniquely determined by $d\theta$ and a certain element $a(\theta) \in H^2(G; H_1(X)) = H^1(G; T_0(X)/H_1(X;\mathbb{Z}))$, the latter depending only on the equivalence class of the extension $1 \to H_1(X) \to \mathcal{G} \to G \to 1$, where $\mathcal{G}$ is the lift of G to the group of (anti-)holomorphic transformations of the universal covering T_0X of X. In particular, $a(\theta)$ is a topological invariant.

Clearly, both the homological type of a Klein action θ and its fundamental representation ρ depend only on the linearization $d\theta$. Since the group $H^2(G; H_1(X))$ is finite and $a(\theta)$ is a topological invariant, the general case of A.1.1 and A.1.2 reduces to the case of linear actions. Thus, from now on, **we consider only actions preserving** 0**.** All (anti-)automorphisms preserving 0 are group homomorphisms, and they all commute with the automorphism $-\operatorname{id}\colon X \to X$. For simplicity, we **always assume that** $-\operatorname{id} \in G$. For such actions, we prove theorems A.1.4 and A.1.5 below, which imply A.1.1 and A.1.2.

A.1.4. Theorem. *The number of equivariant deformation classes of complex* 2*-tori with faithful linear Klein actions of finite groups preserving* 0 *is finite.*

A.1.5. Theorem. *Let X and Y be two complex* 2*-tori with linear finite group G Klein actions of the same homological type. Then either X or $\bar X$ is G-equivariantly deformation equivalent to Y. If the associate fundamental representation is trivial, then X and $\bar X$ are G-equivariantly deformation equivalent.*

These theorems are proved at the end of Section A.3.

Remark. Note that, speaking about linear actions, Theorem A.1.5 is somewhat stronger than A.1.2, as it also asserts that the diffeomorphism type of a linear action is determined by its homological type.

Remark. In the case of real actions (see 1.7), the surfaces X and $\bar X$ are obviously equivariantly isomorphic. The same remark applies to A.1.3, which gives us *gratis* the following generalization of the corresponding result by F. Catanese and P. Frediani [CF] for real structures on hyperelliptic surfaces: Let X and Y be two complex 2-tori with real structures and with real holomorphic G^0-actions, so that the extended Klein actions of $G = G^0 \times \mathbb{Z}_2$ have the same homological type and the same value of $a(\theta)$. Then X and Y are G-equivariantly deformation equivalent.

A.2. Periods of marked 2-tori. Let Λ be an oriented free abelian group of rank 4. Put $L = \bigwedge^2 \Lambda^\vee$. The orientation of Λ defines an identification $\bigwedge^4 \Lambda^\vee = \mathbb{Z}$ and turns L into a lattice via per: $L \otimes L \to \bigwedge^4 \Lambda^\vee = \mathbb{Z}$. It is isomorphic to $3U$. Denote $\operatorname{Aut}^+ L = \operatorname{Aut} L \cap SO^+(L \otimes \mathbb{R})$.

Let $\mathcal{J}$ be the set of complex structures on $\Lambda \otimes \mathbb{R}$ compatible with the orientation of Λ. Let, further, Ω be the set of oriented positive definite 2-subspaces in $L \otimes \mathbb{R}$. As in (4.1.1), one can identify Ω with the space $\{\omega \in L \otimes \mathbb{C} \mid \omega^2 = 0,\ \omega \cdot \bar\omega > 0\}/\mathbb{C}^*$. Both $\mathcal{J}$ and Ω have natural structures of smooth manifolds. Let per: $\mathcal{J} \to \Omega$ be the map defined via $J \mapsto (x^1 + iJ^*x^1) \wedge (x^2 + iJ^*x^2)$, where $J \in \mathcal{J}$, J^* is the adjoint operator on $L^\vee$, and $x^1, x^2 \in L^\vee \otimes \mathbb{R}$ are any two vectors generating $L^\vee \otimes \mathbb{R}$ over $\mathbb{C}$ (with respect to the complex structure J^*).

The following statement is essentially contained in [PSS] and [Shi].

A.2.1. Proposition. *The map* per: $\mathcal{J} \to \Omega$ *is a well defined diffeomorphism. The map* $SL(\Lambda) \to \operatorname{Aut}^+ L$, $\varphi \mapsto \wedge^2\varphi^*$, *is an epimorphism*; *its kernel is the center* $\{\pm 1\} \subset SL(\Lambda)$. *An element* $\varphi \in SL(\Lambda)$ *commutes with a complex structure* $J \in \mathcal{J}$ *if and only if its image* $\wedge^2\varphi^*$ *preserves* per J.

Proof. We will briefly indicate the proof. A simple calculation in coordinates shows that the map per: $\mathcal{J} \to \Omega$ is an immersion and generically one-to-one. (Remarkably, the equations involved are partially linear.) Since $\mathcal{J}$ and Ω are connected homogeneous spaces of the same dimension, per is a diffeomorphism.

The map $SL(\Lambda \otimes \mathbb{R}) \to O(L \otimes \mathbb{R})$, $\varphi \mapsto \wedge^2\varphi^*$, is a homomorphism of Lie groups of the same dimension. Hence, it takes the connected group $SL(\Lambda \otimes \mathbb{R})$ to the component of unity $SO^+(L \otimes \mathbb{R})$. The pull-back of $\operatorname{Aut}^+ L \subset SO^+(L \otimes \mathbb{R})$ is a discrete subgroup of $SL(\Lambda \otimes \mathbb{R})$ containing $SL(\Lambda)$; on the other hand, the latter is a maximal discrete subgroup (see [Ra]); hence, it coincides with the pull-back.

The last statement follows from the naturallity of the construction: one has $\operatorname{per}(\varphi J \varphi^{-1}) = \wedge^2\varphi^*(\operatorname{per} J)$. $\square$

A 1-*marking* of a 2-torus X is a group isomorphism $\varphi_1 \colon \Lambda \to H_1(X)$. We call a 1-marking *admissible* if it takes the orientation of Λ to the canonical orientation of $H_1(X)$ (induced from the complex orientation of X). A 2-*marking* of X is a lattice isomorphism $\varphi\colon H^2(X) \to L$. Since $H^2(X) = \bigwedge^2 H^1(X)$, every 1-marking φ_1 defines a 2-marking $\varphi = \wedge^2\varphi_1^*$. A 2-marking is called *admissible* if it has the form $\wedge^2\varphi_1^*$ for some admissible 1-marking φ_1. Any two admissible 1-marking differ by an element of $SL(\Lambda)$; in view of A.2.1, any two admissible 2-markings differ by an element of $\operatorname{Aut}^+ L$ and any admissible 2-marking has the form $\wedge^2\varphi_1^*$ for exactly two admissible 1-markings φ_1.

From now on by a 1- (respectively, 2-) marked torus we mean a 2-torus with a fixed admissible 1- (respectively, 2-) marking. Isomorphisms of marked tori are defined in the obvious way (*cf.* 4.2). Clearly, 1-marked tori have no automorphisms; the group of (marked) automorphisms of a 2-marked torus is $\{\pm \operatorname{id}\}$.

Consider the space $\Phi = \mathcal{J} \times (\Lambda \otimes \mathbb{R})/\Lambda$ and the projection $p\colon \Phi \to \mathcal{J}$. The bundle $\operatorname{Ker} dp$ has a tautological complex structure, which converts $p\colon \Phi \to \mathcal{J}$ to a family of 1-marked tori. This family is obviously universal. In view of A.2.1, this implies the following statement, called the global Torelli theorem for 2-marked tori.

A.2.2. Theorem. *The family* $p\colon \Phi \to \Omega$ *is a universal smooth family of* 2*-marked complex* 2*-tori,* i.e., *any other smooth family* $p'\colon X \to S$ *of* 2*-marked complex* 2*-tori is induced from* p *by a unique smooth map* $S \to \Omega$.

A.3. Equivariant period spaces. The following statement is similar to 4.3.1; it relies on Proposition A.2.1 and on the fact that a finite group action admits an equivariant Kähler metric.

A.3.1. Proposition. *Given a Klein action of a finite group* G *on a complex* 2*-torus* X*, the twisted induced action* $\theta_X\colon G \to \operatorname{Aut} H^2(X)$ *is almost geometric* (*see* 2.6); *its image belongs to* $\operatorname{Aut}^+ H^2(X)$. □

Now, we proceed as in the case of $K3$-surfaces. Let $\theta\colon G \to \operatorname{Aut}^+ L$ be an almost geometric action, and denote by $\Omega^G \subset \Omega$ the fixed point set of the induced action $g\colon \mathfrak{v} \mapsto \kappa(g)g(\mathfrak{v})$, $\mathfrak{v} \in \Omega$. (As before, $-\mathfrak{v}$ stands here for $\mathfrak{v}$ with the opposite orientation.) Then the following holds.

A.3.2. Proposition. *The space* Ω^G *is a fine period space of* 2*-marked complex* 2*-tori with a Klein* G*-action compatible with* θ, i.e., *it is the base of a universal smooth family of* 2*-marked complex* 2*-tori with a Klein* G*-action compatible with* θ. □

A.3.3. Proposition. *Let* $\kappa\colon G \to \{\pm 1\}$ *be the augmentation and* $\rho\colon G^0 \to S^1$ *a fundamental representation associated with* θ*. If* $\rho = 1$*, then the space* Ω^G *is connected. If* $\rho \neq 1$*, then the space* Ω^G *consists of two components, which are transposed by the involution* $\mathfrak{v} \mapsto -\mathfrak{v}$.

Proof. As in the case of $K3$-surfaces, one can consider the contractible space Per^G and sphere bundle $K\Omega^G \to \operatorname{Per}^G$ and use the fibration $K\Omega^G \to \Omega^G$ with contractible fibers. □

Proof of Theorems A.1.4 *and* A.1.5. Theorem A.1.5 follows from A.3.2 and A.3.3. In view of A.1.5, Theorem A.1.4 follows from the finiteness of the number of homological types of faithful actions, *cf.* 4.5. □

A.4. Comparing X and $\bar{X}$. As a refinement of Theorem A.1.2, we show that in most cases the 2-tori X and $\bar{X}$ are not equivariantly deformation equivalent.

A.4.1. Proposition. *Consider a faithful finite group* G *Klein action on a complex* 2*-torus* X*. Assume that* G^0 *has an element of order* > 2 *acting non-trivially on holomorphic* 2*-forms. Then* X *is not* G*-equivariantly deformation equivalent to* $\bar{X}$.

Proof. Let $g \in G$ be an element as in the statement. The assertion is obvious if the associated fundamental representation ρ is non-real. Thus, one can assume that ρ is real and $\rho(g) = -1$. A simple calculation (using the fact that g is orientation preserving, $\operatorname{ord} g > 2$, and $\wedge^2 g^*$ has eigenvalue (-1) of multiplicity $\geqslant 2$) shows that in this case the eigenvalues of the action of g on Λ are of the form ξ, $\bar{\xi}$, $-\bar{\xi}$, $-\xi$ for some $\xi \notin \mathbb{R}$. Hence, there is a distinguished square root $\sqrt{g} \in SL(\Lambda \otimes \mathbb{R})$. (One chooses the arguments of the eigenvalues in the interval $(-\pi, \pi)$ and divides them by 2.) The automorphism $\wedge^2(\sqrt{g})^*$ has order 4 on the (only) g-skew-invariant 2-subspace $\mathfrak{v}$; hence, it defines a distinguished orientation on $\mathfrak{v}$. □

Remark. As a comment to the proof of Proposition A.4.1, we would like to emphasize a difference between $K3$-surfaces and 2-tori. Under the assumptions of A.4.1, if ρ is real, it is still possible that there is an element $a \in \operatorname{Aut}^+_G L$ interchanging the two points $\mathfrak{v}$ and $-\mathfrak{v}$ of Ω^G (representing X and $\bar{X}$). However, unlike the case of $K3$-surfaces, this does not imply that X and $\bar{X}$ are G-isomorphic; an additional requirement is that a lift of a to $SL(\Lambda)$ should commute with G.

A.5. Remarks on symplectic actions. We would like to outline here a simple way to enumerate all symplectic (*i.e.*, identical on the holomorphic 2-forms) finite group actions on 2-tori. (This result is contained in the classification by Fujiki [Fu], who calls symplectic actions special.) Our approach follows that of Kondō [Ko1] to the similar problem for $K3$-surfaces.

In view of A.3.1 and A.3.2, it suffices to consider finite group actions on $L \cong 3U$ identical on a positive definite 3-subspace in $L \otimes \mathbb{R}$. Let $\theta\colon G \to \operatorname{Aut}^+ L$ be such an action and $L^\bullet = (L^G)^\perp$. Then, $L^\bullet$ is a negative definite lattice of rank $\leqslant 3$, and the induced G-action on $L^\bullet$ is orientation preserving and trivial on $\operatorname{discr} L^\bullet$ (as so it is on $\operatorname{discr} L^G$). Standard calculations with discriminant forms (*cf.* [Ko1]) show that $L^\bullet$ can be embedded to E_8 (the only negative definite unimodular even lattice of rank 8), and the G-action on $L^\bullet$ extends to E_8 identically on $E_8^G = (L^\bullet)^\perp \subset E_8$. Since $\operatorname{Aut} E_8 = W(E_8)$, the lattice $L^\bullet$ is the orthogonal complement of a face of a camera of E_8. Hence, $L^\bullet$ is a root system contained in A_3, $A_2 \oplus A_1$, or $3A_1$, and $G/\operatorname{Ker}\theta$ is a subgroup of $W(L^\bullet) \cap SO(L^\bullet \otimes \mathbb{R})$. It remains to observe that any such lattice admits a unique (up to isomorphism) embedding to L and, hence, the pair $L^\bullet$, $G/\operatorname{Ker}\theta \subset W(L^\bullet)$ determines a G-action on L up to automorphism.

In particular, one obtains a complete list of finite groups G acting faithfully and symplectically on 2-tori. One has $\operatorname{Ker}\theta = \{\pm\operatorname{id}\}$ and the group $G/\operatorname{Ker}\theta$ is a subgroup of $W(L^\bullet) \cap SO(L^\bullet \otimes \mathbb{R})$ for $L^\bullet = A_3$, $A_2 \oplus A_1$, or $3A_1$, *i.e.*, of $\mathfrak{A}_4$ (alternating group on 4 elements), $\mathfrak{S}_3$ (symmetric group on 3 elements), or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Lifting the action from L to Λ, see A.2.1, one finds that G is a subgroup of the binary tetrahedral group T_{24}, binary dihedral group Q_{12}, or Klein (quaternion) group Q_8.

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