MAXIMIZING PLANE SEXTICS WITH DOUBLE POINTS

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Abstract. We collect the available information on the defining equations and fundamental groups of maximizing plane sextics with double singular points.

1. Introduction

1.1. **Disclaimer.** This text is not intended as an 'official' publication. Its main purpose is collecting the currently known information on the defining equations (which are often too bulky to appear elsewhere) and the fundamental groups of maximizing plane sextics and to record the details of some computations announced in other papers (mainly in [9]).

All equations and a few further details are available in the machine readable form at http://www.fen.bilkent.edu.tr/~degt/papers/equations.zip.

Sextics with triple or worse singular points are ignored here. Such sextics are completely covered in [7], where a combinatorial approach (which seems much more effective than explicit equations) is used.

Only *irreducible* sextics are considered. The reducible ones seem less interesting, and there are too many of them.

I did my best to collect all information available in the literature. Should there be a priority reference missing, please let me know and I will gladly include it into the next update.

This text is a satellite of [9]. Should you wish to refer to it, please cite [9] instead.

1.2. **Sources.** The defining equations found in these notes are mainly due to Artal, Carmona, Cogolludo [1], Degtyarev [9], Oka, Pho [12], and Orevkov [14]. The fundamental groups are computed in [1, 4, 5, 8, 9] and in these notes. More detailed references are given in Table 1. The classification is due to [3, 16, 15].

1.3. **Simple sextics.** A plane sextic $D \subset \mathbb{P}^2$ is called *simple* if all its singular points are simple. For such a sextic D, one has $\mu(D) \leqslant 19$; if $\mu(D) = 19$, the sextic is called *maximizing*. Maximizing sextics are rigid (*i.e.*, their equisingular moduli spaces are discrete); they are defined over algebraic number fields.

The classification of maximizing sextics can be obtained using [3] (a reduction to an arithmetical problem), [16] (a complete list of the sets of singularities), and [15] (deformation classification). Up to projective equivalence, there are 42 real and 20 pairs of complex conjugate irreducible maximizing sextics with double singular points only; they realize 39 sets of singularities, see Table 1.

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Key words and phrases. Plane sextic, fundamental group, tetragonal curve.

These notes are to be extended should there be any interesting development. They will be available at http://www.fen.bilkent.edu.tr/~degt/papers/papers.htm and on the arXiv.

TABLE 1. Irreducible maximizing sextics with $\mathbf{A}$ type singularities

#	Singularities	(r, c)	π_1	Equation, references, remarks
1.	$\mathbf{A}_{19}$	$(2,0)$	$\mathbb{Z}_6$	§4.1, see [1]
2.	$\mathbf{A}_{18} \oplus \mathbf{A}_1$	$(1,1)$	$\mathbb{Z}_6$	§4.2, see [1, 14]
3.	$(\mathbf{A}_{17} \oplus \mathbf{A}_2)$	$(1,0)^*$	Γ	§2.1, see [1, 4, 12] (torus)
4.	$\mathbf{A}_{16} \oplus \mathbf{A}_3$	$(2,0)$	$\mathbb{Z}_6$	§4.3, §6.1, see [1, 9]
5.	$\mathbf{A}_{16} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,1)$	$\mathbb{Z}_6$	§4.4, §6.2, see [1, 13]
6.	$\mathbf{A}_{15} \oplus \mathbf{A}_4$	$(0,1)^*$	$\mathbb{Z}_6$	§4.5, see [1, 9]
7.	$\mathbf{A}_{14} \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(0,3)$		§4.6, see [9]
8.	$(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$	$(1,0)$	Γ	§2.2, §5.1, see [8, 12] (torus)
9.	$(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,0)$	Γ	§2.3, §5.2, see [8, 12] (torus)
10.	$\mathbf{A}_{13} \oplus \mathbf{A}_6$	$(0,2)$		§4.7, see [9]
11.	$\mathbf{A}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2,0)$	$\mathbb{Z}_6$	§4.8, §6.3, see [9]
12.	$\mathbf{A}_{12} \oplus \mathbf{A}_7$	$(0,1)$		§4.9, see [9]
13.	$\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$	$(1,1)$	$\mathbb{Z}_6$	§4.10, §6.4, see [9], Remark 1.3
14.	$\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	$(1,0)$	$\mathbb{Z}_6$	§4.11, §6.5, see [9]
15.	$\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,1)$	$\mathbb{Z}_6$	§4.12, §6.6, see [14], Remark 1.3
16.	$\mathbf{A}_{11} \oplus 2\mathbf{A}_4$	$(2,0)$	$\mathbb{Z}_6$	§4.13, §6.7, see [9], Remark 1.4
17.	$(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$	$(1,0)$	Γ	§2.4, §5.3, see [8, 12] (torus)
18.	$\mathbf{A}_{10} \oplus \mathbf{A}_9$	$(2,0)^*$	$\mathbb{Z}_6$	§4.14, §6.8, see [9]
19.	$\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$	$(1,1)$	$\mathbb{Z}_6$	§4.15, §6.9, see [9], Remark 1.3
20.	$\mathbf{A}_{10} \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$	$(2,0)$	$\mathbb{Z}_6$	§4.16, §6.10, see [9]
21.	$\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_3$	$(0,1)$		§4.17, see [9]
22.	$\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,1)$	$\mathbb{Z}_6$	§4.18, §6.11, see [14], Remark 1.3
23.	$\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$	$(2,0)$	$\mathbb{Z}_6$	§4.19, §6.12, see [9]
24.	$\mathbf{A}_{10} \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$	$(1,1)$		§4.20, see [9]
25.	$\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(1,0)$	$\mathbb{Z}_6$	§4.21, §6.14, see [9]
26.	$\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	$(2,0)$	$\mathbb{Z}_6$	§4.22, §6.15, see [14]
27.	$\mathbf{A}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$	$(1,1)^*$	$\mathbb{Z}_6$	§4.23, §6.16, see [9], Remark 1.3
28.	$\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$	$(1,0)^*$	(3.2)	§3.1, see [5] ($\mathbb{D}_{10}$)
29.	$(2\mathbf{A}_8) \oplus \mathbf{A}_3$	$(1,0)$	Γ	§2.5, §5.4, see [4, 12] (torus)
30.	$\mathbf{A}_8 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4$	$(0,1)$		§4.24, see [9]
31.	$\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1,1)$	$\mathbb{Z}_6$	§4.25, §6.17, see [9], Remark 1.3
32.	$(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_4$	$(0,1)$		§2.6, see [12] (torus)
33.	$(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1,0)$	(2.1)	§2.7, §5.5, see [8, 12] (torus)
34.	$\mathbf{A}_7 \oplus 2\mathbf{A}_6$	$(0,1)$		§4.26, see [9]
35.	$\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2,0)$	$\mathbb{Z}_6$	§4.27, §6.18, see [9]
36.	$\mathbf{A}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$	$(1,0)$	$\mathbb{Z}_6$	§4.28, §6.19, see [14]
37.	$3\mathbf{A}_6 \oplus \mathbf{A}_1$	$(1,0)$	$\mathbb{Z}_3 \times \mathbb{D}_{14}$	§3.2, §6.20, see [10, 8] ($\mathbb{D}_{14}$)
38.	$2\mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(2,0)$	$\mathbb{Z}_6$	§4.29, §6.21, see [14]
39.	$\mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$	$(2,0)$	$\mathbb{Z}_6$	§4.30, §6.22, see [14]

Marked with a * are the sets of singularities realized by reducible sextics as well
There are 42 real and 20 pairs of complex conjugate curves

1.4. **Statements.** The results and references are collected in Table 1, which lists all irreducible maximizing sextics with $\mathbf{A}$ type singularities only.

The number of (projective equivalence classes of) sextics realizing each set of singularities is $n := r + 2c$, where r is the number of real curves and c is the number of pairs of complex conjugate ones (listed in the table).

Theorem 1.1. *Let* $\mathbf{S}$ *be a set of singularities as in Table 1, and let* $n := r + 2c$ *be the number of projective equivalence classes of irreducible sextics realizing* $\mathbf{S}$*. Then, all these sextics are defined and Galois conjugate over a certain algebraic number field* $\Bbbk$*,* minimal *in the sense that* $\Bbbk$ *is contained in the coefficient field of any defining equation in any coordinate system. Furthermore, the degree of* $\Bbbk$ *equals* n *and, if* $n > 2$*, the Galois group of the Galois closure of* $\Bbbk$ *is* $\mathbb{D}_{2n}$*.* ◁

In other words, there is a single algebraic curve defined over $\Bbbk$, and the complex curves merely differ by the embeddings $\Bbbk \hookrightarrow \mathbb{C}$. This statement is experimental; given the equations, the fact that the field is minimal is proved as in [9]. (See Remark 4.2 for the set of singularities $\mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$, line 39.)

Remark 1.2. The conclusion of Theorem 1.1 extends to all maximizing simple sets of singularities for which the defining equations of irreducible curves are known to the author. At the moment, these are

$\mathbf{E}_6 \oplus \mathbf{A}_{13}$,	$\mathbf{D}_9 \oplus \mathbf{A}_{10}$,	$\mathbf{D}_5 \oplus \mathbf{A}_{14}$,
$\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_3$,	$\mathbf{D}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$,	$\mathbf{D}_5 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_4$,
$\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_6$,		$\mathbf{D}_5 \oplus \mathbf{A}_8 \oplus \mathbf{A}_6$,

see [9].

The following notation is used in the description of the fundamental groups:

- $\mathbb{Z}_n$ is the cyclic group of order n;
- $\mathbb{D}_{2n}$ is the dihedral group of order $2n$;
- $\Gamma = PSL(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ is the modular group.

Most fundamental groups are computed as suggested in [8]; the diagrams used in the computation are listed below (see [8] for the explanation). Unfortunately, this approach works for real curves only. As a by-product, the diagrams describe the topology of the real parts of the curves; they may prove useful in the topology of real algebraic curves.

Remark 1.3. The sets of singularities $\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$, line 13, $\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, line 15, $\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$, line 19, $\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, line 22, $\mathbf{A}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$, line 27, and $\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, line 31 are realized by three Galois conjugate curves each. In each case, only one of the three curves is real, and *only for this real curve* the fundamental group $\pi_1 = \mathbb{Z}_6$ has been computed.

Remark 1.4. The set of singularities $\mathbf{A}_{11} \oplus 2\mathbf{A}_4$, line 16 is realized by two Galois conjugate curves. Both curves are real, but the group $\pi_1 = \mathbb{Z}_6$ is computed for one of them only; for the other curve, the presentation obtained is incomplete and I cannot assert that the group is finite.

2. Sextics of torus type

A plane sextic is said to be of *torus type* if its defining polynomial is of the form $f_2^3 + f_3^2$ for some polynomials f_2, f_3 of degree 2 and 3, respectively. The fundamental group of such a sextic always factors to Γ; in particular, it is infinite.

Explicit equations of all maximizing sextics of torus type are found in [12], and cited below, in the form of pairs (f_2, f_3), are the results of this paper, sometimes with references to other approaches.

2.1. **The set of singularities $(\mathbf{A}_{17} \oplus \mathbf{A}_2)$, line 3** (see [12, 1, 4])**.** The curve is `nt145` in [12]. It is defined over $\mathbb{Q}$, the defining polynomials being

$$f_2 = y^2 - \frac{73}{64}x^2 + x,$$
$$f_3 = -\frac{91}{64}xy^2 + y^2 + \frac{827}{512}x^3 - \frac{41}{16}x^2 + x.$$

An alternative equation of this curve is found in [1] and [4]. The fundamental group $\pi_1 = \Gamma$ is also computed in [1] and [4].

2.2. **The set of singularities $(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$, line 8** (see [12, 8])**.** The curve is `nt139` in [12]. It is defined over $\mathbb{Q}$, the defining polynomials being

$$f_2 = 80\left(-36y^2 + 120xy - 82x^2 + 2x\right),$$
$$f_3 = 100\left(-1512y^3 + 7794y^2x - 18y^2 - 11664yx^2 + 144xy + 5313x^3 - 194x^2 + x\right).$$

The fundamental group $\pi_1 = \Gamma$ is computed in [8], see §5.1.

2.3. **The set of singularities $(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, line 9** (see [12, 8])**.** The curve is `nt142` in [12]. It is defined over $\mathbb{Q}$, the defining polynomials being

$$f_2 = -45y^2 - 240yx - 106x^2 + 90x,$$
$$f_3 = 1025y^3 + 6045y^2x - 375y^2 + 5490yx^2 - 4050yx + 1354x^3 - 2040x^2 + 750x.$$

The fundamental group $\pi_1 = \Gamma$ is computed in [8], see §5.2.

2.4. **The set of singularities $(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$, line 17** (see [12, 8])**.** The curve is `nt118` in [12]. It is defined over $\mathbb{Q}$, the defining polynomials being

$$f_2 = \frac{1}{5}\left(-3456y^2 + 1200yx - 3005x^2 + 240x\right),$$
$$f_3 = \frac{1}{5}\left(-89856y^3 + 130464y^2x - 6912y^2 - 112680yx^2 + 8640yx + 91345x^3 - 13320x^2 + 480x\right).$$

The fundamental group $\pi_1 = \Gamma$ is computed in [8], see §5.3.

2.5. **The set of singularities $(2\mathbf{A}_8) \oplus \mathbf{A}_3$, line 29** (see [4, 12])**.** This curve is `nt128` in [12]. It is defined over $\mathbb{Q}$, the defining polynomials being

$$f_2 = -3y^2 - 6xy - x^2 + 6x + 3,$$
$$f_3 = \frac{1}{16}\left(81y^3 + 252xy^2 + 207x^2y - 162xy - 81y + 38x^3 - 180x^2 - 90x\right).$$

An alternative equation of this curve is found in [4]. The fundamental group $\pi_1 = \Gamma$ is computed in [4], see also §5.4.

2.6. **The set of singularities $(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_4$, line 32** (see [12]). These curves are `nt104` in [12]. The two curves are defined over $\mathbb{Q}(i)$; they are complex conjugate. Letting $\epsilon = \pm i$, the defining polynomials are

$$f_2 = (112 + 1566\epsilon)\left((15 + 15\epsilon)\, y^2 + (27 + 39\epsilon)\, xy - (27 + 39\epsilon)\, y - 5x^2 - (17 + 39\epsilon)\, x + 7 + 24\epsilon\right),$$

$$\begin{aligned} f_3 = \left(\frac{43848}{785} + \frac{609953}{1570}\epsilon\right) &\left((14148 + 6561\epsilon)\, y^3 + (41895 + 27990\epsilon)\, xy^2 - (42597 + 29529\epsilon)\, y^2\right. \\ &+ (21546 + 25497\epsilon)\, x^2 y + (36828 + 42696\epsilon)\, y - (72522 + 74754\epsilon)\, xy \\ &\left. - 3925x^3 - (12849 + 24093\epsilon)\, x^2 + (25008 + 43956\epsilon)\, x - 7532 - 18324\epsilon\right). \end{aligned}$$

The fundamental group is unknown.

2.7. **The set of singularities $(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, line 33** (see [12, 8]). This curve is `nt83` in [12]. It is defined over $\mathbb{Q}$, the defining polynomials being

$$f_2 = -565y^2 - 14yx + 176y - 5x^2 + 104x - 16,$$

$$f_3 = 13321y^3 + 3135y^2x - 6294y^2 + 207yx^2 - 3516yx + 1056y + 25x^3 - 558x^2 + 624x - 64.$$

The fundamental group π_1 is computed in [8], see §5.5:

$$\begin{aligned} \pi_1 = \langle \alpha_2, \alpha_3, \alpha_4 \mid\, & [\alpha_3, \alpha_4] = \{\alpha_2, \alpha_3\}_3 = \{\alpha_2, \alpha_4\}_9 = 1, \\ & \alpha_4\alpha_2\alpha_3^{-1}\alpha_4\alpha_2\alpha_4(\alpha_4\alpha_2)^{-2}\alpha_3 = (\alpha_2\alpha_4)^2\alpha_3^{-1}\alpha_2\alpha_4\alpha_3\alpha_2\rangle, \end{aligned} \tag{2.1}$$

where $\{\alpha, \beta\}_{2k+1} := (\alpha\beta)^k\alpha(\alpha\beta)^{-k}\beta^{-1}$. It is not known whether the group given by (2.1) is isomorphic to Γ.

2.8. **Other sextics of torus type.** To do

3. Other special sextics

An irreducible plane sextic D is called *special* (more precisely, $\mathbb{D}_{2n}$-*special*) if its fundamental group $\pi_1 := \pi_1(\mathbb{P}^2 \smallsetminus D)$ factors to the dihedral group $\mathbb{D}_{2n}$. The $\mathbb{D}_6$-special sextics are precisely those of torus type, see [2]; they are discussed in §2. Besides, there are $\mathbb{D}_{10}$ and $\mathbb{D}_{14}$ sextics, see §3.1 and §3.2, respectively.

3.1. **The set of singularities $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$, line 28** (see [5]). The curve is defined over $\mathbb{Q}$, the defining polynomial being

$$y^4 + \left(\frac{5}{3}x^2 - \frac{384}{5}\right)y^3 + \left(\frac{25}{36}x^4 + 483x^2\right)y^2 \\ + \left(\frac{5165}{6}x^4 - 72000x^2\right)y + \frac{675}{2}x^6 - \frac{224375}{4}x^4 + 2304000x^2.$$

Altogether, there are eight deformation families of $\mathbb{D}_{10}$-special sextics with double singular points only, one family for each of the following sets of singularities:

$$\begin{gathered} 4\mathbf{A}_4, \quad 4\mathbf{A}_4 \oplus \mathbf{A}_1, \quad 4\mathbf{A}_4 \oplus 2\mathbf{A}_1, \quad 4\mathbf{A}_4 \oplus \mathbf{A}_2, \\ \mathbf{A}_9 \oplus 2\mathbf{A}_4, \quad \mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1, \quad \mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2, \quad 2\mathbf{A}_9. \end{gathered} \tag{3.1}$$

Their equations and fundamental groups π_1 are computed in [5]. One always has $\pi_1/\pi_1'' = \mathbb{Z}_3 \times \mathbb{D}_{10}$, and $\pi_1'' = 0$ with the exception of the following two cases:

- $\pi_1'' = SL(2, \Bbbk_9)$ for $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$, line 28, and
- $\pi_1'' = (\mathbb{Z}_2)^4$ for $4\mathbf{A}_4 \oplus 2\mathbf{A}_1$.

The former group can be described as follows:

$$\begin{aligned} \bigl\langle \alpha_1, \alpha_2, \alpha_3 \bigm| [\alpha_1, \alpha_2\alpha_3\alpha_2^{-1}] = [\alpha_1\alpha_2, \beta] = \alpha_3^2 = (\alpha_1\alpha_2^2\alpha_3)^2 = 1, \\ (\alpha_2\alpha_3)^3 = (\alpha_3\alpha_2)^3,\ \beta(\alpha_1\alpha_2)^2\alpha_1 = (\alpha_2\alpha_1)^2\alpha_2\beta, \\ \{\alpha_2, (\alpha_1\alpha_2\alpha_3)\alpha_2(\alpha_1\alpha_2\alpha_3)^{-1}\}_5 = 1\bigr\rangle, \end{aligned} \tag{3.2}$$

where $\beta := (\alpha_2\alpha_3\alpha_2)\alpha_3(\alpha_2\alpha_3\alpha_2)^{-1}$. The latter is

$$\begin{aligned} \bigl\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \bigm| [\alpha_2, \alpha_4] = [\alpha_3, \alpha_4] = \{\alpha_1, \alpha_2\}_5 = (\alpha_1\alpha_2\alpha_3\alpha_4)^2 = \alpha_4^2 = 1, \\ (\alpha_1\alpha_4)^2 = (\alpha_4\alpha_1)^2,\ (\beta\alpha_2)^2 = (\alpha_2\beta)^2,\ \beta^{-1}\alpha_2\beta = \alpha_3, \\ \alpha_3^{-1}\alpha_1\alpha_2\alpha_1^{-1}\alpha_3 = (\alpha_1\alpha_2)\alpha_1(\alpha_1\alpha_2)^{-1}\bigr\rangle, \end{aligned} \tag{3.3}$$

where $\beta := \alpha_1^{-1}\alpha_4\alpha_1$.

3.2. **The set of singularities $3\mathbf{A}_6 \oplus \mathbf{A}_1$, line 37** (see [10, 8]). The curve is defined over $\mathbb{Q}$, the defining polynomial being

$$y^4 + \left(\frac{2}{27}x^2 - \frac{50}{9}x - \frac{13}{21}\right)y^3 + \left(\frac{1}{729}x^4 - \frac{158}{243}x^3 + \frac{1562}{189}x^2 + \frac{703}{189}x + \frac{1}{9}\right)y^2 \\ + \left(-\frac{4}{243}x^5 - \frac{17}{567}x^4 - \frac{38}{21}x^3 - \frac{367}{63}x^2 - \frac{2}{3}x\right)y + \frac{1}{567}x^6 + \frac{1}{63}x^5 + \frac{10}{63}x^4 + \frac{5}{7}x^3 + x^2.$$

The only other $\mathbb{D}_{14}$-sextic has the set of singularities $3\mathbf{A}_6$; it degenerates to the maximizing one. The fundamental groups $\pi_1 = \mathbb{Z}_3 \times \mathbb{D}_{14}$ are computed in [8], see §6.20 (see also [10] for the non-maximizing case).

4. Non-special irreducible sextics

Most equations listed below are those found in [9], using the Artal–Carmona–Cogolludo construction, see [1], and E. Moody's explicit formulas for the Bertini involution, see [11, 6]. For all explanations, see [9]; further details are in my web site: `http://www.fen.bilkent.edu.tr/~degt/papers/equations.zip`. For quite a few curves, parametric equations found in [14] are given; in these cases, implicit equations are immediately obtained *via*

$$f(x,y) = \operatorname{resultant}_t(xz_3(t) - z_1(t), yz_3(t) - z_2(t)). \tag{4.1}$$

Alternative references and fundamental groups are listed whenever available.

4.1. **The set of singularities $\mathbf{A}_{19}$, line 1** (see [1])**.** The two curves are defined over $\mathbb{Q}(\sqrt{5})$. Letting $\epsilon = \pm\sqrt{5}$, the defining polynomials are

$$\begin{aligned}
&y^4 + \left(-2x^2 + \left(-\frac{1}{4}\epsilon + \frac{7}{4}\right)x - \frac{1}{16}\epsilon - \frac{7}{16}\right)y^3 \\
&\quad + \left(x^4 + \left(\frac{1}{2}\epsilon - \frac{7}{2}\right)x^3 + \left(\frac{9}{16}\epsilon + \frac{51}{16}\right)x^2 + \left(-\frac{3}{8}\epsilon - \frac{7}{8}\right)x - \frac{5}{32}\epsilon + \frac{1}{32}\right)y^2 \\
&\quad + \left(\left(-\frac{1}{4}\epsilon + \frac{7}{4}\right)x^5 + \left(-\frac{15}{16}\epsilon - \frac{81}{16}\right)x^4 + \left(\frac{1}{2}\epsilon + \frac{3}{2}\right)x^3\right. \\
&\qquad \left. + \left(-\frac{1}{16}\epsilon - \frac{3}{16}\right)x^2 + \left(-\frac{1}{2}\epsilon - \frac{3}{4}\right)x + \frac{3}{32}\epsilon + \frac{5}{32}\right)y \\
&\quad + \left(\frac{7}{16}\epsilon + \frac{37}{16}\right)x^6 + \left(-\frac{1}{8}\epsilon - \frac{5}{8}\right)x^5 + \left(\frac{7}{32}\epsilon + \frac{5}{32}\right)x^4 \\
&\qquad + \left(\frac{1}{2}\epsilon + \frac{3}{4}\right)x^3 + \left(-\frac{1}{16}\epsilon - \frac{1}{16}\right)x^2 + \left(\frac{1}{32}\epsilon + \frac{1}{8}\right)x + \frac{1}{16}\epsilon + \frac{11}{64}.
\end{aligned}$$

The fundamental group $\pi_1 = \mathbb{Z}_6$ is computed in [1].

4.2. **The set of singularities $\mathbf{A}_{18} \oplus \mathbf{A}_1$, line 2** (see [1, 14])**.** The three curves are defined over $\mathbb{Q}(\epsilon)$, where $576\epsilon^3 + 1520\epsilon^2 + 700\epsilon + 125 = 0$. Their parametric equations (*cf.* (4.1)) are found in [14]:

$$\begin{aligned}
z_1 &= \left(\frac{24}{5}\epsilon^2 + \frac{91}{15}\epsilon + \frac{1}{6}\right)t^5 + t^4\epsilon + t^3 + t^2, \\
z_2 &= \left(\frac{816}{25}\epsilon^2 + \frac{52}{3}\epsilon + \frac{10}{3}\right)t^6 + \left(-\frac{384}{25}\epsilon^2 - \frac{16}{3}\epsilon - \frac{4}{3}\right)t^5 + \left(\frac{448}{25}\epsilon^2 + \frac{77}{5}\epsilon + 2\right)t^4 \\
&\quad + \left(\frac{144}{25}\epsilon^2 + 10\epsilon + 1\right)t^3 + \left(-\frac{48}{25}\epsilon^2 - \frac{1}{15}\epsilon + \frac{4}{3}\right)t^2 + 2t + 1, \\
z_3 &= \left(\frac{48}{25}\epsilon^2 + \frac{31}{15}\epsilon - \frac{1}{3}\right)t^6 + t^4.
\end{aligned}$$

The fundamental group $\pi_1 = \mathbb{Z}_6$ is computed in [1].

4.3. **The set of singularities $\mathbf{A}_{16}\oplus\mathbf{A}_3$, line 4** (see [1, 9]). The two curves are defined over $\mathbb{Q}(\sqrt{17})$. Letting $\epsilon=(59\pm3\sqrt{17})/32$, the defining polynomials are

$$\begin{aligned}
&y^4+\left(-2x^2+\left(\frac{61}{16}\epsilon-\frac{11}{4}\right)x+\frac{8153}{4096}\epsilon-\frac{4479}{1024}\right)y^3\\
&\quad+\left(x^4+\left(-\frac{35}{4}\epsilon+10\right)x^3+\left(\frac{25191}{4096}\epsilon-\frac{5697}{1024}\right)x^2+\left(\frac{283749}{524288}\epsilon-\frac{374291}{131072}\right)x-\epsilon+2\right)y^2\\
&\quad+\left(\left(\frac{79}{16}\epsilon-\frac{29}{4}\right)x^5+\left(-\frac{43197}{4096}\epsilon+\frac{15867}{1024}\right)x^4+\left(\frac{7144365}{1048576}\epsilon-\frac{2629003}{262144}\right)x^3\right.\\
&\qquad\left.+\left(-\frac{567357439}{268435456}\epsilon+\frac{208814825}{67108864}\right)x^2\right)y\\
&\quad+\left(\frac{533}{512}\epsilon-\frac{195}{128}\right)x^6+\left(-\frac{66993}{65536}\epsilon+\frac{24551}{16384}\right)x^5+\left(\frac{15872029}{33554432}\epsilon-\frac{5819291}{8388608}\right)x^4.
\end{aligned}$$

An alternative equation was first obtained in [1]. The fundamental group $\pi_1=\mathbb{Z}_6$ is computed in [1], see also §6.1.

4.4. **The set of singularities $\mathbf{A}_{16}\oplus\mathbf{A}_2\oplus\mathbf{A}_1$, line 5** (see [1, 13]). The three curves are defined over $\mathbb{Q}(\epsilon)$, where $311\epsilon^3-293\epsilon^2+85\epsilon-7=0$. Their parametric equations (*cf.* (4.1)) are found in [13]:

$$\begin{aligned}
z_1&=\left(-\frac{317}{56}\epsilon^2+\frac{221}{56}\epsilon-\frac{24}{56}\right)t^4+t^3+t^2,\\
z_2&=\frac{256}{140883}\left(-58843\epsilon^2+46797\epsilon-9236\right)t^3\\
&\qquad+\left(-\frac{398431}{3624}\epsilon^2+\frac{312615}{3624}\epsilon-\frac{58304}{3624}\right)t^2+(2-\epsilon)\,t+1,\\
z_3&=\left(\frac{13}{8}\epsilon^2-\frac{5}{8}\epsilon\right)t^6+\epsilon t^5+t^4.
\end{aligned}$$

The fundamental group $\pi_1=\mathbb{Z}_6$ is computed in [1]; a computation for the only *real* curve is also found in §6.2.

4.5. **The set of singularities $\mathbf{A}_{15}\oplus\mathbf{A}_4$, line 6** (see [1, 9]). The two curves are defined over $\mathbb{Q}(i)$; they are complex conjugate. Letting $\epsilon=\pm i$, the defining polynomials are

$$\begin{aligned}
&y^4+\left(-2x^2-\left(\frac{2002}{145}+\frac{186}{145}\epsilon\right)x+\frac{316}{5}-\frac{312}{5}\epsilon\right)y^3\\
&\quad+\left(x^4+\left(\frac{4598}{145}-\frac{66}{29}\epsilon\right)x^3+\left(-\frac{699084}{4205}+\frac{720414}{4205}\epsilon\right)x^2-\left(4+\frac{1332}{5}\epsilon\right)x-8-\frac{42}{5}\epsilon\right)y^2\\
&\quad+\left(\left(-\frac{2596}{145}+\frac{516}{145}\epsilon\right)x^5+\left(\frac{341862}{4205}-\frac{74970}{841}\epsilon\right)x^4+\left(-\frac{116576}{24389}+\frac{14155488}{121945}\epsilon\right)x^3-\left(\frac{2992}{29}+\frac{13872}{145}\epsilon\right)x^2\right)y\\
&\quad+\left(\frac{422338}{21025}-\frac{624354}{21025}\epsilon\right)x^6+\left(\frac{20023696}{609725}+\frac{180597132}{609725}\epsilon\right)x^5-\left(\frac{233585984}{707281}+\frac{955281408}{3536405}\epsilon\right)x^4.
\end{aligned}$$

The fundamental group $\pi_1=\mathbb{Z}_6$ is computed in [1], where an alternative equation of this curve is also found.

4.6. **The set of singularities $\mathbf{A}_{14} \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, line 7** (see [9]). The six curves are defined over $\mathbb{Q}(\epsilon)$, where $9\epsilon^6 - 27\epsilon^5 - 45\epsilon^4 + 195\epsilon^3 - 20\epsilon^2 - 372\epsilon + 276 = 0$. The extension is purely imaginary, with the Galois group $\mathbb{D}_{12}$. The defining polynomials are

$$
\begin{aligned}
y^4 &+ \Bigg(-2x^2 + \left(\frac{168}{55}\epsilon^5 - \frac{72}{11}\epsilon^4 - \frac{90}{11}\epsilon^3 + \frac{212}{11}\epsilon^2 + \frac{166}{33}\epsilon - \frac{604}{55}\right)x \\
&\qquad - \frac{636}{25}\epsilon^5 - \frac{761}{5}\epsilon^2 + \frac{1414}{3}\epsilon + \frac{657}{5}\epsilon^4 - \frac{852}{5}\epsilon^3 - \frac{6572}{25}\Bigg)y^3 \\
&+ \Bigg(x^4 + \left(-\frac{252}{55}\epsilon^5 + \frac{606}{55}\epsilon^4 + \frac{378}{55}\epsilon^3 - \frac{1326}{55}\epsilon^2 - \frac{118}{55}\epsilon + \frac{1816}{165}\right)x^3 \\
&\qquad + \left(\frac{202698}{3025}\epsilon^5 - \frac{212202}{605}\epsilon^4 + \frac{283608}{605}\epsilon^3 + \frac{21354}{55}\epsilon^2 - \frac{778462}{605}\epsilon + \frac{2252816}{3025}\right)x^2 \\
&\qquad + \left(-\frac{1674}{5}\epsilon^5 + \frac{9054}{5}\epsilon^4 - \frac{13038}{5}\epsilon^3 - \frac{7974}{5}\epsilon^2 + \frac{31908}{5}\epsilon - \frac{19084}{5}\right)x \\
&\qquad - \frac{342}{5}\epsilon^5 + \frac{1886}{5}\epsilon^4 - \frac{2872}{5}\epsilon^3 - \frac{1211}{5}\epsilon^2 + \frac{6192}{5}\epsilon - \frac{3828}{5}\Bigg)y^2 \\
&+ \Bigg(\left(\frac{84}{55}\epsilon^5 - \frac{246}{55}\epsilon^4 + \frac{72}{55}\epsilon^3 + \frac{266}{55}\epsilon^2 - \frac{476}{165}\epsilon - \frac{4}{165}\right)x^5 \\
&\qquad + \left(-\frac{93908}{3025}\epsilon^5 + \frac{20271}{121}\epsilon^4 - \frac{144634}{605}\epsilon^3 - \frac{25771}{165}\epsilon^2 + \frac{3304592}{5445}\epsilon - \frac{3310388}{9075}\right)x^4 \\
&\qquad + \left(\frac{199748}{1331}\epsilon^5 - \frac{5454564}{6655}\epsilon^4 + \frac{8005868}{6655}\epsilon^3 + \frac{13583132}{19965}\epsilon^2 - \frac{172685912}{59895}\epsilon + \frac{34873768}{19965}\right)x^3 \\
&\qquad + \left(-\frac{4014}{5}\epsilon^5 + \frac{48538}{11}\epsilon^4 - \frac{73550}{11}\epsilon^3 - \frac{31356}{11}\epsilon^2 + \frac{158044}{11}\epsilon - \frac{485688}{55}\right)x^2\Bigg)y \\
&+ \left(-\frac{23879}{3025}\epsilon^5 + \frac{5185}{121}\epsilon^4 - \frac{38917}{605}\epsilon^3 - \frac{4393}{165}\epsilon^2 + \frac{734636}{5445}\epsilon - \frac{2246132}{27225}\right)x^6 \\
&\qquad + \left(\frac{2367472}{6655}\epsilon^5 - \frac{2573224}{1331}\epsilon^4 + \frac{3743632}{1331}\epsilon^3 + \frac{6570808}{3993}\epsilon^2 - \frac{81413236}{11979}\epsilon + \frac{245146156}{59895}\right)x^5 \\
&\qquad + \left(-\frac{171558021}{73205}\epsilon^5 + \frac{943211199}{73205}\epsilon^4 - \frac{1430821093}{73205}\epsilon^3 - \frac{54857644}{6655}\epsilon^2 + \frac{9188546584}{219615}\epsilon - \frac{16950921512}{658845}\right)x^4.
\end{aligned}
$$

The fundamental group is unknown.

4.7. **The set of singularities $\mathbf{A}_{13}\oplus\mathbf{A}_6$, line 10** (see [9])**.** The four curves are defined over $\mathbb{Q}(\epsilon)$, where $9\epsilon^4-63\epsilon^3+175\epsilon^2-224\epsilon+112=0$. In other words,

$$\epsilon=\frac{1}{12}\left(21+\alpha_1\sqrt{16\sqrt{7}-35}+\left(\alpha_1\alpha_2\sqrt{7}+\alpha_2\sqrt{16\sqrt{7}+35}\right)i\right),$$

$\alpha_1,\alpha_2=\pm1$. The extension is purely imaginary, with the Galois group $\mathbb{D}_8$. The defining polynomials are

$$\begin{aligned}
&y^4+\left(-2x^2+\left(\frac{9}{2}\epsilon^3-\frac{21}{2}\epsilon^2-\frac{1}{2}\epsilon+16\right)x+\frac{15}{2}\epsilon^2-\frac{23}{2}\epsilon-\frac{27}{14}\epsilon^3+6\right)y^3\\
&\quad+\left(x^4+\left(-\frac{27}{4}\epsilon^3+\frac{63}{4}\epsilon^2+\frac{1}{4}\epsilon-\frac{43}{2}\right)x^3+\left(\frac{351}{28}\epsilon^3-\frac{261}{4}\epsilon^2+\frac{441}{4}\epsilon-\frac{117}{2}\right)x^2\right.\\
&\qquad\left.+\left(-\frac{33}{4}\epsilon^3+43\epsilon^2-\frac{157}{2}\epsilon+\frac{195}{4}\right)x+\frac{14}{9}\epsilon^2-\frac{1}{4}\epsilon^3+\frac{47}{36}-\frac{47}{18}\epsilon\right)y^2\\
&\quad+\left(\left(\frac{9}{4}\epsilon^3-\frac{21}{4}\epsilon^2+\frac{1}{4}\epsilon+\frac{11}{2}\right)x^5+\left(-\frac{387}{56}\epsilon^3+\frac{327}{8}\epsilon^2-\frac{603}{8}\epsilon+\frac{93}{2}\right)x^4\right.\\
&\qquad\left.+\left(-\frac{35}{4}\epsilon^3+\frac{515}{12}\epsilon^2-\frac{895}{12}\epsilon+\frac{545}{12}\right)x^3+\left(-\frac{7}{16}\epsilon^3+\frac{373}{144}\epsilon^2-\frac{701}{144}\epsilon+\frac{109}{36}\right)x^2\right)y\\
&\quad+\left(-\frac{15}{14}\epsilon^3+\frac{99}{16}\epsilon^2-\frac{185}{16}\epsilon+\frac{117}{16}\right)x^6\\
&\qquad+\left(-2\epsilon^3+\frac{223}{24}\epsilon^2-\frac{365}{24}\epsilon+\frac{211}{24}\right)x^5+\left(-\frac{3}{32}\epsilon^3+\frac{163}{288}\epsilon^2-\frac{317}{288}\epsilon+\frac{13}{18}\right)x^4.
\end{aligned}$$

The fundamental group is unknown.

4.8. **The set of singularities $\mathbf{A}_{13}\oplus\mathbf{A}_4\oplus\mathbf{A}_2$, line 11** (see [9])**.** The two curves are defined over $\mathbb{Q}(\sqrt{21})$. Letting $\epsilon=(7\pm\sqrt{21})/2$, the defining polynomials are

$$\begin{aligned}
&y^4+\left(-2x^2+\left(\frac{37}{7}\epsilon-\frac{29}{7}\right)x+\frac{11784}{245}\epsilon-\frac{14376}{245}\right)y^3\\
&\quad+\left(x^4+\left(-\frac{83}{7}\epsilon+\frac{85}{7}\right)x^3+\left(-\frac{27432}{245}\epsilon+\frac{33588}{245}\right)x^2+\left(-\frac{45960}{343}\epsilon+\frac{55344}{343}\right)x+\frac{2880}{343}\epsilon-\frac{3456}{343}\right)y^2\\
&\quad+\left(\left(\frac{46}{7}\epsilon-8\right)x^5+\left(\frac{27549}{490}\epsilon-\frac{16623}{245}\right)x^4+\left(\frac{28011}{343}\epsilon-\frac{135519}{1372}\right)x^3+\left(\frac{154512}{2401}\epsilon-\frac{26676}{343}\right)x^2\right)y\\
&\quad+\left(\frac{7899}{980}\epsilon-\frac{4773}{490}\right)x^6+\left(\frac{51531}{686}\epsilon-\frac{124575}{1372}\right)x^5+\left(\frac{4728285}{38416}\epsilon-\frac{5715117}{38416}\right)x^4.
\end{aligned}$$

The fundamental group $\pi_1=\mathbb{Z}_6$ is computed in [9], see §6.3.

4.9. **The set of singularities $\mathbf{A}_{12} \oplus \mathbf{A}_7$, line 12** (see [9])**.** The two curves are defined over $\mathbb{Q}(i)$; they are complex conjugate. Letting $\epsilon = \pm i$, the defining polynomials are

$$\begin{aligned} y^4 &+ \left(\left(\frac{46}{13} - \frac{9}{13}\epsilon \right) x^2 + \left(-4 + \frac{84}{13}\epsilon \right) x + \frac{48608}{28561} - \frac{123456}{28561}\epsilon \right) y^3 \\ &+ \left(\left(\frac{2035}{676} - \frac{207}{169}\epsilon \right) x^4 + \left(-\frac{880}{169} + \frac{1605}{169}\epsilon \right) x^3 \right. \\ &\qquad \left. + \left(\frac{54}{169} - \frac{3078}{169}\epsilon \right) x^2 + \left(\frac{27680}{2197} + \frac{30240}{2197}\epsilon \right) x - \frac{7424}{2197} - \frac{8832}{2197}\epsilon \right) y^2 \\ &+ \left(\left(-\frac{7901}{4394} - \frac{12624}{2197}\epsilon \right) x^5 + \left(\frac{46380}{2197} + \frac{21015}{2197}\epsilon \right) x^4 + \left(-\frac{328}{13} + \frac{1548}{169}\epsilon \right) x^3 + \left(\frac{1264}{169} - \frac{48}{169}\epsilon \right) x^2 \right) y \\ &+ \left(-\frac{31745}{8788} + \frac{9195}{4394}\epsilon \right) x^6 + \left(\frac{116}{169} - \frac{1737}{169}\epsilon \right) x^5 + \left(-\frac{263}{169} + \frac{366}{169}\epsilon \right) x^4. \end{aligned}$$

The fundamental group is unknown.

4.10. **The set of singularities $\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$, line 13** (see [9])**.** The three curves are defined over $\mathbb{Q}(\epsilon)$, where $441\epsilon^3 + 315\epsilon^2 + 79\epsilon + 7 = 0$. The defining polynomials are

$$\begin{aligned} y^4 &+ \left(-2x^2 + \left(-\frac{10360}{9}\epsilon^2 - \frac{5464}{9}\epsilon - \frac{6376}{81} \right) x - 28\epsilon^2 - 72\epsilon - \frac{1066}{63} \right) y^3 \\ &+ \left(x^4 + \left(\frac{5936}{3}\epsilon^2 + \frac{3152}{3}\epsilon + \frac{3728}{27} \right) x^3 + \left(-\frac{700168}{243}\epsilon^2 - \frac{381136}{243}\epsilon - \frac{3253864}{15309} \right) x^2 \right. \\ &\qquad \left. + \left(-\frac{3808}{81}\epsilon^2 - \frac{12232}{81}\epsilon - \frac{26392}{729} \right) x - \frac{980}{3}\epsilon^2 - \frac{2087}{81} - 184\epsilon \right) y^2 \\ &+ \left(\left(-\frac{7448}{9}\epsilon^2 - \frac{3992}{9}\epsilon - \frac{4808}{81} \right) x^5 + \left(\frac{532012}{243}\epsilon^2 + \frac{2121688}{1701}\epsilon + \frac{2699338}{15309} \right) x^4 \right. \\ &\qquad \left. + \left(-\frac{75739744}{59049}\epsilon^2 - \frac{340672336}{413343}\epsilon - \frac{67846576}{531441} \right) x^3 + \left(-\frac{1843816}{6561}\epsilon^2 - \frac{7259824}{45927}\epsilon - \frac{1305538}{59049} \right) x^2 \right) y \\ &+ \left(\frac{221776}{729}\epsilon^2 + \frac{123712}{729}\epsilon + \frac{1082356}{45927} \right) x^6 + \left(-\frac{32133568}{59049}\epsilon^2 - \frac{135035128}{413343}\epsilon - \frac{25605880}{531441} \right) x^5 \\ &\qquad + \left(-\frac{288948764}{4782969}\epsilon^2 - \frac{1136232968}{33480783}\epsilon - \frac{204097751}{43046721} \right) x^4. \end{aligned}$$

The fundamental group $\pi_1 = \mathbb{Z}_6$ of the only *real* curve is computed in [9], see §6.4.

4.11. **The set of singularities $\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$, line 14** (see [9])**.** The curve is defined over $\mathbb{Q}$, the defining polynomial being

$$\begin{aligned} y^4 &+ \left(-\frac{2548}{45}x^2 + \frac{2620}{39}x + \frac{9885625}{107653} \right) y^3 \\ &+ \left(\frac{1623076}{2025}x^4 - \frac{118384}{27}x^3 - \frac{82000}{13}x^2 - \frac{33781250}{24843}x + \frac{1953125}{124852} \right) y^2 \\ &+ \left(\frac{9488752}{135}x^5 + \frac{32335100}{351}x^4 + \frac{109225000}{4563}x^3 + \frac{6640625}{1911}x^2 \right) y \\ &+ \frac{1344560}{3}x^6 + \frac{1012585000}{1521}x^5 + \frac{22578125}{117}x^4. \end{aligned}$$

The fundamental group $\pi_1 = \mathbb{Z}_6$ is computed in [9], see §6.5.

4.12. **The set of singularities $\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, line 15** (see [14]). The three curves are defined over $\mathbb{Q}(\epsilon)$, where $15\epsilon^3 - 48\epsilon^2 + 40\epsilon - 10 = 0$. Their parametric equations (*cf.* (4.1)) are found in [14]:

$$\begin{aligned}
z_1 &= \left(-\frac{8}{27}\epsilon^2 + \frac{8}{27}\epsilon + \frac{5}{81}\right) t^4 + t^3 + t^2, \\
z_2 &= \left(\frac{28}{27}\epsilon^2 - \frac{55}{27}\epsilon + \frac{50}{81}\right) t^6 + t^5\epsilon + t^4, \\
z_3 &= \left(-\frac{17}{27}\epsilon^2 + \frac{8}{27}\epsilon + \frac{41}{81}\right) t^2 + (-\epsilon + 2)\, t + 1.
\end{aligned}$$

The fundamental group of the only *real* curve is $\pi_1 = \mathbb{Z}_6$, see §6.6.

4.13. **The set of singularities $\mathbf{A}_{11} \oplus 2\mathbf{A}_4$, line 16** (see [9]). The two curves are defined over $\mathbb{Q}(\sqrt{2})$. Letting $\epsilon = \pm\sqrt{2}$, the defining polynomials are

$$\begin{aligned}
y^4 &+ \left(x^2 + \left(\frac{176}{5}\epsilon - \frac{266}{5}\right) x - \frac{624}{25}\epsilon + \frac{1673}{25}\right) y^3 \\
&+ \left(\frac{1}{4}x^4 + (24\epsilon - 37)\, x^3 + \left(-\frac{37896}{25}\epsilon + \frac{108771}{50}\right) x^2 + \left(\frac{51208}{25}\epsilon - \frac{73573}{25}\right) x - \frac{48248}{5}\epsilon + \frac{281301}{20}\right) y^2 \\
&+ \left(\left(\frac{16}{5}\epsilon - \frac{26}{5}\right) x^5 + \left(-\frac{17712}{25}\epsilon + \frac{25326}{25}\right) x^4 + \left(\frac{764832}{25}\epsilon - \frac{1082148}{25}\right) x^3\right. \\
&\qquad \left. + \left(-\frac{1027296}{25}\epsilon + \frac{1461132}{25}\right) x^2 + \left(\frac{2290896}{5}\epsilon - \frac{3240594}{5}\right) x - \frac{2284848}{5}\epsilon + \frac{3243726}{5}\right) y \\
&\qquad + \left(-\frac{1728}{25}\epsilon + \frac{2484}{25}\right) x^6 + \left(\frac{1029888}{125}\epsilon - \frac{1456488}{125}\right) x^5 + \left(-\frac{6153408}{25}\epsilon + \frac{8704044}{25}\right) x^4 \\
&\qquad + \left(\frac{6179328}{25}\epsilon - \frac{8738928}{25}\right) x^3 + \left(-\frac{36687168}{5}\epsilon + \frac{51888924}{5}\right) x^2 \\
&\qquad + \left(\frac{9268992}{5}\epsilon - \frac{13108392}{5}\right) x - 54914112\epsilon + 77665716.
\end{aligned}$$

For one of the curves, the fundamental group is $\pi_1 = \mathbb{Z}_6$, see [9] and §6.7. The other group is not known to be finite.

4.14. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_9$, line 18** (see [9]). The two curves are defined over $\mathbb{Q}(\sqrt{5})$. Letting $\epsilon = (-11 \pm 3\sqrt{5})/2$, the defining polynomials are

$$\begin{aligned}
y^4 &+ \left(\left(-\frac{3}{22}\epsilon - \frac{29}{22}\right) x^2 + \left(\frac{35}{22}\epsilon + \frac{1}{22}\right) x - 5\epsilon - \frac{237}{22}\right) y^3 \\
&+ \left(\left(\frac{75}{1936}\epsilon + \frac{335}{968}\right) x^4 + \left(\frac{283}{968}\epsilon + \frac{749}{242}\right) x^3 + \left(-\frac{8667}{1936}\epsilon - \frac{2457}{484}\right) x^2\right. \\
&\qquad \left. + \left(-\frac{2325}{968}\epsilon - \frac{4919}{968}\right) x + \frac{5}{44}\epsilon + \frac{43}{176}\right) y^2 \\
&+ \left(\left(-\frac{1253}{21296}\epsilon - \frac{5387}{10648}\right) x^5 + \left(-\frac{6}{11}\epsilon - \frac{153}{44}\right) x^4 + \left(-\frac{21}{4}\epsilon - \frac{113}{8}\right) x^3 + \left(\frac{1}{4}\epsilon + \frac{1}{2}\right) x^2\right) y \\
&+ \left(\frac{1}{44}\epsilon + \frac{51}{176}\right) x^6 + \left(-\frac{9}{4}\epsilon - 4\right) x^5 + \left(\frac{1}{4}\epsilon + \frac{5}{4}\right) x^4.
\end{aligned}$$

The fundamental group $\pi_1 = \mathbb{Z}_6$ is computed in [9], see §6.8.

4.15. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$, line 19** (see [9])**.** The three curves are defined over $\mathbb{Q}(\epsilon)$, where $\epsilon^3 + 17\epsilon^2 + 51\epsilon + 43 = 0$. The defining polynomials are

$$\begin{aligned}
y^4 &+ \left(\left(\frac{54}{11}\epsilon^2 + \frac{828}{11}\epsilon + \frac{1368}{11}\right) x^2 + \left(-\frac{262}{33}\epsilon^2 - \frac{4040}{33}\epsilon - \frac{2390}{11}\right) x + \frac{65938}{891}\epsilon + \frac{118675}{891} + \frac{4261}{891}\epsilon^2\right) y^3 \\
&+ \left(\left(\frac{162}{11}\epsilon^2 + \frac{26892}{121}\epsilon + \frac{39447}{121}\right) x^4 + \left(-\frac{9996}{121}\epsilon^2 - \frac{151776}{121}\epsilon - \frac{233652}{121}\right) x^3 \right.\\
&\qquad + \left(\frac{167278}{1089}\epsilon^2 + \frac{2549852}{1089}\epsilon + \frac{4052902}{1089}\right) x^2 + \left(-\frac{135448}{1089}\epsilon^2 - \frac{2071772}{1089}\epsilon - \frac{3370324}{1089}\right) x \\
&\qquad \left. + \frac{133565}{3267}\epsilon^2 + \frac{2044138}{3267}\epsilon + \frac{3339872}{3267}\right) y^2 \\
&+ \left(\left(\frac{38070}{1331}\epsilon^2 + \frac{579960}{1331}\epsilon + \frac{916002}{1331}\right) x^5 + \left(-\frac{120423}{1331}\epsilon^2 - \frac{1845894}{1331}\epsilon - \frac{3049025}{1331}\right) x^4 \right.\\
&\qquad \left. + \left(\frac{32680}{363}\epsilon^2 + \frac{1509256}{1089}\epsilon + \frac{233968}{99}\right) x^3 + \left(-\frac{15124}{363}\epsilon^2 - \frac{696968}{1089}\epsilon - \frac{1169206}{1089}\right) x^2\right) y \\
&+ \left(\frac{1152}{121}\epsilon^2 + \frac{17712}{121}\epsilon + \frac{29916}{121}\right) x^6 + \left(-\frac{3488}{363}\epsilon^2 - \frac{55012}{363}\epsilon - \frac{35868}{121}\right) x^5 + \left(\frac{325}{33}\epsilon^2 + \frac{5018}{33}\epsilon + \frac{2906}{11}\right) x^4.
\end{aligned}$$

The fundamental group $\pi_1 = \mathbb{Z}_6$ of the only *real* curve is computed in [9], see §6.9.

4.16. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$, line 20** (see [9])**.** The two curves are defined over $\mathbb{Q}(\sqrt{3})$. Letting $\epsilon = (-6 \pm \sqrt{3})/11$, the defining polynomials are

$$\begin{aligned}
y^4 &+ \left((-231\epsilon - 164)\, x^2 + (964\epsilon + 676)\, x - \frac{1018720}{1331}\epsilon - \frac{715712}{1331}\right) y^3 \\
&+ \left(\left(4389\epsilon + \frac{12343}{4}\right) x^4 + (-36113\epsilon - 25382)\, x^3 + \left(101502\epsilon + \frac{784818}{11}\right) x^2 \right.\\
&\qquad \left. - \left(\frac{13956576}{121}\epsilon + \frac{9810272}{121}\right) x + \frac{5574784}{121}\epsilon + \frac{3918592}{121}\right) y^2 \\
&+ \left(\left(16654\epsilon + \frac{23411}{2}\right) x^5 + \left(-92955\epsilon - \frac{718734}{11}\right) x^4 \right.\\
&\qquad \left. + \left(\frac{1691724}{11}\epsilon + \frac{1189136}{11}\right) x^3 + \left(-\frac{772624}{11}\epsilon - \frac{543088}{11}\right) x^2\right) y \\
&+ \left(\frac{29713}{2}\epsilon + \frac{459485}{44}\right) x^6 + (-50211\epsilon - 35294)\, x^5 + (26770\epsilon + 18817)\, x^4.
\end{aligned}$$

The fundamental group $\pi_1 = \mathbb{Z}_6$ is computed in [9], see §6.10.

4.17. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_3$, line 21** (see [9])**.** The two curves are defined over $\mathbb{Q}(\sqrt{-7})$; they are complex conjugate. Letting $\epsilon = \pm i\sqrt{7}$, the defining polynomials are

$$\begin{aligned}
y^4 &+ \left(\left(\frac{1695}{616}\epsilon + \frac{643}{88}\right)x^2 + \left(\frac{4420}{33}\epsilon + \frac{19180}{33}\right)x + \frac{4096}{33}\epsilon - \frac{28672}{33}\right)y^3 \\
&+ \left(\left(\frac{1089885}{108416}\epsilon + \frac{10559}{108416}\right)x^4 + \left(\frac{1836433}{2541} + \frac{1662385}{847}\epsilon\right)x^3 + \left(\frac{37145792}{847}\epsilon + \frac{61946944}{847}\right)x^2\right. \\
&\qquad \left. + \left(-\frac{249036800}{7623}\epsilon - \frac{2303197184}{7623}\right)x + \frac{914358272}{6237} - \frac{109051904}{2079}\epsilon\right)y^2 \\
&+ \left(\left(-\frac{1027528757}{149072} + \frac{7228106645}{3130512}\epsilon\right)x^5 + \left(\frac{2754946112}{17787}\epsilon - \frac{294225088}{2541}\right)x^4\right. \\
&\qquad \left. + \left(\frac{37738900480}{14553}\epsilon + \frac{3698441216}{2079}\right)x^3 + \left(-\frac{218103808}{81} - \frac{486539264}{1323}\epsilon\right)x^2\right)y \\
&+ \left(-\frac{419451757}{4851}\epsilon + \frac{418532809}{539}\right)x^6 \\
&\qquad + \left(-\frac{1680525568}{189}\epsilon + \frac{33403489024}{1323}\right)x^5 + \left(\frac{5040029696}{1323}\epsilon + \frac{4795187200}{3969}\right)x^4.
\end{aligned}$$

The fundamental group is unknown.

4.18. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, line 22** (see [1, 14])**.** The three curves are defined over $\mathbb{Q}(\epsilon)$, where $3\epsilon^3 + 9\epsilon^2 + 5\epsilon + 7 = 0$. Their parametric equations (*cf.* (4.1)) are found in [14]:

$$\begin{aligned}
z_1 &= \left(-\frac{2}{27}\epsilon^2 + \frac{7}{27}\epsilon + \frac{56}{81}\right)t^4 + t^3 + t^2, \\
z_2 &= \left(\frac{7}{27}\epsilon^2 - \frac{11}{27}\epsilon - \frac{7}{81}\right)t^6 + t^5\epsilon + t^4, \\
z_3 &= \left(-\frac{11}{27}\epsilon^2 - \frac{11}{27}\epsilon + \frac{11}{81}\right)t^2 + (-\epsilon + 2)\,t + 1.
\end{aligned}$$

The fundamental group of the only *real* curve is $\pi_1 = \mathbb{Z}_6$, see §6.11.

4.19. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$, line 23** (see [9]). The two curves are defined over $\mathbb{Q}(\sqrt{15})$. Letting $\epsilon = (-11 \pm 3\sqrt{15})/7$, the defining polynomials are

$$\begin{aligned}
y^4 &+ \left(\left(\frac{1904}{3} + \frac{595}{3}\epsilon\right)x^2 + \left(\frac{20482}{5} + 931\epsilon\right)x - \frac{520625}{132}\epsilon - \frac{291550}{33}\right)y^3 \\
&+ \left(\left(\frac{621061}{6} + \frac{192185}{6}\epsilon\right)x^4 + \left(\frac{38155327}{15} + \frac{2363578}{3}\epsilon\right)x^3 + \left(-\frac{6009531423}{1100} - \frac{385022883}{220}\epsilon\right)x^2\right. \\
&\qquad \left. + \left(\frac{3995425}{6}\epsilon + \frac{23173465}{12}\right)x - \frac{13038235}{96} - \frac{3287375}{96}\epsilon\right)y^2 \\
&+ \left(\left(\frac{1984902864}{5} + 122856790\epsilon\right)x^5 + \left(-\frac{44924331172}{165}\epsilon - \frac{725383892716}{825}\right)x^4\right. \\
&\qquad \left. + \left(\frac{31917917731}{150}\epsilon + \frac{250263024452}{375}\right)x^3 + \left(-82943000 - \frac{49816375}{2}\epsilon\right)x^2\right)y \\
&+ \left(-\frac{41376305712}{11}\epsilon - \frac{133678448176}{11}\right)x^6 + \left(\frac{35791529088}{5}\epsilon + \frac{113349772424}{5}\right)x^5 \\
&\qquad + \left(-\frac{11742500000}{3}\epsilon - \frac{38123312500}{3}\right)x^4.
\end{aligned}$$

The fundamental group $\pi_1 = \mathbb{Z}_6$ is computed in [9], see §6.12.

4.20. **The set of singularities $\mathbf{A}_{10}\oplus 2\mathbf{A}_4\oplus\mathbf{A}_1$, line 24** (see [9])**.** The three curves are defined over $\mathbb{Q}(\epsilon)$, where $7\epsilon^3+43\epsilon^2+77\epsilon+49=0$. The defining polynomials are

$$\begin{aligned}
&y^4+\left(\left(\frac{37309}{178304}\epsilon^2+\frac{26111}{44576}\epsilon-\frac{14311}{25472}\right)x^2+1\right)y^3\\
&\quad+\left(\left(-\frac{88525258533}{1557823504384}\epsilon^2-\frac{19547591059}{111273107456}\epsilon+\frac{3833989567}{31792316416}\right)x^4\right.\\
&\qquad+\left(-\frac{2611518593}{13909138432}\epsilon^2-\frac{802416009}{993509888}\epsilon-\frac{115885817}{283859968}\right)x^3+\left(\frac{124037257}{496754944}\epsilon^2+\frac{395561}{8870624}\epsilon-\frac{18827545}{10137856}\right)x^2\\
&\qquad\left.+\left(\frac{458483}{1108828}\epsilon^2+\frac{6312189}{4435312}\epsilon+\frac{1441041}{633616}\right)x-\frac{5464}{39601}\epsilon^2-\frac{74496}{277207}\epsilon-\frac{13295}{158404}\right)y^2\\
&\quad+\left(\left(\frac{3195588747347967}{30380673982496768}\epsilon^2+\frac{490287145164577}{1085024070803456}\epsilon+\frac{152797699711529}{620013754744832}\right)x^5\right.\\
&\qquad+\left(\frac{122537326169765}{542512035401728}\epsilon^2+\frac{54239062570643}{38750859671552}\epsilon+\frac{25080232500109}{11071674191872}\right)x^4\\
&\qquad+\left(-\frac{7704399837271}{19375429835776}\epsilon^2-\frac{3086075117007}{1383959273984}\epsilon-\frac{1382988778847}{395416935424}\right)x^3\\
&\qquad+\left(\frac{280990228017}{691979636992}\epsilon^2+\frac{59090447007}{49427116928}\epsilon+\frac{20491288029}{14122033408}\right)x^2\\
&\qquad\left.+\left(-\frac{22854715}{441313544}\epsilon^2-\frac{4857155}{252179168}\epsilon+\frac{21181495}{252179168}\right)x\right)y\\
&\quad+\left(-\frac{566739757458334863413}{473987123205321575628 8}\epsilon^2-\frac{6823914272055613507}{10580069714404499456}\epsilon-\frac{72890565412789579675}{96732065960269709312}\right)x^6\\
&\qquad+\left(\frac{3985340521419752219}{21160139428808998912}\epsilon^2+\frac{3389724487695743913}{3022877061258428416}\epsilon+\frac{329761923581611341}{215919790089887744}\right)x^5\\
&\qquad+\left(\frac{4897016303132771}{377859632657303552}\epsilon^2-\frac{27017672358973381}{107959895044943872}\epsilon-\frac{10332281942217573}{15422842149277696}\right)x^4\\
&\qquad+\left(\frac{1547244742317313}{38557105373194 24}\epsilon^2+\frac{129706927860283}{137703947761408}\epsilon+\frac{61295661362303}{78687970149376}\right)x^3\\
&\qquad+\left(-\frac{14738246306113}{78687970149376}\epsilon^2-\frac{2400930361203}{5620569296384}\epsilon-\frac{475931747445}{1605876941824}\right)x^2
\end{aligned}$$

The fundamental group is unknown, *cf.* §6.13.

4.21. **The set of singularities $\mathbf{A}_{10}\oplus\mathbf{A}_4\oplus\mathbf{A}_3\oplus\mathbf{A}_2$, line 25** (see [9])**.** The curve is defined over $\mathbb{Q}$, the defining polynomial being

$$\begin{aligned}
&y^4+\left(-\frac{1}{7}x^2-\frac{72579}{3388}x+\frac{1397294028}{512435}\right)y^3\\
&\quad+\left(\frac{1}{196}x^4+\frac{83109}{23716}x^3-\frac{1305916911}{2608760}x^2+\frac{8478820584}{717409}x+\frac{1184481792}{65219}\right)y^2\\
&\quad+\left(-\frac{273}{1936}x^5+\frac{2842749}{146410}x^4-\frac{1586605293}{2576816}x^3+\frac{129552696}{14641}x^2\right)y\\
&\quad+\frac{521703}{2342560}x^6-\frac{956281599}{28344976}x^5+\frac{5554571841}{5153632}x^4.
\end{aligned}$$

The fundamental group $\pi_1=\mathbb{Z}_6$ is computed in [9], see §6.14.

4.22. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$, line 26** (see [14])**.** The two curves are defined over $\mathbb{Q}(\sqrt{5})$. Letting $\epsilon = \pm\sqrt{5}$, the parametric equations (*cf.* (4.1)) are

$$\begin{aligned} z_1 &= \left(\frac{3}{2}\epsilon - \frac{25}{6}\right) t^4 + t^3 + t^2, \\ z_2 &= \left(\frac{5}{6}\epsilon - \frac{5}{6}\right) t^6 + t^5\epsilon + t^4, \\ z_3 &= \left(-\frac{11}{6}\epsilon + \frac{11}{6}\right) t^2 + (-\epsilon + 2)\, t + 1. \end{aligned}$$

Both fundamental groups are $\pi_1 = \mathbb{Z}_6$, see §6.15.

4.23. **The set of singularities $\mathbf{A}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$, line 27** (see [9])**.** The three curves are defined over $\mathbb{Q}(\epsilon)$, where $57\epsilon^3 - 3196\epsilon^2 + 221\epsilon - 7 = 0$. The defining polynomials are

$$\begin{aligned} y^4 &+ \Bigg(\left(-\frac{71488}{37765}\epsilon^2 + \frac{29114368}{113295}\epsilon - \frac{256192}{16185} \right) x^2 \\ &\qquad + \left(-\frac{908504}{113295}\epsilon^2 + \frac{137387864}{339885}\epsilon - \frac{1774616}{48555} \right) x - \frac{3496}{7553}\epsilon^2 + \frac{2410904}{113295}\epsilon + \frac{1063288}{16185} \Bigg) y^3 \\ &+ \Bigg(\left(\frac{77703984128}{13633165}\epsilon^2 - \frac{7510651904}{5842785}\epsilon + \frac{227505152}{5842785} \right) x^4 \\ &\qquad + \left(-\frac{12933947776}{2152605}\epsilon^2 - \frac{63030771584}{6457815}\epsilon + \frac{375078656}{922545} \right) x^3 + \left(-\frac{27896832}{37765}\epsilon^2 + \frac{172695312}{7553}\epsilon - \frac{7529904}{5395} \right) x^2 \\ &\qquad + \left(\frac{2362688}{22659}\epsilon^2 - \frac{402316256}{67977}\epsilon + \frac{6227744}{9711} \right) x - \frac{957232}{113295}\epsilon + \frac{5472}{37765}\epsilon^2 + \frac{300208}{16185} \Bigg) y^2 \\ &+ \Bigg(\left(-\frac{1291810154184704}{111012915}\epsilon^2 + \frac{276706956775424}{333038745}\epsilon - \frac{8591281836032}{333038745} \right) x^5 \\ &\qquad + \left(-\frac{808745183232}{1947595}\epsilon^2 - \frac{115421092864}{1947595}\epsilon + \frac{3162643456}{1947595} \right) x^4 \\ &\qquad + \left(-\frac{3580103296}{307515}\epsilon^2 + \frac{58021363456}{922545}\epsilon - \frac{2477778688}{922545} \right) x^3 + \left(\frac{1283968}{5395}\epsilon^2 - \frac{220127488}{16185}\epsilon + \frac{14124544}{16185} \right) x^2 \Bigg) y \\ &+ \left(-\frac{113917996542328832}{703081795}\epsilon^2 + \frac{4712810747760640}{421849077}\epsilon - \frac{750594787303424}{2109245385} \right) x^6 \\ &\quad + \left(-\frac{510923510687744}{111012915}\epsilon^2 + \frac{169884655560704}{333038745}\epsilon - \frac{4434795072512}{333038745} \right) x^5 \\ &\quad + \left(-\frac{148571136}{1805}\epsilon^2 - \frac{259999488}{1805}\epsilon + \frac{8605184}{1805} \right) x^4; \end{aligned}$$

The fundamental group $\pi_1 = \mathbb{Z}_6$ of the only *real* curve is computed in [9], see §6.16.

4.24. **The set of singularities $\mathbf{A}_8 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4$, line 30** (see [9])**.** The two curves are defined over $\mathbb{Q}(i)$ and are complex conjugate. Letting $\epsilon = \pm i$, the defining polynomials are

$$\begin{aligned}
y^4 &+ \left(\left(\frac{1966}{4995} + \frac{3229}{1665}\epsilon\right)x^2 - \left(\frac{934}{555} + \frac{1238}{555}\epsilon\right)x + \frac{726}{185} - \frac{104}{555}\epsilon\right)y^3 \\
&+ \left(\left(-\frac{89972813}{99800100} + \frac{3174107}{8316675}\epsilon\right)x^4 + \left(\frac{2624566}{1663335} - \frac{9630238}{1663335}\epsilon\right)x^3\right. \\
&\qquad \left. + \left(-\frac{175604}{308025} + \frac{9728966}{924075}\epsilon\right)x^2 - \left(\frac{209666}{34225} + \frac{107962}{34225}\epsilon\right)x + \frac{124821}{34225} + \frac{3672}{34225}\epsilon\right)y^2 \\
&+ \left(\left(\frac{295456349909}{83083583250} - \frac{81260812187}{83083583250}\epsilon\right)x^5 + \left(-\frac{540008023}{41028930} + \frac{407759096}{61543395}\epsilon\right)x^4\right. \\
&\qquad \left. + \left(\frac{30763298}{6331625} - \frac{266726626}{56984625}\epsilon\right)x^3 + \left(\frac{60967122}{6331625} + \frac{95254329}{6331625}\epsilon\right)x^2\right)y \\
&- \left(\frac{18957786554864}{2561743816875} + \frac{7849012952123}{2561743816875}\epsilon\right)x^6 \\
&\qquad + \left(\frac{16506409877}{189758801250} + \frac{226520022721}{21084311250}\epsilon\right)x^5 + \left(-\frac{120421479563}{14056207500} + \frac{23558056107}{1171350625}\epsilon\right)x^4.
\end{aligned}$$

The fundamental group is unknown.

4.25. **The set of singularities $\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, line 31** (see [9]). The three curves are defined over $\mathbb{Q}(\epsilon)$, where $5\epsilon^3 - 3495\epsilon^2 + 8047\epsilon - 10925 = 0$. The defining polynomials are

$$\begin{aligned}
y^4 + \Bigg(&\frac{10995719975}{12545612828} + \left(\frac{206210}{88119549}\epsilon^2 + \frac{2777926}{88119549}\epsilon - \frac{6513650}{88119549}\right)x^2 \\
&+ \left(-\frac{8038965}{1069556854}\epsilon^2 - \frac{3527268028}{11230346967}\epsilon + \frac{4491270675}{7486897978}\right)x \\
&- \frac{179785535}{37636838484}\epsilon^2 - \frac{2186603299}{18818419242}\epsilon\Bigg)y^3 \\
+ \Bigg(&\left(\frac{24353508629}{35159700051}\epsilon^2 - \frac{505560092368}{316437300459}\epsilon + \frac{687589871950}{316437300459}\right)x^4 \\
&+ \left(-\frac{1095417020222450}{120984527875491}\epsilon^2 + \frac{842641645287125}{40328175958497}\epsilon - \frac{3436500857753125}{120984527875491}\right)x^3 \\
&+ \left(-\frac{64569368357938121}{15418805941465353}\epsilon^2 + \frac{49591539256638499}{5139601980488451}\epsilon - \frac{67639211203442600}{5139601980488451}\right)x^2 \\
&+ \left(-\frac{3642336074569}{25862780844922}\epsilon^2 + \frac{2630108623463}{12931390422461}\epsilon - \frac{6212909320525}{25862780844922}\right)x \\
&- \frac{14513794137}{1603492412385164}\epsilon^2 - \frac{32516845353819}{801746206192582}\epsilon + \frac{286132118320575}{1603492412385164}\Bigg)y^2 \\
+ \Bigg(&\left(-\frac{6492996907022460085}{2533267869392934}\epsilon^2 + \frac{3742481940974968815 1}{6333169673482335}\epsilon - \frac{6787862782538738195}{844422623130978}\right)x^5 \\
&+ \left(-\frac{20583538190394242005}{23914882684721772}\epsilon^2 + \frac{21355366675884909159 7}{107616972081247974}\epsilon - \frac{58099369300967268372 5}{215233944162495948}\right)x^4 \\
&+ \left(-\frac{441649933022498337617}{2285864203281322707}\epsilon^2 + \frac{509125917476367825769 4}{11429321016406613535}\epsilon - \frac{138512231222150927924 5}{2285864203281322707}\right)x^3 \\
&+ \left(-\frac{4492773285046212}{5751301866871277}\epsilon^2 + \frac{51838191846097386}{28756509334356385}\epsilon - \frac{14106870468876030}{5751301866871277}\right)x^2\Bigg)y \\
+ \Bigg(&-\frac{9561816751401874230821081}{184025022258934035540}\epsilon^2 + \frac{11022622425993408593722373}{92012511129467017770}\epsilon - \frac{6664030757785116029162 45}{4089444939087423012}\Bigg)x^6 \\
&+ \left(-\frac{8943221863808894296081741}{651471297935176971495}\epsilon^2 + \frac{6185713056561977935250472 4}{195441389380553091448 5}\epsilon - \frac{16828846363786267744312645}{390882778761106182897}\right)x^5 \\
&+ \left(-\frac{18871249808243393430933265 3}{830263976368497762560 85}\epsilon^2 + \frac{87017212928788761618001324}{16605279527369955251217}\epsilon - \frac{39456494202758606509140325}{5535093175789985083739}\right)x^4.
\end{aligned}$$

The fundamental group $\pi_1 = \mathbb{Z}_6$ of the only *real* curve is computed in [9], see §6.17.

4.26. **The set of singularities $\mathbf{A}_7 \oplus 2\mathbf{A}_6$, line 34** (see [9])**.** The two curves are defined over $\mathbb{Q}(\sqrt{-7})$ and are complex conjugate. However, the defining equations look much simpler over $\mathbb{Q}(\epsilon)$, where $49\epsilon^4 + 245\epsilon^3 + 357\epsilon^2 + 56\epsilon + 22$. They are

$$\begin{aligned}
y^4 &+ \Bigg(\Big(-\frac{7}{9}\epsilon^3 - \frac{14}{9}\epsilon^2 - \frac{14}{9}\epsilon - \frac{1}{9}\Big)x^2 + \Big(-\frac{2359}{891}\epsilon^3 - \frac{602}{99}\epsilon^2 + \frac{322}{297}\epsilon + \frac{89}{81}\Big)x + \frac{119}{324}\epsilon^3 + \frac{77}{36}\epsilon^2 + \frac{427}{108}\epsilon + \frac{371}{162}\Bigg)y^3 \\
&+ \Bigg(\Big(-\frac{77}{36}\epsilon^3 - \frac{11}{2}\epsilon^2 - \frac{5}{6}\epsilon - \frac{97}{252}\Big)x^4 + \Big(-\frac{7511}{1782}\epsilon^3 - \frac{5878}{297}\epsilon^2 - \frac{460}{99}\epsilon - \frac{1307}{1134}\Big)x^3 \\
&\qquad + \Big(\frac{154357}{26136}\epsilon^3 + \frac{14627}{2904}\epsilon^2 - \frac{22069}{8712}\epsilon + \frac{1175}{2079}\Big)x^2 + \Big(-\frac{637}{1944}\epsilon^3 - \frac{343}{648}\epsilon^2 + \frac{91}{72}\epsilon + \frac{455}{972}\Big)x \\
&\qquad + \frac{19355}{46656}\epsilon^3 + \frac{12397}{5832}\epsilon^2 + \frac{37625}{11664}\epsilon + \frac{10493}{15552}\Bigg)y^2 \\
&+ \Bigg(\Big(\frac{3955}{594}\epsilon^3 + \frac{3265}{198}\epsilon^2 + \frac{443}{198}\epsilon + \frac{191}{189}\Big)x^5 + \Big(\frac{86525}{39204}\epsilon^3 + \frac{206837}{13068}\epsilon^2 + \frac{55987}{30492}\epsilon + \frac{5921}{6237}\Big)x^4 \\
&\qquad + \Big(\frac{50707013}{7762392}\epsilon^3 + \frac{24270925}{862488}\epsilon^2 + \frac{9653245}{2587464}\epsilon + \frac{641599}{352836}\Big)x^3 + \Big(\frac{11123}{128304}\epsilon^3 + \frac{13517}{128304}\epsilon^2 + \frac{3479}{128304}\epsilon + \frac{11}{486}\Big)x^2\Bigg)y \\
&+ \Big(-\frac{713627}{29403}\epsilon^3 - \frac{13460885}{137214}\epsilon^2 - \frac{1826971}{137214}\epsilon - \frac{1690177}{261954}\Big)x^6 + \Big(\frac{72492749}{1940598}\epsilon^3 + \frac{756238}{35937}\epsilon^2 + \frac{1472864}{323433}\epsilon + \frac{196919}{176418}\Big)x^5 \\
&\qquad + \Big(-\frac{8192884}{10673289}\epsilon^3 - \frac{158628029}{85386312}\epsilon^2 - \frac{22937339}{85386312}\epsilon - \frac{309409}{2587464}\Big)x^4.
\end{aligned}$$

The fundamental group is unknown.

4.27. **The set of singularities $\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$, line 35** (see [9])**.** The two curves are defined over $\mathbb{Q}(\sqrt{21})$. Letting $\epsilon = (57 \pm 13\sqrt{21})/50$, the defining polynomials are

$$\begin{aligned}
y^4 &+ \Bigg(\Big(-\frac{2096}{36855}\epsilon - \frac{16}{4095}\Big)x^2 + \Big(\frac{1064792}{4641}\epsilon + \frac{164672}{13923}\Big)x - \frac{15332575}{546}\epsilon - \frac{1315297}{910}\Bigg)y^3 \\
&+ \Bigg(\Big(\frac{1792}{17769375} + \frac{243136}{124385625}\epsilon\Big)x^4 + \Big(-\frac{1246208128}{140970375}\epsilon - \frac{7127168}{15663375}\Big)x^3 + \Big(\frac{1616108248}{1183455} + \frac{697760564}{26299}\epsilon\Big)x^2 \\
&\qquad + \Big(-\frac{28379828}{819} - \frac{1654167700}{2457}\epsilon\Big)x - \frac{2360846}{273} - \frac{45868675}{273}\epsilon\Bigg)y^2 \\
&+ \Bigg(\Big(-\frac{1628150272}{67967859375} - \frac{31633197568}{67967859375}\epsilon\Big)x^5 + \Big(\frac{341154688}{1521585} + \frac{178963157632}{41082795}\epsilon\Big)x^4 \\
&\qquad + \Big(-\frac{1313841345376}{46560501} - \frac{8508836838880}{15520167}\epsilon\Big)x^3 + \Big(-\frac{158092264400}{53703}\epsilon - \frac{2712319456}{17901}\Big)x^2\Bigg)y \\
&+ \Big(-\frac{46153511699456}{415963299375}\epsilon - \frac{2375506509824}{415963299375}\Big)x^6 + \Big(\frac{57116410887488512}{94285014525}\epsilon + \frac{36293375684608}{1164012525}\Big)x^5 \\
&\qquad + \Big(-\frac{4727931285666176}{7123756653} - \frac{30619523458081600}{2374585551}\epsilon\Big)x^4.
\end{aligned}$$

The fundamental group $\pi_1 = \mathbb{Z}_6$ is computed in [9], see §6.18.

4.28. **The set of singularities $\mathbf{A}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$, line 36** (see [14]). The curve is defined over $\mathbb{Q}$. Letting $\epsilon^2 - \epsilon + 1 = 0$, the parametric equations are

$$\begin{aligned} z_1 &= t\epsilon\left(-2t^2 + 2t\epsilon + 1 - \epsilon\right)\left(4t^2 - 5t + 2\right), \\ z_2 &= \left(2t^2 + t + 1\right)\left(2t^2 - (2-\epsilon)\,t + 1 - \epsilon\right)^2, \\ z_3 &= t^2\epsilon^2\left(2t^2 - 2t + 1\right), \end{aligned}$$

and the implicit equation, *cf.* (4.1), is rational:

$$\begin{aligned} 32y^4 &+ \left(-16x^2 + 288x + 248\right)y^3 + \left(2x^4 - 96x^3 + 570x^2 + 2816x - 198\right)y^2 \\ &+ \left(6x^5 - 50x^4 - 1130x^3 + 10650x^2 - 1670x - 238\right)y \\ &- 9x^6 + 303x^5 - 3430x^4 + 13375x^3 - 3530x^2 - 973x - 49. \end{aligned}$$

The fundamental group is $\pi_1 = \mathbb{Z}_6$, see §6.19.

4.29. **The set of singularities $2\mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, line 38** (see [14]). The two curves are defined over $\mathbb{Q}(\sqrt{21})$. Letting $\epsilon = \pm\sqrt{21}$, the parametric equations (*cf.* (4.1)) are

$$\begin{aligned} z_1 &= \left(-\frac{989}{578} - \frac{401}{578}\epsilon\right)t^4 + \left(\frac{86}{17} + \frac{9}{17}\epsilon\right)t^3 + t^2, \\ z_2 &= \left(\frac{4359665}{48275138} - \frac{3144015}{48275138}\epsilon\right)t^6 + \left(\frac{72220}{83521} - \frac{23645}{83521}\epsilon\right)t^5 + \left(\frac{1020961}{167042} + \frac{30749}{167042}\epsilon\right)t^4 \\ &\quad + \left(-\frac{12791}{9826} - \frac{18219}{9826}\epsilon\right)t^3 + \left(\frac{53}{289} - \frac{373}{289}\epsilon\right)t^2 + \left(\frac{47}{17} - \frac{2}{17}\epsilon\right)t + 1, \\ z_3 &= \left(\frac{191}{578} + \frac{19}{578}\epsilon\right)t^6 + t^5 + t^4. \end{aligned}$$

Both fundamental groups are $\pi_1 = \mathbb{Z}_6$, see §6.21.

4.30. **The set of singularities $\mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$, line 39** (see [14]). The two curves are defined over $\mathbb{Q}(\sqrt{7})$. Letting $\epsilon = (-231 \pm 92\sqrt{7})/7$, the parametric equations (see (4.1)) are

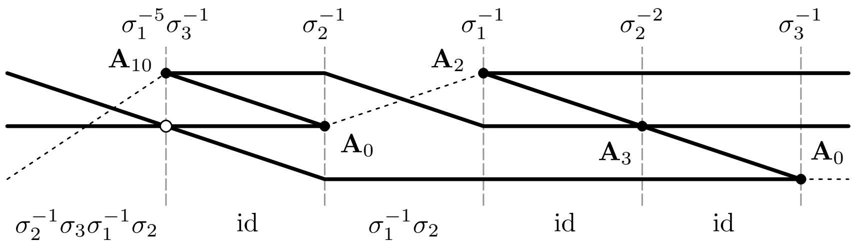
$$\begin{aligned} z_1 &= \left(\frac{237}{46}\epsilon - \frac{2523}{322}\right)t^6 + \left(\frac{11}{46}\epsilon + \frac{87}{46}\right)t^5 + \left(\frac{3}{46}\epsilon + \frac{7}{46}\right)t^4 \\ &\quad + \left(-\frac{21}{1334}\epsilon - \frac{1705}{1334}\right)t^3 + \left(-\frac{7}{841}\epsilon - \frac{462}{841}\right)t^2, \\ z_2 &= \left(-\frac{102125}{69}\epsilon + \frac{1267387}{483}\right)t^6 + \left(\frac{7}{69}\epsilon + \frac{1}{69}\right)t^4 + \left(\frac{1456}{69}\epsilon - \frac{2552}{69}\right)t^5, \\ z_3 &= \left(\frac{26766461}{161}\epsilon - \frac{332142858}{1127}\right)t^6 + \left(\frac{2578167}{161}\epsilon - \frac{4569153}{161}\right)t^5 + \left(\frac{779250}{161}\epsilon - \frac{1381763}{161}\right)t^4 \\ &\quad + \left(\frac{10866}{23}\epsilon - \frac{136242}{161}\right)t^3 + \left(\frac{787}{23}\epsilon - \frac{10092}{161}\right)t^2 + \left(\frac{81}{23}\epsilon - \frac{87}{23}\right)t + 1. \end{aligned}$$

Both fundamental groups are $\pi_1 = \mathbb{Z}_6$, see §6.22.

Remark 4.2. In the above representation, the point $(0, \infty)$ is of type $\mathbf{A}_5$ and, after rescaling, the y-discriminant is defined over $\mathbb{Q}$. To establish Theorem 1.1 for this curve, *i.e.*, to prove that $\mathbb{Q}(\epsilon^2)$ is the minimal field of definition, one should place to $(0, \infty)$ the $\mathbf{A}_6$-type point; then, one can proceed as in [9].

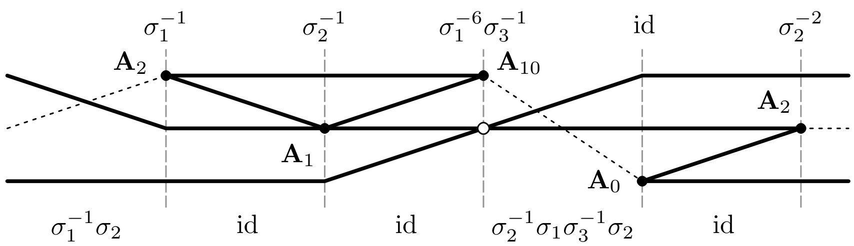
5. The fundamental groups: torus type

5.1. **The set of singularities $(\mathbf{A}_{14}\oplus\mathbf{A}_2)\oplus\mathbf{A}_3$, line 8** (see [8])**.** The curve is described in §2.2, and the diagram is



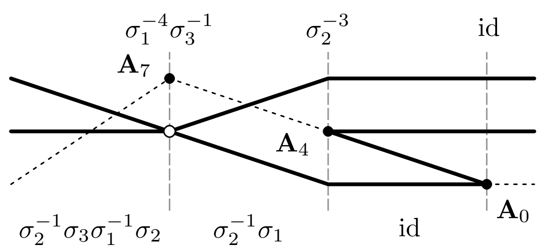
The fundamental group is Γ.

5.2. **The set of singularities $(\mathbf{A}_{14}\oplus\mathbf{A}_2)\oplus\mathbf{A}_2\oplus\mathbf{A}_1$, line 9** (see [8])**.** The curve is described in §2.3, and the diagram is



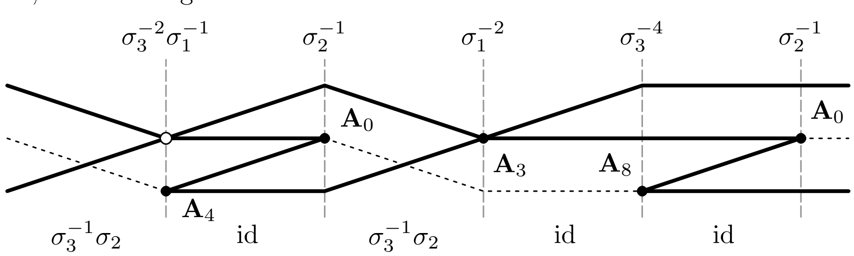
The fundamental group is Γ.

5.3. **The set of singularities $(\mathbf{A}_{11}\oplus 2\mathbf{A}_2)\oplus\mathbf{A}_4$, line 17** (see [8])**.** The curve is described in §2.4, and the diagram is



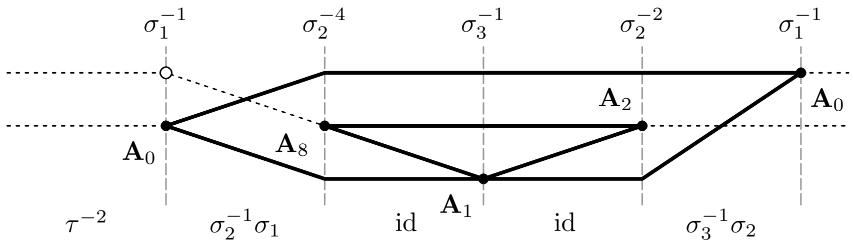
The fundamental group is Γ.

5.4. **The set of singularities $(2\mathbf{A}_8)\oplus\mathbf{A}_3$, line 29** (see [8])**.** The curve is described in §2.4, and the diagram is

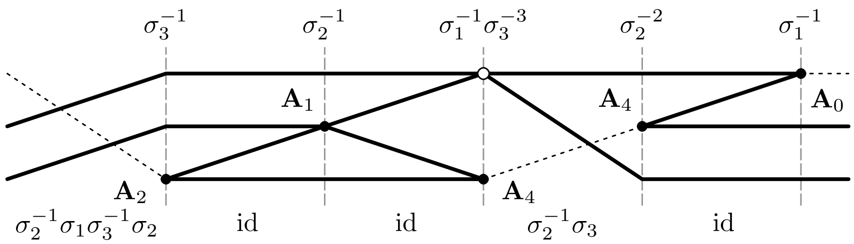


The fundamental group is Γ.

5.5. **The set of singularities $(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, line 33** (see [8])**.** The curve is described in §2.7, and the diagram is



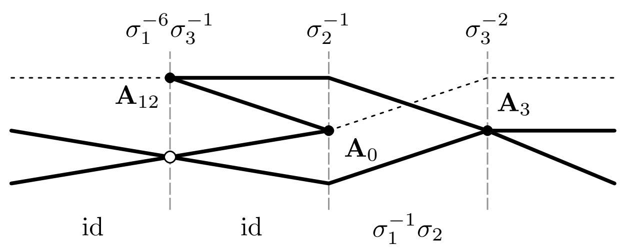
Alternatively, projecting from the $\mathbf{A}_8$ type point, we obtain the diagram



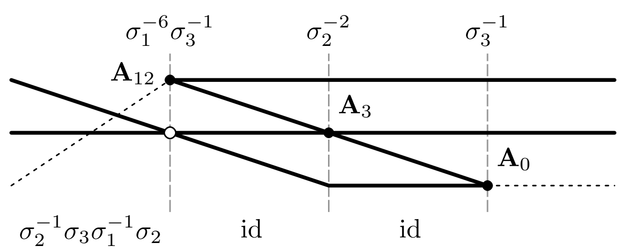
Both presentations are complete (there is a single pair of complex conjugate fibers), and the group obtained is (2.1).

6. The fundamental groups: other curves

6.1. **The set of singularities $\mathbf{A}_{16} \oplus \mathbf{A}_3$, line 4** (see [1, 9])**.** The two curves are described in §4.3. For $\epsilon = (59 - 3\sqrt{17})/32$, the diagram looks as follows:

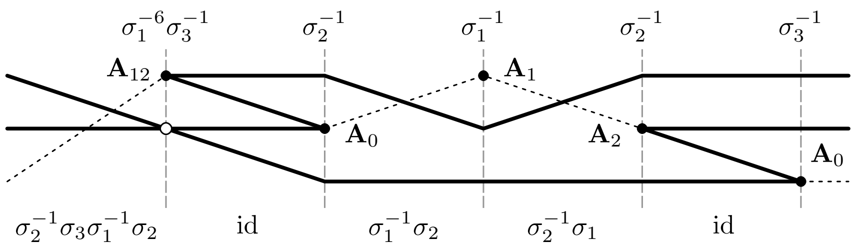


For $\epsilon = (59 + 3\sqrt{17})/32$, the diagram looks as follows:



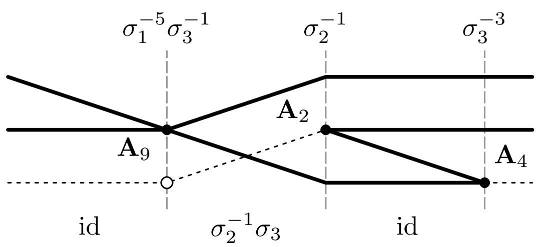
In both cases, $\pi_1 = \mathbb{Z}_6$.

6.2. **The set of singularities $\mathbf{A}_{16} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, line 5** (see [1])**.** The three curves are described in §4.4. For the only real value of ϵ, the diagram looks as follows:

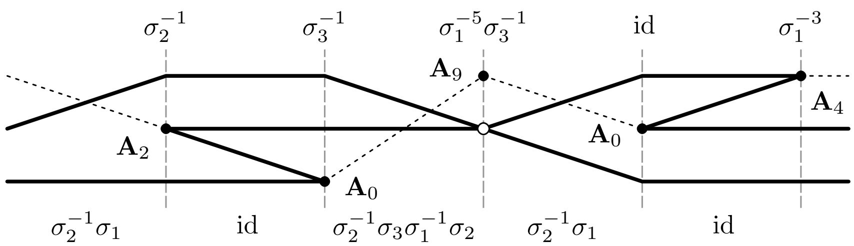


The fundamental group of this real curve is $\pi_1 = \mathbb{Z}_6$.

6.3. **The set of singularities $\mathbf{A}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$, line 11** (see [9])**.** The two curves are described in §4.8. For $\epsilon = (7 - \sqrt{21})/2$, the diagram looks as follows:

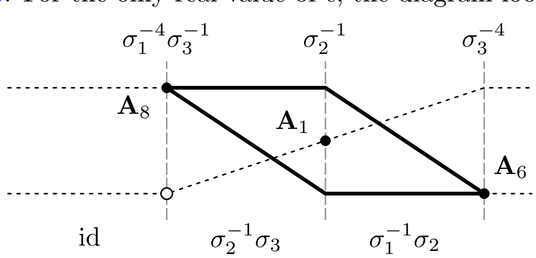


For $\epsilon = (7 + \sqrt{21})/2$, the diagram looks as follows:



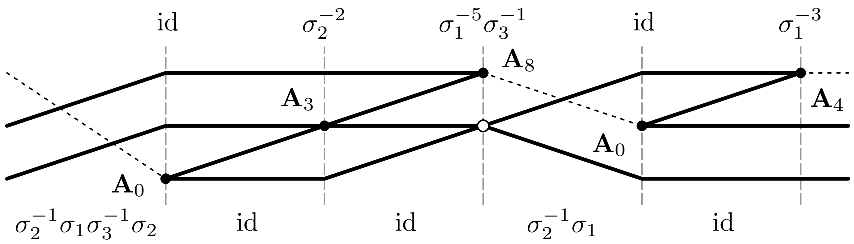
In both cases, $\pi_1 = \mathbb{Z}_6$.

6.4. **The set of singularities $\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$, line 13** (see [9])**.** The curves are described in §4.10. For the only real value of ϵ, the diagram looks as follows:



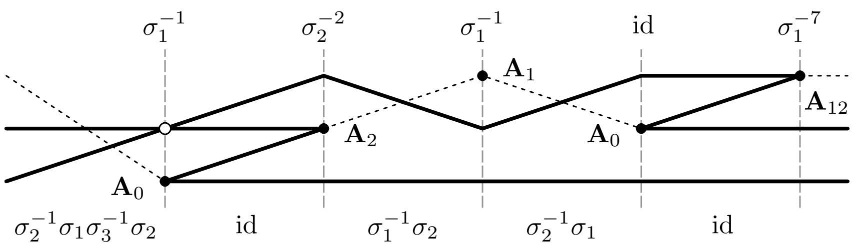
In this real case, $\pi_1 = \mathbb{Z}_6$.

6.5. **The set of singularities $\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$, line 14** (see [9])**.** The curve is described in §4.11, and the diagram looks as follows:



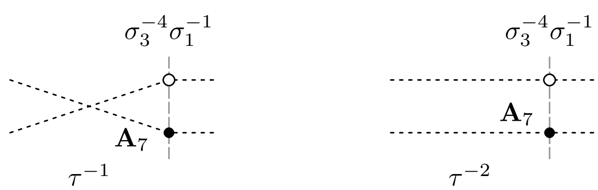
One has $\pi_1 = \mathbb{Z}_6$.

6.6. **The set of singularities $\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, line 15.** The three curves are described in §4.12. For the only real value of ϵ, the diagram looks as follows:



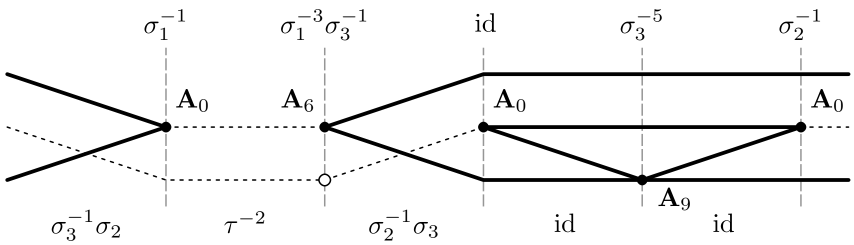
The fundamental group of this real curve is $\pi_1 = \mathbb{Z}_6$.

6.7. **The set of singularities $\mathbf{A}_{11} \oplus 2\mathbf{A}_4$, line 16** (see [9])**.** The two curves are described in §4.13. For $\epsilon = \sqrt{2}$ and $\epsilon = -\sqrt{2}$, we obtain the following diagrams:

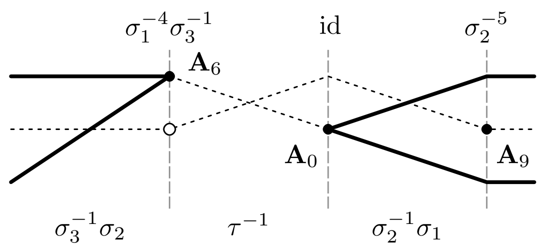


In the former case, $\pi_1 = \mathbb{Z}_6$. In the latter case, the presentation is incomplete (there are three pairs of complex conjugate fibers) and we cannot even confirm that the group is finite (*cf.* §6.10).

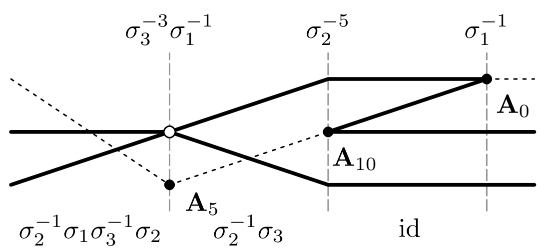
6.8. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_9$, line 18** (see [9])**.** The two curves are described in §4.14. For $\epsilon = (-11 + 3\sqrt{15})/2$, the diagram looks as follows:



For $\epsilon = (-11 - 3\sqrt{15})/2$, the diagram looks as follows:

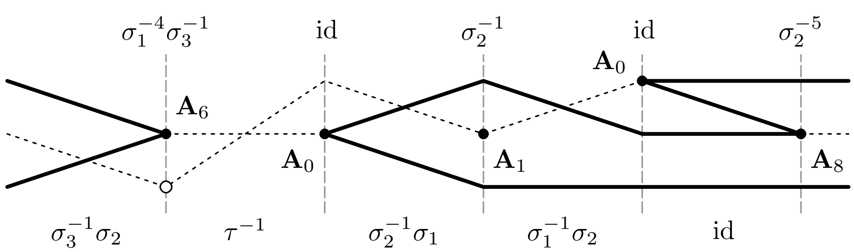


In the former case, $\pi_1 = \mathbb{Z}_6$. In the latter case, the presentation is incomplete (there are two pairs of complex conjugate fibers) and we cannot even confirm that the group is finite. However, projecting the same curve from its $\mathbf{A}_9$ type point, we obtain the diagram



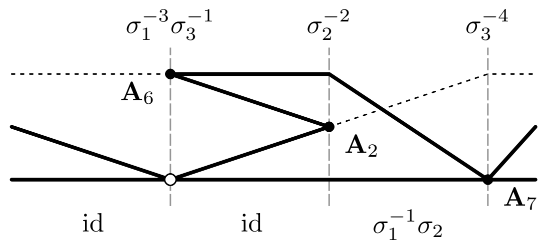
The corresponding presentation of the group yields $\pi_1 = \mathbb{Z}_6$.

6.9. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$, line 19** (see [9])**.** The curves are described in §4.15. For the only real value of ϵ, the diagram is as follows:

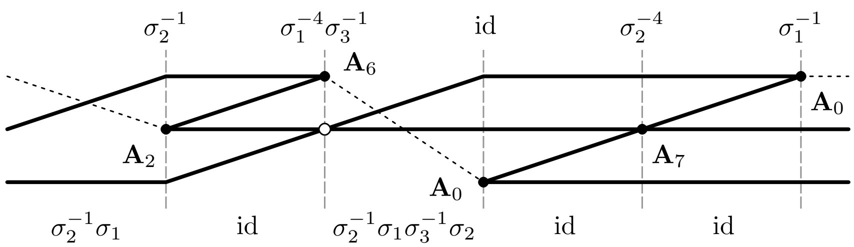


In this real case, $\pi_1 = \mathbb{Z}_6$.

6.10. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$, line 20** (see [9])**.** The two curves are described in §4.16. For $\epsilon = (-6+\sqrt{3})/11$, the diagram looks as follows:

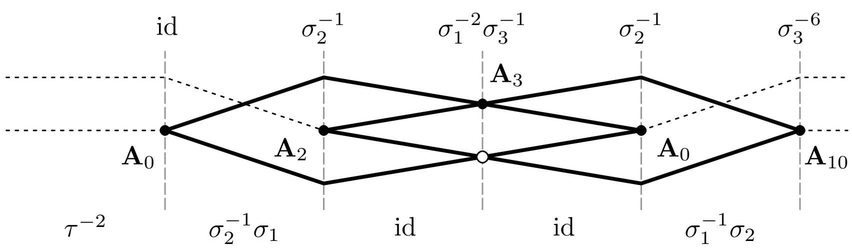


For $\epsilon = (-6-\sqrt{3})/11$, the diagram looks as follows:



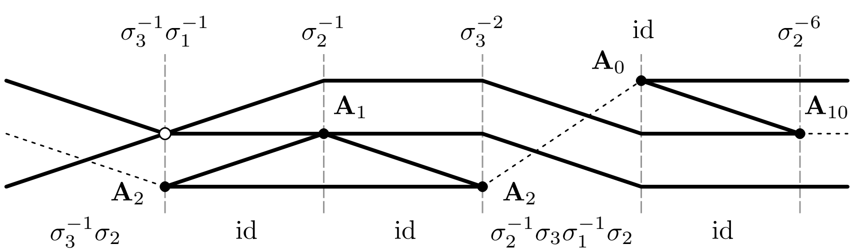
For the second diagram, one has $\pi_1 = \mathbb{Z}_6$. In the first case, the computation gives a group G with $[G, G] \cong SL(2, \mathbb{F}_5)$. However, the presentation given by this diagram does not need to be complete, as there are two pairs of complex conjugate singular fibers. Since the two curves are Galois conjugate and both fundamental groups are known to be *finite*, they must be isomorphic. Thus, in both cases, $\pi_1 = \mathbb{Z}_6$.

We can also project the first curve from its $\mathbf{A}_7$ type point to get the diagram



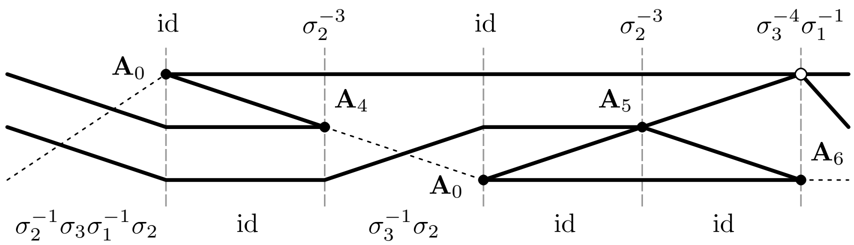
The presentation obtained shows immediately that $\pi_1 = \mathbb{Z}_6$.

6.11. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, line 22.** The three curves are described in §4.18. For the only real value of ϵ, the diagram looks as follows:

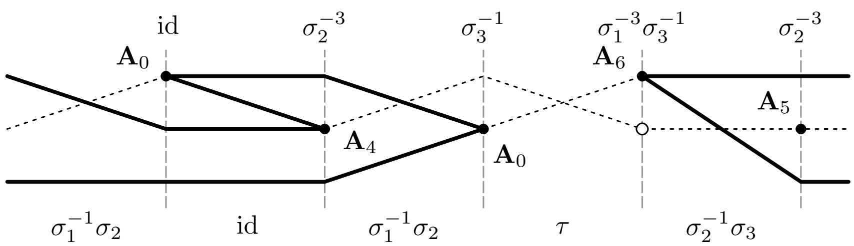


One has $\pi_1 = \mathbb{Z}_6$.

6.12. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$, line 23** (see [9])**.** The two curves are described in §4.19. For $\epsilon = (-11 + 3\sqrt{15})/7$, the diagram looks as follows:

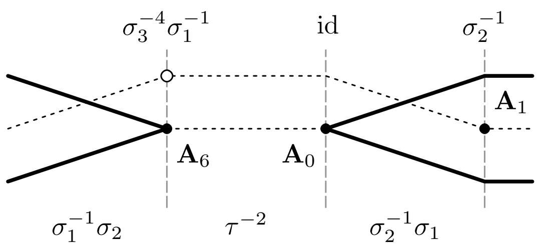


For $\epsilon = (-11 - 3\sqrt{15})/7$, the diagram looks as follows:



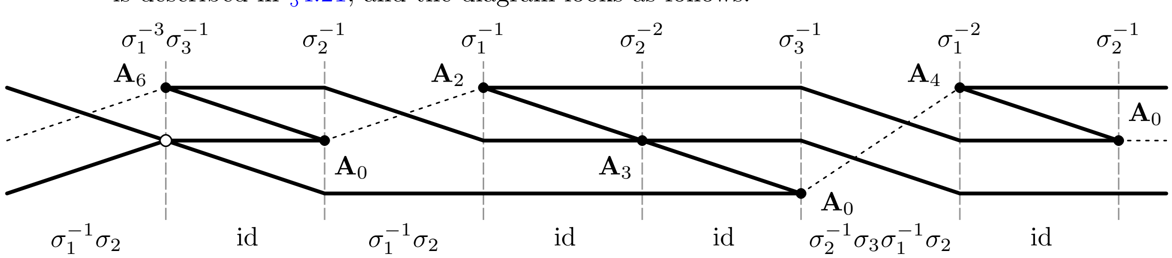
In both cases, $\pi_1 = \mathbb{Z}_6$.

6.13. **The set of singularities $\mathbf{A}_{10} \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$, line 24** (see [9])**.** The curves are described in §4.20. For the only real value of ϵ, the diagram looks as follows:



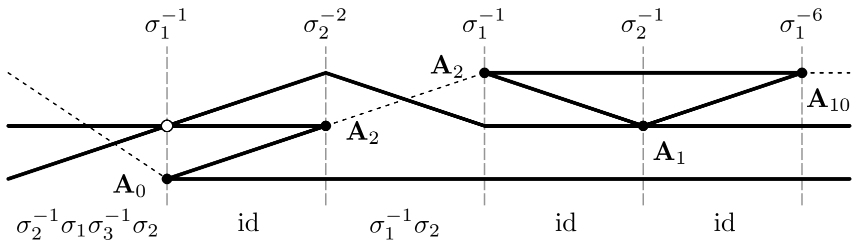
The presentation obtained is incomplete (there are two pairs of complex conjugate fibers), and we cannot confirm that the group is finite.

6.14. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$, line 25** (see [9])**.** The curve is described in §4.21, and the diagram looks as follows:

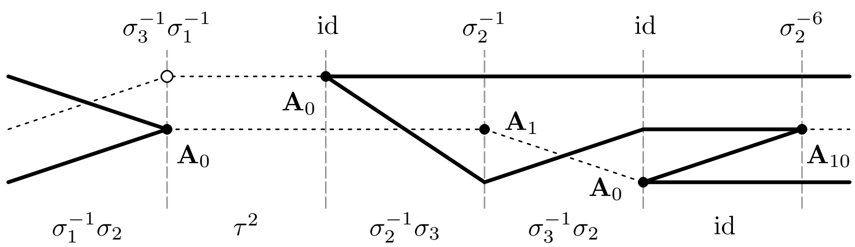


One has $\pi_1 = \mathbb{Z}_6$.

6.15. **The set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$, line 26.** The two curves are described in §4.22. For $\epsilon = \sqrt{5}$, the diagram looks as follows:

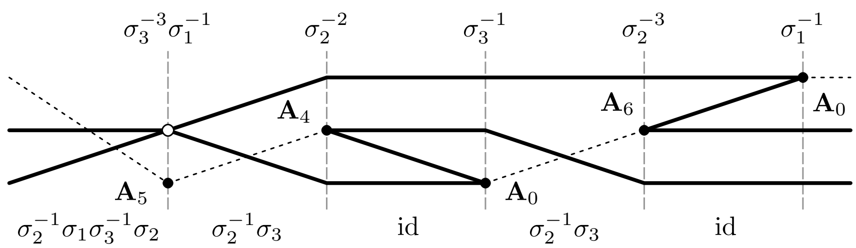


For $\epsilon = -\sqrt{5}$, the diagram looks as follows:



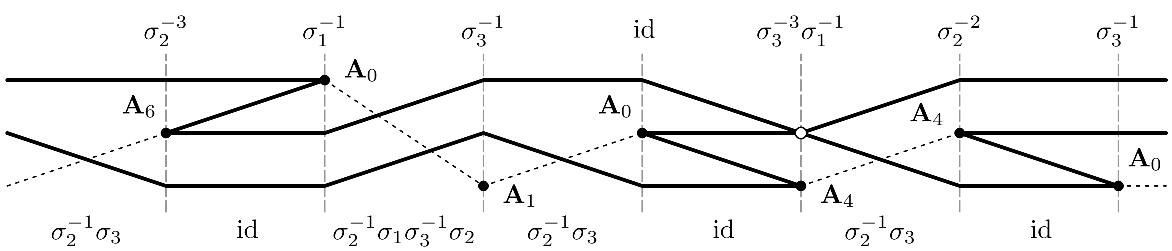
In both cases, $\pi_1 = \mathbb{Z}_6$.

6.16. **The set of singularities $\mathbf{A}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$, line 27** (see [9])**.** The curves are described in §4.23. For the only real value of ϵ, the diagram is as follows:



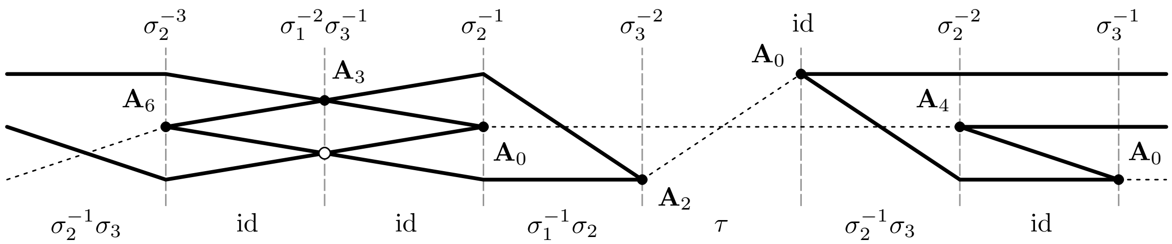
In this real case, $\pi_1 = \mathbb{Z}_6$.

6.17. **The set of singularities $\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, line 31** (see [9])**.** The curves are described in §4.25. For the only real value of ϵ, the diagram is as follows:

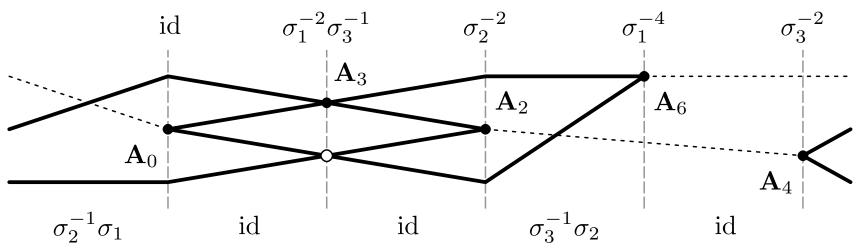


In this real case, $\pi_1 = \mathbb{Z}_6$.

6.18. **The set of singularities $\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$, line 35** (see [9])**.** The two curves are described in §4.25. For $\epsilon = (57 + 13\sqrt{21})/50$, the diagram is as follows:

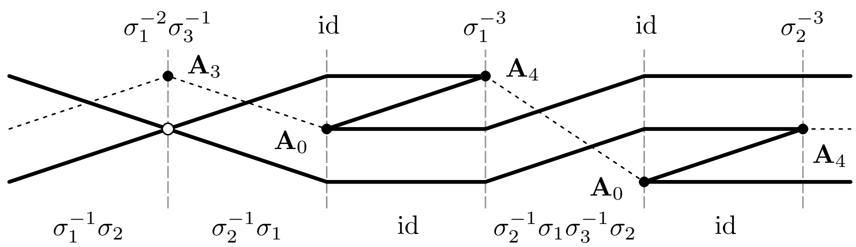


For $\epsilon = (57 - 13\sqrt{21})/50$, the diagram looks as follows:



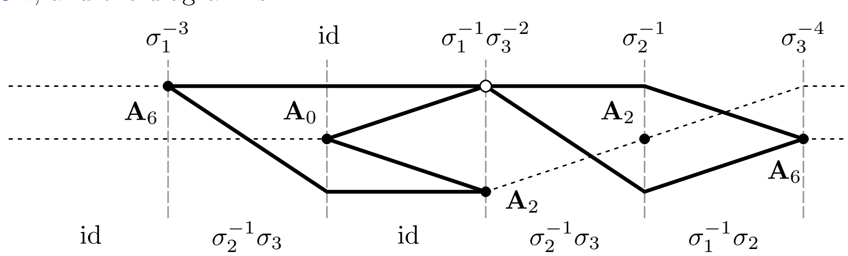
In both cases, $\pi_1 = \mathbb{Z}_6$.

6.19. **The set of singularities $\mathbf{A}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$, line 36.** The curve is described in §4.28, and the diagram looks as follows:



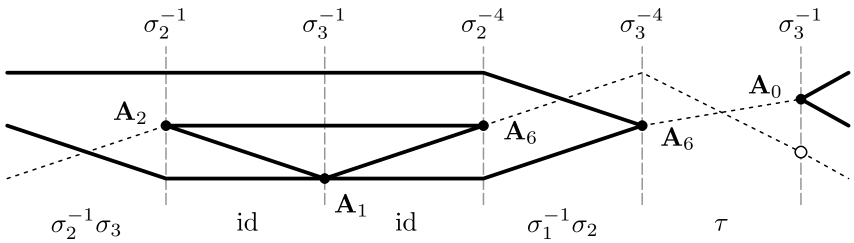
One has $\pi_1 = \mathbb{Z}_6$.

6.20. **The set of singularities $3\mathbf{A}_6 \oplus \mathbf{A}_1$, line 37** (see [8])**.** The curve is described in §3.2, and the diagram is

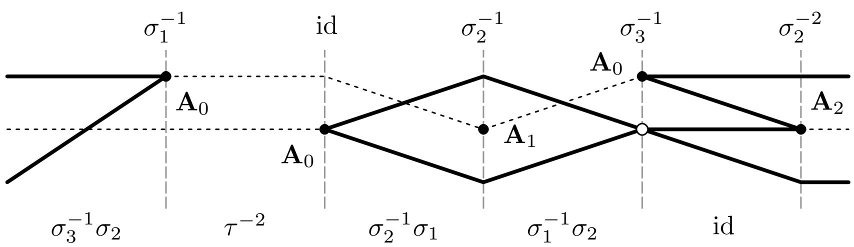


The fundamental group is $\mathbb{Z}_3 \times \mathbb{D}_{14}$, as expected for a $\mathbb{D}_{14}$-special sextic.

6.21. **The set of singularities $2\mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, line 38.** The two curves are described in §4.29. For $\epsilon = \sqrt{21}$, the diagram looks as follows:

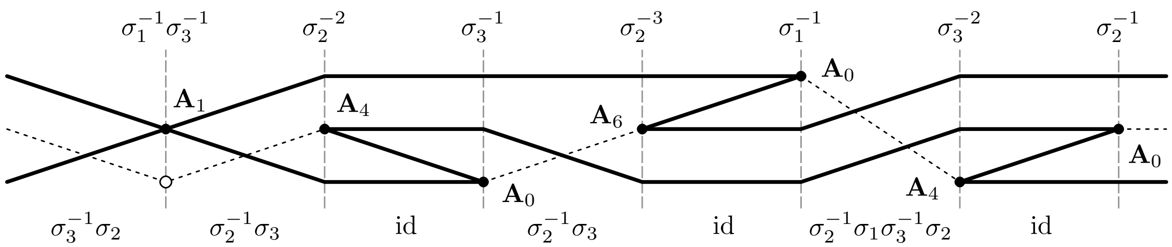


For $\epsilon = -\sqrt{21}$, the diagram looks as follows:

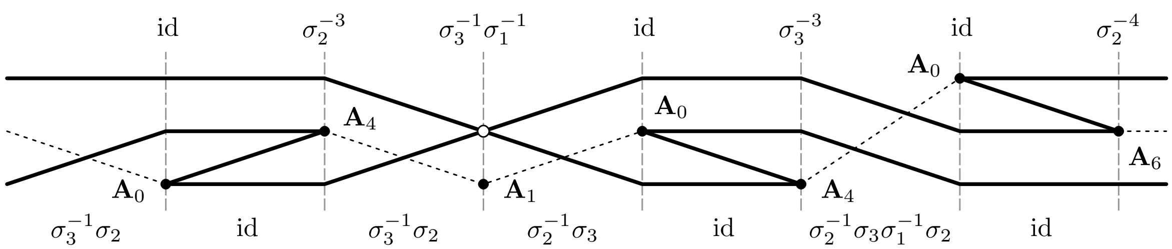


In both cases, $\pi_1 = \mathbb{Z}_6$.

6.22. **The set of singularities $\mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$, line 39.** The two curves are described in §4.30. For $\epsilon = (-231 + 92\sqrt{7})/7$, the diagram looks as follows:



For $\epsilon = (-231 - 92\sqrt{7})/7$, the diagram looks as follows:



In both cases, $\pi_1 = \mathbb{Z}_6$.

References

1. Enrique Artal Bartolo, Jorge Carmona Ruber, and José Ignacio Cogolludo Agustín, *On sextic curves with big Milnor number*, Trends in singularities, Trends Math., Birkhäuser, Basel, 2002, pp. 1–29. MR 1900779 (2003d:14034)
2. Alex Degtyarev, *Oka's conjecture on irreducible plane sextics*, J. Lond. Math. Soc. (2) **78** (2008), no. 2, 329–351. MR 2439628 (2009f:14054)
3. ———, *On deformations of singular plane sextics*, J. Algebraic Geom. **17** (2008), no. 1, 101–135. MR 2357681 (2008j:14061)
4. ———, *Fundamental groups of symmetric sextics. II*, Proc. Lond. Math. Soc. (3) **99** (2009), no. 2, 353–385. MR 2533669
5. ———, *On irreducible sextics with non-abelian fundamental group*, Singularities—Niigata–Toyama 2007, Adv. Stud. Pure Math., vol. 56, Math. Soc. Japan, Tokyo, 2009, pp. 65–91. MR 2604077
6. ———, *The Bertini involution*, `arXiv:1212.0991`, 2012.
7. ———, *Topology of algebraic curves: An approach via dessins d'enfants*, De Gruyter Studies in Mathematics, vol. 44, Walter de Gruyter & Co., Berlin, 2012. MR 2952675
8. ———, *On plane sextics with double singular points*, Pacific J. Math. **265** (2013), no. 2, 327–348. MR 3096504
9. ———, *On the Artal–Carmona–Cogolludo construction*, to appear, `arXiv:1301.2105`, 2013.
10. Alex Degtyarev and Mutsuo Oka, *A plane sextic with finite fundamental group*, Singularities —Niigata–Toyama 2007, Adv. Stud. Pure Math., vol. 56, Math. Soc. Japan, Tokyo, 2009, pp. 93–108. MR 2604078
11. Ethel I. Moody, *Notes on the Bertini involution*, Bull. Amer. Math. Soc. **49** (1943), 433–436. MR 0008163 (4,253c)
12. Mutsuo Oka and Duc Tai Pho, *Classification of sextics of torus type*, Tokyo J. Math. **25** (2002), no. 2, 399–433. MR 1948673 (2003k:14030)
13. S. Yu. Orevkov, *A new affine M-sextic*, Funktsional. Anal. i Prilozhen. **32** (1998), no. 2, 91–94, English translation: Funct. Anal. Appl. 32 (1998), no. 2, 141–143. MR 1647852 (99j:14053)
14. Stepan Yu. Orevkov, *Parametric equations of plane sextic curves with a maximal set of double points*, `http://www.math.univ-toulouse.fr/~orevkov/abh.pdf`, 2013.
15. Ichiro Shimada, *On the connected components of the moduli of polarized $K3$ surfaces*, `http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3.html`, 2007.
16. Jin-Gen Yang, *Sextic curves with simple singularities*, Tohoku Math. J. (2) **48** (1996), no. 2, 203–227. MR 1387816 (98e:14026)

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