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TOPOLOGICAL CLASSIFICATION OF REAL ENRIQUES SURFACES

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Abstract. We complete the classification of the topological types of real Enriques surfaces started by V. Nikulin. The resulting list contains 87 topological types.

1. Introduction

Enriques surfaces constitute one of special classes in the classification of nonsingular algebraic surfaces. Over the field of complex numbers an Enriques surface can be defined as the quotient of a $K3$-surface by a fixed point free involution. Complex Enriques surfaces considered as smooth 4-manifolds are all diffeomorphic. Moreover, their moduli space is irreducible.

A *real Enriques surface* is a complex Enriques surface equipped with an antiholomorphic involution (called *complex conjugation*). The fixed point set of this involution is called the *real part* of the surface, or its *set of real points*. If this set is nonempty, the involution can be lifted to the covering $K3$-surface. Thus the study of real Enriques surfaces with nonempty real part can be reduced to the study of real $K3$-surfaces supplied with a holomorphic fixed point free involution (and whose real part is nonempty; note by the way that there are real Enriques surfaces, with empty real part, whose real structure does not lift to an involution).

Contrary to the complex case, the moduli space of real Enriques surfaces is not connected. Real Enriques surfaces considered as smooth 4-manifolds with a smooth involution, and even their real parts considered as smooth 2-manifolds, turn out to be of several different types. The present paper is devoted to classification of the topological types of the real parts of real Enriques surfaces. The main result, which gives this classification, is Theorem 2.2 (see Section 2).

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The only, to our knowledge, published observation concerning this classification problem is due to R. Silhol [13], who found one of the so-called *maximal* real Enriques surfaces (briefly, M-surfaces). A significant progress in this direction was recently achieved by V. Nikulin. It is his preprint [11] that originated our present work.

Nikulin's paper is devoted to a somewhat more detailed study of the real part. Namely, there is a natural decomposition of the set of the components of the real part of an Enriques surface into two groups (which is due to the two different liftings of the real structure to the covering $K3$-surface, see 3.2 below for details), and this decomposition is included in the classification problem. To solve this problem, he studies $(\mathbb{Z}/2\times\mathbb{Z}/2)$-actions in the $K3$-lattice, the two $\mathbb{Z}/2$-factors corresponding to the deck translation of the covering (Enriques involution) and to one of the two real structures respectively. Interpreting topologically the arithmetical information obtained, he produced two lists which bound the collection of realizable topological types, along with the decomposition, from above and below. For the topological types (without decomposition) he left a lacuna of 21 elements: his lower and upper lists contain 59 and 80 elements respectively.

Influenced by Nikulin's result, we tried to understand it and to complete the classification. For that purpose, we chose a different, simpler, approach, which had already proved its efficiency in the case of $K3$-surfaces (see [6], [16]): first, we use purely topological methods to prohibit most of the types, and then we give an explicit construction for the rest. We succeeded to complete Nikulin's classification (the full list contains 87 elements: Nikulin's tables turned out to contain a few mistakes). Besides, our construction gives all the existing topological types, which makes the proof of completeness self-contained.

Our approach shows, in addition, that the problem of enumeration of the topological types of Enriques surfaces belongs, in fact, to topology of smooth involutions. Namely, transforming the properties used in the proof into axioms, one can introduce a much more relaxed notion of *flexible real Enriques surface* and still obtain the same list of topological types of real parts. More precisely, we give the following definition:

Definition. A *flexible real Enriques surface* is *a flexible real $K3$-surface* with a flexible Enriques involution, i.e., a triple (X, t, τ), where X is a closed smooth oriented 4-dimensional manifold homotopy equivalent to a $K3$-surface equipped with the complex orientation and t and τ are two commuting smooth orientation preserving involutions $X \to X$ so that:

(1) τ is fixed point free, and
(2) the normal and tangent bundles of the fixed point sets of both t and $t\circ\tau$ are antiisomorphic (equivalently, the Euler characteristic of each fixed point set is equal to minus its normal Euler number).

Naturally, by the real part of a flexible real Enriques surface one means the fixed point set of the involution induced by t on X/τ.

The paper is organized as follows. In Section 2 we formulate the main result; it is proved in Sections 3 (prohibitions) and 4, 5 (constructions). In the Appendix we prove certain auxiliary results, which are known or almost known but whose proof is not published elsewhere, and give an alternative, flexible, proof of some results of the main text to show that, as it is stated above, the classification obtained extends to a wider class of objects. In conclusion, in A.5, we briefly discuss the problem of extended classification of real Enriques surfaces, with the splitting of the real part into two halves taken into account.

2. Listing of the topological types

2.1. Notation. In what follows we use the notation S_g and V_p to stand, respectively, for the connected sum of g copies of a 2-torus and the connected sum of p copies of a real projective plane. It is convenient for us to consider the 2-sphere S to belong to both the families, $S = S_0 = V_0$.

To describe the topological types of real Enriques surfaces, we use the notion of *Morse simplification*, i.e., Morse transformation which decreases the total Betti number. There are two types of such simplifications:

(1) removing a spherical component ($S \to \varnothing$), and
(2) contracting a handle ($S_{g+1} \to S_g$ or $V_{p+2} \to V_p$).

By *topological type* we mean a class of surfaces with homeomorphic real parts. A topological type of an Enriques surface is called *extremal* if it cannot be obtained from the topological type of another Enriques surface by a Morse simplification.

Remark. Note that a Morse simplification may not correspond to a Morse simplification in a continuous family of Enriques surfaces. As a result, the notions of extremal topological type and extremal (in the obvious sense) surface may be different: *a priori*, the topological type of an extremal surface may not be extremal.

2.2. Theorem. *There are* 87 *topological types of real Enriques surfaces. Each of them can be obtained by a sequence of Morse simplifications from one of the* 22 *extremal types listed below. Conversely, with the exception of the two types* $6S$ *and* $S_1 \sqcup 5S$*, any topological type obtained in this way is realized by a real Enriques surface.*

The 22 *extremal types are*:

(1) *M-surfaces*:
(a) $\chi(E_{\mathbb{R}}) = 8$: (b) $\chi(E_{\mathbb{R}}) = -8$:

$4V_1 \sqcup 2S$,	$V_{11} \sqcup V_1$,
$V_2 \sqcup 2V_1 \sqcup 3S$,	$V_{10} \sqcup V_2$,
$V_3 \sqcup V_1 \sqcup 4S$,	$V_9 \sqcup V_3$,
$2V_2 \sqcup 4S$,	$V_8 \sqcup V_4$,
$V_4 \sqcup 5S$,	$V_7 \sqcup V_5$,
$V_2 \sqcup S_1 \sqcup 4S$,	$2V_6$,
	$V_{10} \sqcup S_1$;

(2) *$(M-2)$-surfaces with $\chi(E_{\mathbb{R}}) = 0$*:

$V_4 \sqcup 2V_1$,	$V_5 \sqcup V_1 \sqcup S$,
$V_3 \sqcup V_2 \sqcup V_1$,	$V_4 \sqcup V_2 \sqcup S$,
$V_6 \sqcup 2S$,	$2V_3 \sqcup S$,
$V_4 \sqcup S_1 \sqcup S$,	$2V_2 \sqcup S_1$;

(3) *Pair of tori*: $2S_1$.

3. Prohibitions

In this Section we study topological properties of the real part of a real Enriques surface. We use the following notation: E is the Enriques surface under consideration, as well as its set of complex points, X and τ are the covering $K3$ surface and the Enriques (deck translation) involution on it respectively, conj is the antiholomorphic involution on E which defines the real structure, and $E_{\mathbb{R}}$ is the real part $\operatorname{Fix}\operatorname{conj}$ of E. The real part is supposed to be nonempty (the case $E_{\mathbb{R}} = \varnothing$ is trivial, and we do not consider it).

3.1. Topology of the set of complex points. The following properties, which do not require a real structure on the surface, are well known (most of them are immediate consequences of the definition of an Enriques surface; see, for example, [1]):

(3.2.1) $\pi_1(E) = H_1(E;\mathbb{Z}) = H_3(E;\mathbb{Z}) = \mathbb{Z}/2$;

(3.2.2) $H_2(E;\mathbb{Z}) \cong \mathbb{Z}^{\oplus 10} \oplus \mathbb{Z}/2$; the nontrivial element of $\operatorname{Tors} H_2(E;\mathbb{Z})$ is the Chern class $c_1(E)$;

(3.2.3) $\sigma(E) = -8$ and the intersection form on $H_2(E;\mathbb{Z})/\operatorname{Tors}$ is even;

(3.2.4) $H_2(E;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{\oplus 12}$;

(3.2.5) the kernel of $\operatorname{pr}^!\colon H_2(E;\mathbb{Z}/2) \to H_2(X;\mathbb{Z}/2)$ is nontrivial; it coincides with the image of $\operatorname{Tors} H_2(E;\mathbb{Z}) \cong \mathbb{Z}/2$ in $H_2(E;\mathbb{Z}/2)$; the only nontrivial element of this kernel is $w_2(E)$.

In particular, this implies that $H_2(E;\mathbb{Z})/\operatorname{Tors} \cong -E_8 \oplus U$ as a lattice, $h^{0,2}(E) = h^{2,0}(E) = 0$, $h^{1,1}(E) = 10$, and $\beta_*(E) \overset{\text{def}}{=} \sum \dim H_i(E;\mathbb{Z}/2) = 16$.

3.2. Decomposition of the real part. Since $E_{\mathbb{R}} \neq \varnothing$, the $\mathbb{Z}/2$-action on E given by conj lifts to a $(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$-action on X: there are two antiholomorphic involutions $t^{(1)}, t^{(2)} \colon X \to X$, which commute with each other and whose composition is τ (the proof is obvious as soon as the points of X are represented by homotopy classes of paths in E starting at a fixed point of $E_{\mathbb{R}}$).

Since $\operatorname{Fix} \tau = \varnothing$, both the two real parts $X_{\mathbb{R}}^{(i)} = \operatorname{Fix} t^{(i)}$, $i = 1, 2$, and their images in E are disjoint. Thus, $E_{\mathbb{R}}$ canonically splits into two disjoint parts, which we will refer to as the *halves* of $E_{\mathbb{R}}$ and denote by $E_{\mathbb{R}}^{(1)}$, $E_{\mathbb{R}}^{(2)}$. Note that both the halves consist of whole components of $E_{\mathbb{R}}$, and that $X_{\mathbb{R}}^{(1)}$ and $X_{\mathbb{R}}^{(2)}$ are unramified double coverings of $E_{\mathbb{R}}^{(1)}$ and $E_{\mathbb{R}}^{(2)}$ respectively. (In fact, these are the orientation coverings, see 3.4).

3.3. Eigenspaces of the complex conjugation. Intersecting the eigenspaces of τ_* and $t_*^{(1)}$, one obtains the orthogonal decomposition

$$H_2(X; \mathbb{R}) = H^{++} \oplus H^{+-} \oplus H^{-+} \oplus H^{--}, \tag{3.3.1}$$

where $H^{\delta\varepsilon}$ is the bi-eigenspace $\bigl\{x \in H_2(X; \mathbb{R}) \bigm| \tau_* x = \delta x,\ t_*^{(1)} x = \varepsilon x\bigr\}$. The dimensions and signatures of these spaces can be found using, respectively, Lefschetz fixed point theorem and Hirzebruch signature formula applied to τ, $t^{(1)}$, and $t^{(2)}$. They are

$$\begin{aligned}
\dim H^{++} &= 4 + \tfrac{1}{2}\chi, & \sigma(H^{++}) &= -4 - \tfrac{1}{2}\chi,\\
\dim H^{+-} &= 6 - \tfrac{1}{2}\chi, & \sigma(H^{+-}) &= -4 + \tfrac{1}{2}\chi,\\
\dim H^{-+} &= 6 + \tfrac{1}{2}(\chi_1 - \chi_2), & \sigma(H^{-+}) &= -4 - \tfrac{1}{2}(\chi_1 - \chi_2),\\
\dim H^{--} &= 6 + \tfrac{1}{2}(\chi_2 - \chi_1), & \sigma(H^{--}) &= -4 - \tfrac{1}{2}(\chi_2 - \chi_1),
\end{aligned} \tag{3.3.2}$$

where $\chi = \chi(E_{\mathbb{R}})$ and $\chi_i = \chi(E_{\mathbb{R}}^{(i)})$.

3.4. Orientability of real components. *The projection $X_{\mathbb{R}}^{(i)} \to E_{\mathbb{R}}^{(i)}$ is the orientation covering of $E_{\mathbb{R}}^{(i)}$. Thus, there are only the following two possibilities for a component C of $E_{\mathbb{R}}^{(i)}$:*

1. *it is covered by a pair T_1, T_2 of the components of $X_{\mathbb{R}}^{(i)}$ interchanged by τ; then C is orientable;*
2. *it is covered by a connected component T of $X_{\mathbb{R}}^{(i)}$ invariant in respect to τ; then C is nonorientable.*

Here we reproduce Nikulin's proof ([11], cf. also [14]) which appeals (via holomorphic forms) to the complex structure. A flexible (in the sense of Introduction) proof is given in A.2.

Proof. There is a unique, up to multiplication by nonzero reals, nontrivial holomorphic 2-form ω on X such that $t^{(i)}\omega = \overline{\omega}$. When restricted to $X_{\mathbb{R}}^{(i)}$, this form defines an orientation on it. Thus, the result follows from the fact that ω, like any other holomorphic 2-form on X, is skew-invariant in respect to τ. (This skew-invariance property is well-known. It follows, for example, from the fact that $h^{2,0}(E) = 0$; see, e.g., [1]; cf. 3.1.) □

Remark. The above orientation of $X_{\mathbb{R}}^{(i)}$ given by a holomorphic form is well defined up to total reversing. With certain ambiguity, though commonly accepted, we call it the *canonical orientation* of $X_{\mathbb{R}}^{(i)}$. (A somewhat more proper name would be the *canonical semiorientation*.) Note that an orientation of one half constructed in such a way determines the orientation of the other half: if $t^{(1)}\omega = \overline{\omega}$, then $t^{(2)}i\omega = \overline{i\omega}$. This correspondence is not an involution, it has order 4. It gives a structure a little bit finer than just the union of the semiorientations of the halves.

3.5. Components of high genus. The real part $X_{\mathbb{R}}$ of a $K3$-surface X can have at most one nonspherical component, with the exception of the case $X_{\mathbb{R}} = 2S_1$ (see [6]). In view of 3.4 this implies that each of the two halves $E_{\mathbb{R}}^{(i)}$ is of one of the following three forms:

(1) $\alpha V_g \sqcup aV_1 \sqcup bS$, $g > 1$, $a \geqslant 0$, $b \geqslant 0$, $\alpha = 0, 1$;
(2) $2V_2$;
(3) S_1.

The next two assertions also follow from [6] (for completeness we give their proof, which is flexible, in A.3 and A.4).

3.5.1. *If $X_{\mathbb{R}}^{(1)} = 2S_1$, then the two components T_1 and T_2 of $X_{\mathbb{R}}^{(1)}$ realize proportional nontrivial elements in $H_2(X;\mathbb{R})$.*

Remark. In fact, these elements are equal if the surface is algebraic; this follows, e.g., from [10]. They may not be equal in the case of flexible surfaces.

3.5.2. *If $X_{\mathbb{R}} = S_1$, then $X_{\mathbb{R}}$ realizes a nontrivial element in $H_2(X;\mathbb{Z}/2)$.*

3.6. Proposition. *The real part of any Enriques surface is of one of the* 87 *topological types given in Theorem* 2.2.

Proof. To prohibit almost all the topological types not included in Theorem 2.2, it suffices to combine 3.5 with the following known inequalities and congruences:

(1) Smith-Thom inequality [15]: $\beta_*(E_{\mathbb{R}}) \leqslant \beta_*(E) = 16$ (where β_* is the total Betti number over $\mathbb{Z}/2$);
(2) Gudkov-Rokhlin congruence [12]: if E is an M-surface, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) = -8 \pmod{16}$;
(3) Gudkov-Kharlamov congruence [7]: if E is an $(M-1)$-surface, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pm 2 \equiv \pm 10 \pmod{16}$.

Thereafter, there are only three types, $6S$, $S_1 \sqcup 5S$, and $3V_2$, and one series, $S_1 \sqcup V_1 \sqcup \dots$, still left to be prohibited. This is done in the rest of this section.

3.7. Prohibition of the type $6S$. Existence of such a surface contradicts to Comessatti-Severi inequality [2]. Indeed, applying it to an Enriques surface and using the fact that the complex conjugation always interchanges the two halves of the cone $\{x \in H^2(E;\mathbb{R}) \mid x^2 > 0\}$, one gets the inequality $\chi(E_{\mathbb{R}}) \leqslant h^{1,1}(E) = 10$ which is not satisfied in the case $X_{\mathbb{R}} = 6S$. $\square$

3.8. Prohibition of the type $S_1 \sqcup 5S$. According to 3.4 and 3.5, if such a surface exists, the component S_1 should constitute one of the halves, say $E_{\mathbb{R}}^{(1)}$. Let T_1, T_2 be the two components of $X_{\mathbb{R}}^{(1)}$ with their canonical orientations induced by ω (see 3.4), and let $[T_1], [T_2] \in H_2(X;\mathbb{R})$ be their fundamental classes. Then $[T_i]^2 = -\chi(T_i) = 0$ and $[T_2] = r[T_1] \neq 0$ for some $r \in \mathbb{Q}$, see 3.5.1 . In addition, $\tau_*[T_1] = -[T_2]$, since τ reverses the canonical orientation, see 3.4. Hence, $-r$ is an eigenvalue of τ_*, i.e., $r = \pm 1$. Thus, $[T_1]$ is a nontrivial isotropic class, which is invariant in respect to $t_*^{(1)}$ and either skew-invariant (case $r = 1$) or invariant (case $r = -1$) in respect to τ_*, i.e., $[T_1]$ belongs to one of the subspaces H^{-+}, H^{++} of the decomposition (3.3.1). On the other hand, from (3.3.2) it follows that $\sigma(H^{-+}) = \dim H^{-+} = 1$ and $\sigma(H^{++}) = -\dim H^{++} = -4$, i.e., H^{-+} is positive definite and H^{++} is negative definite. Thus, neither of these subspaces can contain an isotropic class. $\square$

Remark. In fact, in the algebraic case the above r is equal to 1, i.e., $[T_1] = [T_2]$ (see remark in 3.5.1). In order to give a flexible proof we do not use this information.

3.9. Prohibition of the type $3V_2$. The two halves should be $E_{\mathbb{R}}^{(1)} = 2V_2$ and $E_{\mathbb{R}}^{(2)} = V_2$. Since E is an $(M-2)$-surface and $\chi(E_{\mathbb{R}}) = 0 \equiv \sigma(E) + 8 \pmod{16}$, from Theorem A.1 (see Appendix) it follows that the fundamental class $[E_{\mathbb{R}}]$ equals $w_2(E)$ in $H_2(E;\mathbb{Z}/2)$, and, hence, $[X_{\mathbb{R}}]$ vanishes in $H_2(X;\mathbb{Z}/2)$, see (3.2.5). On the other hand, this class cannot be trivial since $[X_{\mathbb{R}}^{(1)}]$ vanishes in $H_2(X;\mathbb{Z}/2)$, but $[X_{\mathbb{R}}^{(2)}]$ does not, see 3.5.2. $\square$

3.10. Prohibition of the types $S_1 \sqcup V_1 \sqcup \dots$. To avoid the realizability problem, let us work with flexible surfaces (see Introduction), which form a wider class. The results cited and obtained in 3.5 extend to flexible real $K3$- and Enriques surfaces (see the definition in Introduction); the proofs given in [6] and in A.3, A.4 work without change.

From Lemma 3.10.1 below it follows that, if there exists a (flexible) real Enriques surface with the real part $S_1 \sqcup V_1 \sqcup \dots$, then there also exists a flexible surface with one of the halves $S_1 \sqcup V_1$; this contradicts to 3.5 (in its flexible setting). $\square$

3.10.1. Lemma. *If there exists a (flexible) real Enriques surface with* $E_{\mathbb{R}}^{(1)} = E'_{\mathbb{R}}$ *and* $E_{\mathbb{R}}^{(2)} = E''_{\mathbb{R}} \sqcup V_1$*, then there also exists a flexible real Enriques surface with* $E_{\mathbb{R}}^{(1)} = E'_{\mathbb{R}} \sqcup V_1$ *and* $E_{\mathbb{R}}^{(2)} = E''_{\mathbb{R}}$*. In other words, a component* V_1 *can be transferred from one half to the other.*

An informal way to think about the desired transformation is the following: The original surface is included into a one-parametric real family which has a single singular fiber with one singular point in it. (This is a nondegenerate double point in the covering family of $K3$-surfaces; the singularity downstairs is somewhat more complicated.) When passing through the singular fiber, the component V_1 collapses to the singular point and then reappears in the other half. As we are working with flexible surfaces and we do not know if the above family can be made algebraic when the original Enriques surface is algebraic, we give below an alternative description of this deformation in terms of modification of the real structure. (In other words, instead of passing through the singular fiber, one may go around it in the complex part of the parameter line; this corresponds to 'one half' of the Picard-Lefschetz transformation.)

Proof of Lemma 3.10.1. Pick a component $V_1 \in E_{\mathbb{R}}^{(2)}$ and denote by $\widetilde{S}$ its pull-back in X. One can identify some both $t^{(1)}$- and τ-invariant tubular neighborhood of $\widetilde{S}$ with the total space of the tangent bundle of $\widetilde{S}$, and then, in this neighborhood, identify $t^{(1)}$ with the differential of $-t^{(1)}\big|_{\widetilde{S}}$. Denote by $\delta\colon X \to X$ the map which is the identity outside the chosen neighborhood of $\widetilde{S}$ and sends a point ξ, $\|\xi\| \leqslant 1$, of the tangent bundle of $\widetilde{S}$ into the point which is obtained from ξ by the geodesic flow at time $\pi(1 - \|\xi\|)$ (assuming that the metric on $\widetilde{S}$ is homogeneous of radius one and that the radius of the tubular neighborhood is also one). Then δ acts on $\widetilde{S}$ as the antipodal map (just like $t^{(1)}$ does) and satisfies the commutative relations $\delta \circ t^{(1)} = t^{(1)} \circ \delta^{-1}$ and $\delta \circ \tau = \tau \circ \delta$. The desired surface is obtained now by replacing $t^{(1)}$ with $t^{(1)} \circ \delta$. □

4. Construction (preliminaries)

4.1. General idea. Let X be the $K3$-surface obtained as the double covering of $Y = \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ branched over a nonsingular curve $C \subset Y$ of bidegree $(4,4)$. Denote by $s\colon Y \to Y$ the Cartesian product of the nontrivial involutions $(u : v) \mapsto (-u : v)$ of the factors. If C is s-symmetric, s lifts to two different involutions on X, which commute with the deck translation d of $X \to Y$. If, besides, C does not pass through the fixed points of s, then exactly one of these two involutions, which we denote by τ, is fixed point free (see, e.g., [5] or [1]), and, hence, the orbit space $E = X/\tau$ is an Enriques surface.

Suppose now that Y is equipped with a real structure conj which commutes with s, and C is a real curve. Then $s \circ \mathrm{conj}$ is another real structure on Y and C. We denote the real point sets of these two structures by $Y_{\mathbb{R}}^{(i)}$ and $C_{\mathbb{R}}^{(i)}$, $i = 1, 2$ ($i = 1$ corresponding to conj) and call them the *halves* of Y and C respectively. The involutions conj and $s \circ \mathrm{conj}$ lift to four different commuting real structures ($t^{(1)}$, $t^{(2)} = \tau \circ t^{(1)}$, $d \circ t^{(1)}$, and $d \circ t^{(2)}$) on X, which, in turn, descend to two real structures on E; we call them the *expositions* of E. A choice of an exposition is determined by a choice of one of the two liftings $t^{(1)}$, $t^{(2)}$ of conj to X.

In the rest of this section we prove some preliminary results, and in Section 5 we show that the real part of $E = X/\tau$ may have any topological type not forbidden by Proposition 3.6. This completes the proof of Theorem 2.2. We use the real models of Y given by real quadrics in $\mathbb{C}\mathrm{P}^3$.

Remark. There are many other constructions of (complex) Enriques surfaces, most of them going back to Enriques himself. Examples are pencils of elliptic curves with two double fibers, order six surfaces in $\mathbb{C}\mathrm{P}^3$ with six double lines forming a tetrahedron, and double projective planes branched over order eight curves with certain singularities. The double plane model was used in the initial version of the proof, in addition to the present construction. The construction via elliptic pencils seems to be general: there is certain evidence that any real Enriques surface is a real pencil of elliptic curves.

Remark. Note that, though the construction used in the paper does give all the *topological* types of $E_{\mathbb{R}}$, there are *rigid isotopy* classes (i.e., components in the moduli space) of real Elliptic surfaces which cannot be obtained in this way. Indeed, if E can be obtained from $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$, it carries two elliptic pencils. The half-fibers (or multiple fibers) of such a pencil are the projections of the pull-backs of two lines in Y which pass through the fixed points of s and both belong to the same family of generatrices of Y. In particular, the two generatrices L_1, L_2 of Y passing through a fixed point of s produce two curves L_1', L_2' in E such that $[L_i']^2 = 0$, $i = 1, 2$, and $[L_1'] \circ [L_2'] = 1$. If Y, E, and the chosen fixed point of s are real, then either L_1', L_2' are both conj-invariant or they are transposed by conj. This gives the following necessary condition: if a real Enriques surface with both the halves nonempty can be obtained from $Y = \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$, then there are some homology classes $[L_1'], [L_2'] \in H_2(E; \mathbb{Z})$ such that $[L_i']^2 = 0$, $i = 1, 2$, $[L_1'] \circ [L_2'] = 1$, and either $\mathrm{conj}_*[L_i'] = -[L_i']$, $i = 1, 2$, or $\mathrm{conj}_*[L_1'] = -[L_2']$. There is at least one lattice which does not satisfy this condition: according to [11], conj_* may act on $H_2(E; \mathbb{Z})/\operatorname{Tors} \cong E_8 \oplus U$ so that the invariant and the skew-invariant parts are D_8 (the lattice generated by the root system D_8) and $U(2)$ (the hyperbolic plane whose form is multiplied by 2) respectively.

4.2 Quadrics to be used. Let Y be a quadric in $\mathbb{C}\mathrm{P}^3$ real in respect to the standard complex conjugation involution and invariant in respect to the symmetry $s\colon \mathbb{C}\mathrm{P}^3 \to \mathbb{C}\mathrm{P}^3$, $(x_0\colon x_1\colon x_2\colon x_3) \mapsto (x_0\colon x_1\colon -x_2\colon -x_3)$. The complex conjugation and s commute with each other. If Y does not contain either of the two axes $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$ of s, then s has four fixed points on Y and exchanges its two families of generatrices.

In our construction Y is one of the following three quadrics:

1. *hyperboloid of type* I, given by the equation $x_0^2+x_3^2 = x_1^2+x_2^2$. Both the halves, $Y_{\mathbb{R}}^{(1)}$ and $Y_{\mathbb{R}}^{(2)}$, are topological tori. They have four common points which are the four fixed points of $s\big|_Y$;
2. *hyperboloid of type* II, given by $x_0^2 + x_1^2 = x_2^2 + x_3^2$. In this case $Y_{\mathbb{R}}^{(1)}$ is a torus, $Y_{\mathbb{R}}^{(2)}$ is empty, and s has no fixed point in $Y_{\mathbb{R}}^{(1)}$;
3. *ellipsoid,* given by $x_0^2 = x_1^2 + x_2^2 + x_3^2$. Both the halves are spheres. They have two common points which constitute the fixed point set of the restriction of s to $Y_{\mathbb{R}}^{(1)}$, as well as to $Y_{\mathbb{R}}^{(2)}$.

Since C is a bi-degree $(4, 4)$ curve, $C_{\mathbb{R}}^{(i)}$ separates $Y_{\mathbb{R}}^{(i)}$ into two parts, which have $C_{\mathbb{R}}^{(i)}$ as their common boundary (at least one of the two parts is nonempty unless $Y_{\mathbb{R}}^{(i)}$ is empty). The fixed point set $X_{\mathbb{R}}^{(i)}$ of $t^{(i)}$ is the pull-back of one of the parts. Thus, a choice of $t^{(1)}$ is equivalent to a choice of one of the two parts of $Y_{\mathbb{R}}^{(1)}$, and, since $t^{(2)} = \tau \circ t^{(1)}$, the latter determines as well the choice of the part of $Y_{\mathbb{R}}^{(2)}$ whose pull-back is $\operatorname{Fix} t^{(2)}$. This correlation is easily controlled due to the fact that $X_{\mathbb{R}}^{(1)}$ and $X_{\mathbb{R}}^{(2)}$ are disjoint:

4.2.1. *The pull-back of a point of $Y_{\mathbb{R}}^{(1)} \cap Y_{\mathbb{R}}^{(2)}$ is contained in exactly one of the sets $X_{\mathbb{R}}^{(1)}$, $X_{\mathbb{R}}^{(2)}$.* (*Note that in all the examples we use the above intersection is not empty unless one of the halves is empty.*)

Another assertion we use is the following:

4.2.2. *If $C_{\mathbb{R}}^{(2)}$ is empty and $X_{\mathbb{R}}^{(2)}$ is nonempty, then $X_{\mathbb{R}}^{(2)} \to Y_{\mathbb{R}}^{(2)}$ is the trivial double covering.*

Proof. The statement needs proof only in the case of hyperboloid of type I. In this case we should show that the images of the generators of $H_1(Y_{\mathbb{R}}^{(2)}; \mathbb{Z}/2)$ in $H_1(Y \smallsetminus C; \mathbb{Z}/2)$ are trivial, i.e., that the generators are not linked $\pmod 2$ with C. The standard generators are cut on $Y_{\mathbb{R}}^{(2)}$ by the real generatrices of Y. Such a generatrix intersects C in four imaginary pairwise conjugate points. Thus, a hemisphere of imaginary points intersects C in two points, which means that the generator realized by the circle of real points is not linked with C. □

4.3. Branch curves. To construct the branch curve $C \in Y$ we start with a singular s-symmetric curve $\widetilde{C} \in Y$, given by an equation $f = 0$, and perturb it to the curve C given by $f + \varepsilon h = 0$; here f and h are homogeneous real bi-degree $(4, 4)$ polynomials either both s-symmetric or both s-skew-symmetric and ε is a small real parameter. The following considerations are used to control the topology of $C_{\mathbb{R}}^{(i)}$:

4.3.1. Nodes of $\widetilde{C}$.

Lemma. *Suppose that all the singularities of $\widetilde{C}$ are nodes which all belong to one of the two halves of $\widetilde{C}$. Then the symmetric pairs of nodes can be perturbed independently with preserving the symmetry of the curve.*

Proof. From the generalized Bruzotti theorem for order two surfaces (see [4]) it follows that one can find a perturbing term h with any prescribed distribution of signs at the nodes of $\widetilde{C}$. (This means that the nodes can be perturbed independently.) If the prescribed distribution of signs is of the same symmetry as f, we can replace h with $h + s^*h$ or $h - s^*h$ to obtain a symmetric perturbation. □

Remark. In fact, the same reasoning covers the case when $\widetilde{C}$ is allowed to have nodes at some of the fixed points of s (while all the other nodes are still concentrated in one half). (Note that in this case $\widetilde{C}$ must be given by a symmetric, not skew-symmetric, polynomial.) The result is the same: the symmetric pairs of nodes and the fixed nodes can be perturbed independently with preserving the symmetry of $\widetilde{C}$.

4.3.2. Double components of $\widetilde{C}$.

In this subsection we assume that:

$\widetilde{C}$ has a symmetric real double component T, the real parts $T_{\mathbb{R}}^{(1)}, T_{\mathbb{R}}^{(2)}$ of both the halves of T are smooth and disjoint from the other components of $\widetilde{C}$, and the zero-set D of the perturbative term h is transversal to $\widetilde{C}$.

Under such a perturbation the topology of the curve changes only in a neighborhood of $T_{\mathbb{R}} = T_{\mathbb{R}}^{(1)} \cup T_{\mathbb{R}}^{(2)}$. This modification is controlled, in an evident way, by the set $D \cap \widetilde{C}$, called *the ramification divisor*, and the signs of f, h, and ε: a portion of $T_{\mathbb{R}}$ doubles while the complementary portion disappears. In particular, if T passes through a fixed point of s, then the portion containing this point doubles in one half and disappears in the other (Figure 1(a)).

The ramification divisors considered are always conj-s-symmetric, and the difference of any two such divisors is the divisor of a rational conj-s-invariant function. Besides, the linear system of the bi-degree $(4, 4)$ sections on T is complete. Thus, to decide whether a conj-s-symmetric set $\mathcal{D} \subset T$ is a ramification divisor (i.e., can be taken for the ramification divisor of a real symmetric perturbation), one should consider the conj-invariant set $\mathcal{D}/s$ in the quotient $T' = T/s$, which is a real algebraic curve; then the problem reduces to existence of a linear equivalence over $\mathbb{R}$ on T' between $\mathcal{D}/s$ and the quotient of an arbitrary chosen particular ramification divisor.

We use this approach in the next lemma to take care about the set of ramification points.

Lemma. *Let T be either a rational curve of bi-degree $(1,1)$ or an elliptic curve of bi-degree $(2,2)$.*

(1) *If T is elliptic and it does not pass through any of the fixed points of s, consider a* conj-s*-invariant set $\Delta \subset T_{\mathbb{R}}$ of $4k$, $k \leqslant 4$, points. It is isotopic in $T_{\mathbb{R}}$ to a set which can be taken for the real part $D \cap T_{\mathbb{R}}$ of the ramification divisor of a symmetric real perturbation if and only if each connected component of $T'_{\mathbb{R}}$ contains an even number of points of Δ/s.*

(2) *In the other two cases, any* conj-s*-invariant set of 8 (if T is rational) or 16 (if T is elliptic and contains a fixed point of s) points is the ramification divisor of a symmetric real perturbation.*

Proof. Statement (2): Under the hypotheses T' is a rational curve; hence, any two conj-invariant sets on T' of the same cardinality are linear equivalent over $\mathbb{R}$.

Statement (1): The parity condition follows from the observation that h has a well defined sign on $T_{\mathbb{R}}$ which descends to $T'_{\mathbb{R}}$ and divides it into two parts ('positive' and 'negative') which have $(D \cap T_{\mathbb{R}})/s$ as their common boundary.

It remains to prove the last part, where T is elliptic, s is fixed point free on T, and Δ satisfies the parity condition. Then T' is also elliptic. Let us take for the basic ramification divisor D_0 the one given by $h = (x_0^2 + x_1^2)(x_2^2 + x_3^2)$; its real part $D_0 \cap T_{\mathbb{R}}$ is empty.

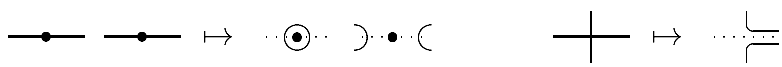
We will distinguish between the following two possibilities: $T'_{\mathbb{R}}$ has one connected component F_1, or $T'_{\mathbb{R}}$ has two connected components F_1, F_2. (If $T'_{\mathbb{R}}$ is empty, the assertion is trivial; D_0 gives an example). It suffices to prove, in each of the two cases, the following assertion: given an even integer $p_1 \leqslant 8$ (respectively, two even integers p_1, p_2 with $p_1 + p_2 \leqslant 8$), there exists a conj-invariant subset $\mathcal{S} \subset T'$ linearly equivalent over $\mathbb{R}$ to $\mathcal{R} = (D_0 \cap T)/s$ and such that $\#(\mathcal{S} \cap F_i) = p_i$.

In both the cases $T' = \mathbb{C}/L$, where $L \in \mathbb{C}$ is a lattice invariant in respect to the standard conjugation $z \mapsto \bar{z}$, and $\mathcal{R}$ and $\mathcal{S}$ can be represented in $\mathbb{C}$ by sets $\{r_1, \bar{r}_1, \ldots, r_4, \bar{r}_4\}$ and $\{s_1, \ldots, s_8\}$ respectively. Then $\mathcal{R}$ and $\mathcal{S}$ are linearly equivalent over $\mathbb{R}$ if and only if $r_1 + \cdots + \bar{r}_4 \equiv s_1 + \cdots + s_8 \pmod{L}$.

In the first case L is generated by $1 \pm ai$, $a \in \mathbb{R}$, the only real component is represented by $\mathbb{R} \subset \mathbb{C}$, and one can replace any pair r_i, $\bar{r}_i$ of conjugate points of $\mathcal{R}$ with a pair $\operatorname{Re} r_i \pm \varepsilon$, $\varepsilon \in \mathbb{R}$, of real points which have the same sum.

In the second case L is generated by 1 and $2ai$, $a \in \mathbb{R}$, the real components are represented by $\mathbb{R} \subset \mathbb{C}$ and $\mathbb{R} + ai \subset \mathbb{C}$, and a pair r_i, $\bar{r}_i$ can be replaced with either $\operatorname{Re} r_i \pm \varepsilon \in \mathbb{R}$ or $\operatorname{Re} r_i + ai \pm \varepsilon \in \mathbb{R} + ai$, $\varepsilon \in \mathbb{R}$, without changing the sum $\mod L$. □

4.3.3.. Other singularities. The only other type of singular curves used in our construction is the union of a smooth curve and the double of another smooth curve intersecting the first one transversally. As in 4.3.2, we consider only perturbations with the zero-set D transversal to $\widetilde{C}$. The only new feature is that the doubling of the multiple component switches to disappearing at a point of intersection with the single one (see Figure 1(b)). Note that here the double curve is always rational, and, due to the lemma in 4.3.2, the ramification points can be chosen arbitrarily.



(a) (b)

FIGURE 1

5. CONSTRUCTION (DESCRIPTION)

5.1. In Table 1 we list all the topological types not prohibited by Proposition 3.6, and for each type give a reference to the paragraph(s) of this section where it is constructed. The branch curve C involved in the construction is obtained by a small perturbation of a singular curve $\widetilde{C}$ shown in Figures 2, 3, and 6–10 below. Each figure consists of two parts which represent the two halves of the real part of Y (only 'visible side' of each half is shown; the rest can be recovered using the symmetry); the bold (light) lines denote double (resp. single) components of $\widetilde{C}$; the black dots and dotted lines represent the fixed points of $s\big|_Y$ and the axis of symmetry of s respectively. A more precise description of each curve is given in 5.2–5.10; existence of the desired perturbation follows each time from the lemmas in 4.3.1 and 4.3.2.

The figures are some sort of coded description of the curves. To decode them and to recover the topology of $E_{\mathbb{R}}$ we use the following conventions and rules.

Projection used. In Figures 2–7, which represent hyperboloids of type I, we show the affine part $x_0 \neq 0$. The left hand piece of each picture shows the 'real' half of $Y_{\mathbb{R}}$ in its projection to the $(x_0 : x_1 : x_3)$-plane; the right hand piece shows the 'imaginary' half in its projection to the $(x_0 : x_1 : ix_2)$-plane. (Actually, these are rather schemes than projections; in particular, the visible counter of the hyperboloid is represented by two vertical lines.) The two left fixed points of s seen in the two pieces of a picture in fact coincide, and so do the two right fixed points. Figures 8–10 show ellipsoids: the affine part $x_0 \neq 0$ of the 'real' half projected to $(x_0 : x_1 : x_3)$ (the left hand piece of the figures) and the affine part $x_1 \neq 0$ of the 'imaginary' half projected to $(x_0 : x_1 : ix_3)$. Finally, Figure 11 represents the affine part $\mathbb{R}^1 \times \mathbb{R}^1$ of the hyperboloid $\mathbb{R}\mathrm{p}^1 \times \mathbb{R}\mathrm{p}^1$.

Choice of an exposition. As it is explained in 4.2, the topological type of $E_{\mathbb{R}}$ depends on the branch curve C **and** on the choice of an exposition, i.e., of one of the two parts into which $C_{\mathbb{R}}$ divides $Y_{\mathbb{R}}$. In Figures 2–7 (hyperboloid of type I) and in the listings related to these figures we mark with a * the surfaces obtained by doubling the part of $Y_{\mathbb{R}} \setminus C_{\mathbb{R}}$ which contains the upper left corner of the picture.

The topology of $E_{\mathbb{R}}$. The exposition being fixed, the topology of $E_{\mathbb{R}}$ can easily be recovered: each s-invariant component of the chosen part of $Y_{\mathbb{R}} \setminus C_{\mathbb{R}}$ produces a nonorientable component of $E_{\mathbb{R}}$ of the same Euler characteristic (or a torus, if the original component coincides with one of the halves of $Y_{\mathbb{R}}$, see Lemma 4.2.2), and each pair of components of $Y_{\mathbb{R}} \setminus C_{\mathbb{R}}$ transposed by s produces an orientable component of $E_{\mathbb{R}}$; the Euler characteristic of the latter is twice that of each of the components of the considered pair.

Remark. In the subsequent paragraphs, describing the figures, we also list the topological types obtained using each branch curve. Note that our list is a little bit excessive: certain topological types appear several times. The reason is that we tried to construct as many surfaces as possible, taking into account the distribution of the components between the two halves of the real part. For that reason for some surfaces we indicate the distribution obtained, using the notation $\{\text{half } E_{\mathbb{R}}^{(1)}\} \sqcup \{\text{half } E_{\mathbb{R}}^{(2)}\}$.

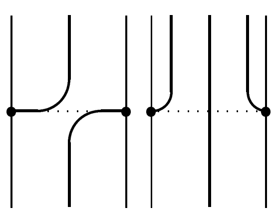
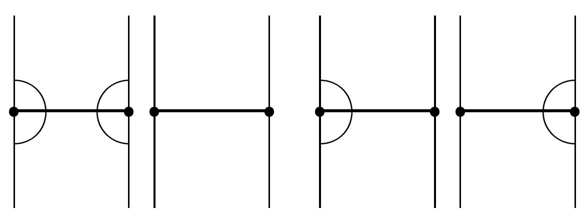


FIGURE 2



(a) (b)

FIGURE 3

5.2 (Figure 2). Y is the hyperboloid of type I, and $\widetilde{C}$ is the double of a bi-degree $(2, 2)$ curve F which has two components in each half of Y and passes through all the four fixed points of $s\big|_Y$. To construct F, we perturb the union of two bi-degree $(1, 1)$ curves on Y through two of the four fixed point each. The latters may be given by $x_1 = 0$ and $x_3 = 0$.

TABLE 1

Types	Paragraph(s)
$4V_1 \sqcup kS$	**5.2** $(k \leqslant 2)$
$V_2 \sqcup 2V_1 \sqcup kS$	**5.2** $(k \leqslant 3)$
$V_4 \sqcup 2V_1$	**5.3**
$V_3 \sqcup V_2 \sqcup V_1$	**5.3**
$V_r \sqcup V_s$	**5.2** $(6 \leqslant r+s \leqslant 12;\ r,s \geqslant 1)$
$V_{2r+1} \sqcup V_1$	**5.4** $(1 \leqslant r \leqslant 5)$
$V_r \sqcup V_{6-r} \sqcup kS$	**5.3** $(r=1,2,3;\ k \leqslant 1)$
$V_3 \sqcup V_1 \sqcup kS$	**5.4** $(k \leqslant 4)$
$2V_2 \sqcup kS$	**5.2** $(k \leqslant 4)$
$2V_1 \sqcup kS$	**5.6** $(k \leqslant 3)$; **5.8** $(k=4)$
$V_{2r} \sqcup kS$	**5.4** $(r \geqslant 2;\ k+r \leqslant 5)$
	5.5 $(2 \leqslant r \leqslant 5;\ k=1)$
	5.5 $(r=2;\ 1 \leqslant k \leqslant 5)$
	5.6, 7, 8 $(r=1;\ k \leqslant 5)$
kS	**5.8, 9** $(k \leqslant 5)$
$V_2 \sqcup S_1 \sqcup kS$	**5.4** $(k \leqslant 4)$
$S_1 \sqcup kS$	**5.4** $(k \leqslant 4)$
$S_1 \sqcup \ldots$	**5.10** (other types)

The types obtained are:

$4V_1 \sqcup kS,\ k \leqslant 2,$

$V_2 \sqcup 2V_1 \sqcup kS,\ k \leqslant 3,$

$\{2V_2\} \sqcup \{kS\},\ k \leqslant 4,$

$^*\ V_r \sqcup V_s,\ 6 \leqslant r+s \leqslant 12,$
$r,s \geqslant 2,\ r+s \equiv 0 \pmod 2.$

To achieve this, we pick an appropriate number of ramification points in an appropriate position (see 4.3.2) to form up to eight ovals and/or double up to two connected components (to obtain V_2). More precisely, the type $4V_1 \sqcup kS$ is obtained by forming $4+2k$ ovals so that four of them contain the fixed points of s; the type $V_2 \sqcup 2V_1 \sqcup kS$ is obtained by forming $2+2k$ ovals (two of them contain the fixed points of s) and doubling one component of $\widetilde{C}$; the type $\{2V_2\} \sqcup \{kS\}$ is obtained by doubling two components (which must belong to the same half of $Y_{\mathbb{R}}$) and forming $2k$ ovals; the type $V_r \sqcup V_s$ is obtained by forming $(r-2)$ ovals in one half of $Y_{\mathbb{R}}$ and $(s-2)$ ovals in the other half.

5.3 (Figure 3). Y is the hyperboloid of type I, and $\widetilde{C}$ is the union of the double of a bi-degree $(1,1)$ curve through two of the four fixed points (given by $x_3=0$) and a bi-degree $(2,2)$ curve. The latter may be given by $x_0^2+x_1^2=4x_2^2+4x_3^2$ in case (a) and by $x_0x_1=x_2^2+x_3^2$ in case (b).

The types obtained are:

(a) ${}^*\ V_5\sqcup V_1\sqcup kS,\ k\leqslant 1,$ ${}^*\ V_4\sqcup 2V_1,$
$V_4\sqcup V_2\sqcup kS,\ k\leqslant 1,$ $V_3\sqcup V_2\sqcup V_1,$
$2V_3\sqcup kS,\ k\leqslant 1,$ (b) $V_3\sqcup V_1\sqcup kS,\ k=1,2.$

The branch curve C (the perturbation of $\widetilde{C}$) in cases (a) and (b) is shown in Figures 4 and 5 respectively, where small circles and semicircles represent the ovals obtained by doubling a portion of the double component of $\widetilde{C}$, and large shapes come from the simple component. Whenever present, a pair of symmetric small ovals may disappear to decrease the number of spherical components of $E_{\mathbb{R}}$.

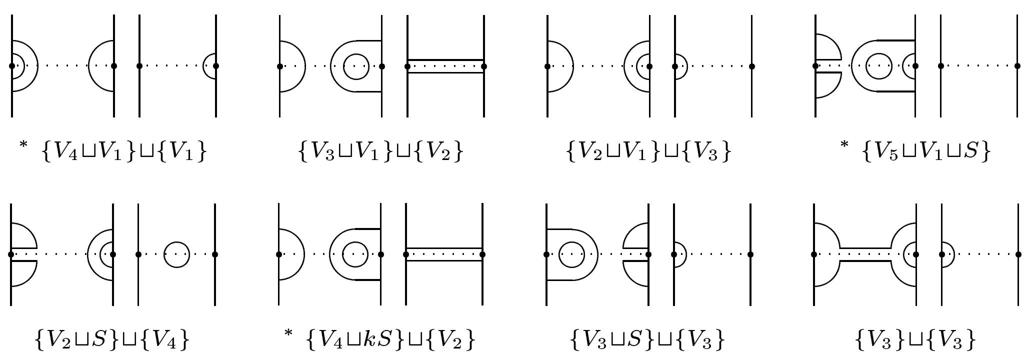


Figure 4

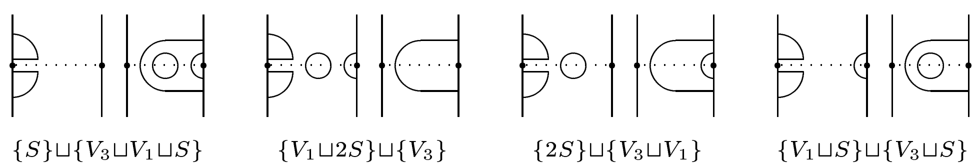


Figure 5

5.4 (Figure 6). Y is the hyperboloid of type I. In cases (a) and (b) $\widetilde{C}$ is the double of a bi-degree $(2,2)$ curve (cf. 5.3); in case (c) it is the union of two such curves with eight common points. (To construct $\widetilde{C}$ in this case, one can

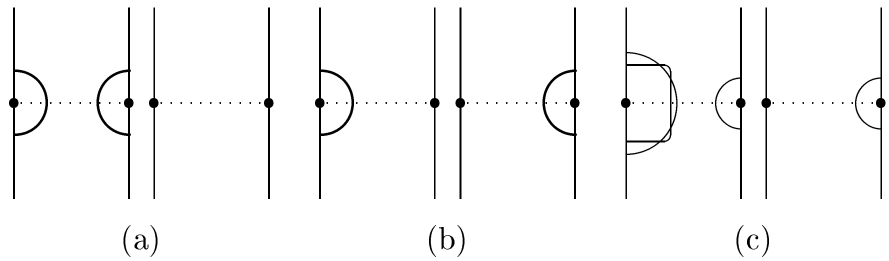


FIGURE 6

start with a curve with an isolated double point at the 'right' fixed point, and then perturb it in the two different ways holding eight points.)

The types obtained are:

(a) $\{V_2 \sqcup kS\} \sqcup \{S_1\},\ k \leqslant 4,$
$\{kS\} \sqcup \{S_1\},\ k \leqslant 4,$
$^*\ \{V_{2r+1} \sqcup V_1\},\ 1 \leqslant r \leqslant 5,$

(b) $\{V_3 \sqcup V_1\} \sqcup \{kS\},\ k \leqslant 4,$
$^*\ \{V_{2r}\} \sqcup \{kS\},\ r \geqslant 2,\ k + r \leqslant 5,$

(c) $\{V_1 \sqcup kS\} \sqcup \{V_3\},\ k \leqslant 4,$
$^*\ \{V_{2r+1}\} \sqcup \{V_1\},\ 1 \leqslant r \leqslant 5.$

The perturbation of $\widetilde{C}$ to C is the following: in case (c), we perturb the nodes to form $2k$ or $(2r-2)$ ovals; for the type $\{kS\} \sqcup \{S_1\}$, we form $2k$ ovals from $\widetilde{C}$; for the type $\{V_{2r}\} \sqcup \{kS\}$, we form $(2r - 2)$ ovals from the left component of $\widetilde{C}$ and $2k$ ovals from the right one; for the other types in cases (a) and (b), we form up to eight ($2k$ or $(2r-2)$) ovals from the left component and double the right one.

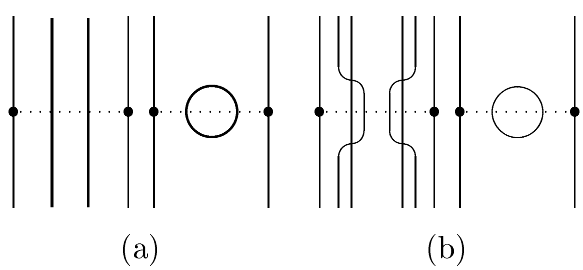


FIGURE 7

5.5 (Figure 7). Y is the hyperboloid of type I. To construct $\widetilde{C}$, we start with a bi-degree $(2, 2)$ curve with two components in one of the halves of Y and two isolated double points in the second half (say, given by the equation $x_0^2 = 4x_1^2 - 4x_2^2 - x_3^2$), and perturb it in the two different ways keeping eight points fixed. In case (a), $\widetilde{C}$ is the double of the perturbation which has two components in each half; in case (b) it is the union of two different perturbations.

We use this construction to obtain two types:

(a) $\{kS\} \sqcup \{V_4 \sqcup S\},\ k \leqslant 4,$ (b) $^*\ \{V_{2r}\} \sqcup \{S\},\ 2 \leqslant r \leqslant 5.$

In case (a), we form $2k$ ovals from the left half of $\widetilde{C}$ and double the right half; in case (b), we perturb the nodes to form $(2r-2)$ ovals.

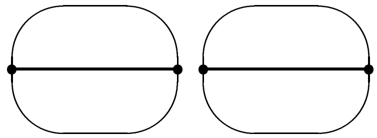


FIGURE 8

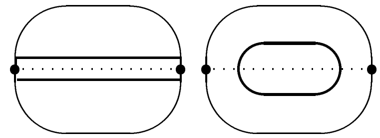


FIGURE 9

5.6 (Figure 8). Y is the ellipsoid; $\widetilde{C}$ is the double of a bi-degree $(2,2)$ curve through both the fixed points of $s\big|_Y$. The two types obtained are:

$2V_1 \sqcup kS,\ k \leqslant 3$ $\{V_2\} \sqcup \{kS\},\ k \leqslant 4.$

The first type is obtained by forming $(2k+2)$ ovals (two of them must contain the fixed points of s in their interior); to obtain the second type, we double the left half of $\widetilde{C}$ and form $2k$ ovals from the right one. The exposition is chosen so that a neighborhood of the leftmost fixed points of s in $Y_{\mathbb{R}}^{(1)}$ (the left half of $Y_{\mathbb{R}}$) is covered by $X_{\mathbb{R}}^{(1)}$.

5.7 (Figure 9). Y is the ellipsoid, $\widetilde{C}$ is the double of a bi-degree $(2,2)$ curve which has two components in each half of Y (it may be given by $x_0^2 + 4x_2^2 = 4x_3^2$). We double the two left connected components of $\widetilde{C}$ and form $2k$ ovals from the two right ones to obtain the type $\{V_2 \sqcup S\} \sqcup \{kS\},\ k \leqslant 4$. The exposition is chosen so that a neighborhood of the leftmost fixed points of s in $Y_{\mathbb{R}}^{(1)}$ (the left half of $Y_{\mathbb{R}}$) is covered by $X_{\mathbb{R}}^{(1)}$.

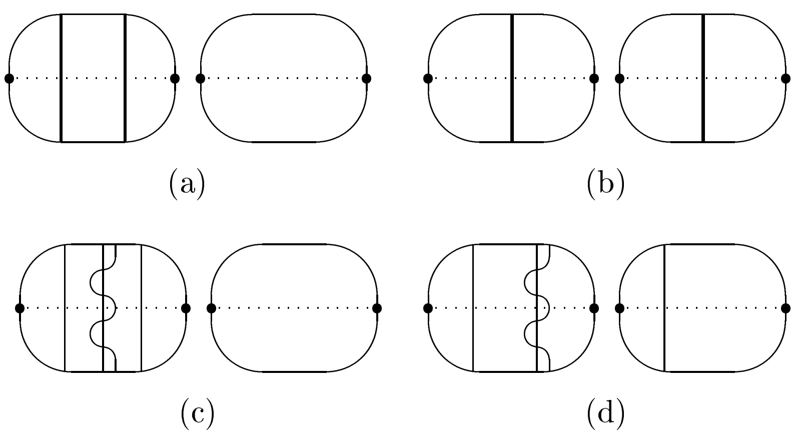


FIGURE 10

5.8 (Figure 10). Y is the ellipsoid. To construct $\widetilde{C}$ in cases (a), (b) and (d), we start with a bi-degree $(2,2)$ curve given by $x_0^2 + x_0x_1 = x_2^2 + x_3^2$ (this curve has one oval in the left half, no ovals in the right half, and an isolated double point at the 'left' fixed point of $s\big|_Y$) and perturb it in the two different ways keeping eight real points on the oval fixed. $\widetilde{C}$ is the double of one of the perturbations (cases (a), (b)) or the union of two different perturbations (case (d)).

To explain the construction of $\widetilde{C}$ in the remaining case (c) let us use the affine coordinates $x = x_1/x_0$, $y = x_2/x_0$, $z = x_3/x_0$. Consider the ellipse F given in the (xy)-plane by the equation $16x^2 + y^2 = 4$, and a tangent L to F. Choose the latter in such a manner that both the contact point and the point of intersection with the x-axis belong to the disk $x^2 + y^2 < 1$ (bounded by the visible counter of $Y_{\mathbb{R}}^{(1)}$ in the (xy)-plane). Let L' be the line symmetric to L against the x-axis. Consider the union $L \cup L'$ and slightly perturb it to a hyperbola H which intersects F in four real points which all belong to the same branch of H. (The perturbation should be so small that all the four points, as well as the two points of intersection of H and the x-axis, still belong to the disk.) Now we can take for $\widetilde{C}$ the curve cut on Y by the cylinder over $H \cup F$.

The types obtained are:

(a) $\{V_2 \sqcup kS\} \sqcup \{S\}$, $k \leqslant 4$, $\{kS\} \sqcup \{S\}$, $k \leqslant 4$,

(b) (c) (d) $2V_1 \sqcup kS$, $k \leqslant 4$.

(We use (b), (c), and (d) to obtain a variety of distributions of the components of the type $2V_1 \sqcup kS$.) In cases (a) and (b) we form $2k$ ovals from the right component of $\widetilde{C}$ and, except the type $\{kS\} \sqcup \{S\}$, double the left component. In cases (c) and (d) we perturb the nodes to form $2k$ ovals. In all the cases except (c) **no** neighborhood of the leftmost fixed point of s in $Y_{\mathbb{R}}^{(1)}$ (the left half of $Y_{\mathbb{R}}$) is covered by $X_{\mathbb{R}}^{(1)}$.

5.9. The empty surface and the distribution $\{kS\}$, $k \leqslant 4$, are obtained from the hyperboloid of type II, $\widetilde{C}$ being the double of any nonempty curve on it, which is perturbed to form $2k$ ovals.

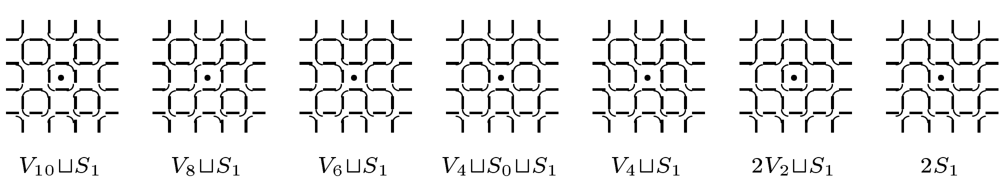


FIGURE 11

5.10 (Figure 11). With the exception of $V_2 \sqcup kS \sqcup S_1$ and $kS \sqcup S_1$ (see 5.4), any topological type which has a torus S_1 can be obtained from the hyperboloid of type I, $\widetilde{C}$ being the union of four lines of bi-degree $(1,0)$ and four lines of bi-degree $(0,1)$. The perturbations are shown in Figure 11, where the affine part $\mathbb{R}^1 \times \mathbb{R}^1$ of $Y_{\mathbb{R}}^{(1)}$ is represented. (The black dot is one of the four fixed points of s; the other three points are at infinity.) The other half of $C_{\mathbb{R}}$ is empty, and the exposition is chosen so that $Y_{\mathbb{R}}^{(2)}$ (not shown in the pictures) is covered by the torus $X_{\mathbb{R}}^{(2)}$ (see 4.2.2).

Appendix

A.1. Theorem. *Let E be a real algebraic $(M-2)$-surface, and $\chi(E_{\mathbb{R}}) \equiv \sigma(E)+8 \pmod{16}$. Then $[E_{\mathbb{R}}] = w_2(E)$, where $[E_{\mathbb{R}}]$ denotes the image of the fundamental class of $E_{\mathbb{R}}$ in $H_2(E;\mathbb{Z}/2)$.*

Proof. Denote $\widetilde{H}_{\mathbb{Z}} = H_2(E;\mathbb{Z})/\operatorname{Tors}$ and $H = H_2(E;\mathbb{Z}/2)$, and let $H_{\mathbb{Z}}$ be the image of $H_2(E;\mathbb{Z})$ in H. Known arguments (sf. e.g., Nikulin [10]; everything follows, in fact, from Smith exact sequence and Van der Blij formula) show that

(A.1.1) $\dim\bigl(\operatorname{Ker}[(1-\operatorname{conj}_*)\colon \widetilde{H}_{\mathbb{Z}} \to \widetilde{H}_{\mathbb{Z}}]/\operatorname{Im}(1+\operatorname{conj}_*)\bigr) = \operatorname{rk}\widetilde{H}_{\mathbb{Z}} - 4$.

(A.1.2) the *twisted* intersection form $x \mapsto x \circ \operatorname{conj}_* x$ on $\widetilde{H}_{\mathbb{Z}}$ and the ordinary square $x \mapsto x \circ x$ are congruent to each other mod2;

Since $[E_{\mathbb{R}}]$ represents the characteristic class of the twisted intersection form $(x,y) \mapsto x \circ \operatorname{conj}_* y$, from (A.1.2) it follows that the difference $[E_{\mathbb{R}}] - w_2(E)$ annihilates integral classes, i.e., belongs to the image of Tors.

Since E is an $(M-2)$-surface, from (A.1.1) and Smith exact sequence it follows that each element $x \in H/H_{\mathbb{Z}}$ has a conj_*-invariant representative $\bar{x} \in H$. Then $\bar{x} \circ [E_{\mathbb{R}}] = \bar{x} \circ \operatorname{conj}_* \bar{x} = \bar{x}^2 = \bar{x} \circ w_2(E)$; thus, $[E_{\mathbb{R}}] - w_2(E)$ annihilates also $H/H_{\mathbb{Z}}$, i.e., is zero. □

A.2. Proof of 3.4. More precisely, we prove here the following result:

Let X be a smooth oriented Spin *simply connected 4-manifold with $\sigma(X) \equiv 16 \pmod{32}$. Let τ be a fixed point free orientation preserving involution on X, and let t be another orientation preserving involution on X which commutes with τ. Suppose that both the fixed point sets $X_{\mathbb{R}} = \operatorname{Fix} t$ and $X'_{\mathbb{R}} = \operatorname{Fix}(t \circ \tau)$ are surfaces (i.e., each of them is either empty or of pure dimension two). Then $X_{\mathbb{R}}$ (as well as $X'_{\mathbb{R}}$) has a canonical orientation which is reversed by τ.*

Remark. Orientability of $X_{\mathbb{R}}$ is a well known fact, see, e.g, [3], [14], or [8]. We need a little bit more: a canonical orientation. We use an idea which goes back to our old discussions with O. Viro, which was inspired by Natanzon's observation about the Spin-orientations of real algebraic curves, see [9].

Proof. The canonical orientation is provided by the Spin-structure on X. Pick a point $x \in X_{\mathbb{R}}$ and an orientation of $X_{\mathbb{R}}$ at x. In order to compare this orientation and an orientation at another point $y \in X_{\mathbb{R}}$, represent them by 2-frames (ξ_1^x, ξ_2^x) and (ξ_1^y, ξ_2^y) and complete these 2-frames to positively oriented 4-frames by some t-skew-invariant vectors (ξ_3^x, ξ_4^x) and (ξ_3^y, ξ_4^y) respectively. Then pick a path γ connecting x and y, extend the 4-frames to a 4-frame field $\Xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ over γ, and evaluate the Spin-structure on the loop $\gamma * t\gamma^{-1}$ framed with $\Xi * \Xi^t$, where $\Xi^t = (dt\,\xi_1, dt\,\xi_2, -dt\,\xi_3, -dt\,\xi_4)$. (The latter framed loop is called a *test loop*.) The two orientations at x and y are considered coherent iff the value obtained is 0.

This construction is consistent since on a simply connected manifold the Spin-structure is unique and, in particular, equivariant, i.e., it takes equal values on any pair of symmetric framed loops.

Note now that the quotient X/τ, as well as the quotient of X by any fixed point free orientation preserving involution, is not Spin since $\sigma(X/\tau) = \frac{1}{2}\sigma(X) \equiv 8 \pmod{16} \not\equiv 0 \pmod{16}$. Hence, given a τ-symmetric loop with a 4-frame field $\Xi = (\xi_1, \xi_2, \xi_3, \xi_4)$, the value of the Spin-structure on it is 1 if Ξ is τ-invariant, and it is 0 if Ξ is τ-skew-invariant, i.e., $d\tau(\xi_1, \xi_2, \xi_3, \xi_4) = (\xi_1, \xi_2, -\xi_3, -\xi_4)$.

To complete the proof it suffices now to construct a τ-invariant test loop. If $\operatorname{Fix}(t \circ \tau) \neq \varnothing$, pick some $x \in X_{\mathbb{R}}$ and $a \in \operatorname{Fix}(t \circ \tau)$, join them by an arc (xa), and let γ be the loop formed by the four arcs (xa), $t(xa)$, $\tau(xa)$, and $t\tau(xa)$. Pick a t-invariant frame (ξ_1^x, ξ_2^x) at x (respectively, a $(t \circ \tau)$-invariant frame (ξ_1^a, ξ_2^a) at a), complete them by t-skew-invariant vectors (ξ_3^x, ξ_4^x) (respectively, $(t \circ \tau)$-skew-invariant vectors (ξ_3^a, ξ_4^a)) to positively oriented 4-frames, and extend these 4-frames to a 4-frame field over (xa). Now simple reflection gives a t-invariant continuous 4-framing over γ.

If $\operatorname{Fix}(t \circ \tau) = \varnothing$, we pick a point $a \in X$ whose orbit a, ta, τa, $t\tau a$ consists of four elements. To form a test loop we take the same four arcs as above and complete them by an arc connecting a and $t\tau a$ and its t-symmetric copy. The test loop obtained in the similar manner as before is not τ-invariant; it is the sum of two loops, one τ-invariant and one $(t \circ \tau)$-skew-invariant. Since both τ and $(t \circ \tau)$ are fixed point free now (and since they both preserve the orientation), the Spin-structure takes value 1 on the former portion and 0 on the latter one, which totals to 1 on γ. □

A.3. Proof of 3.5.1. $[T_1]$ and $[T_2]$ are nontrivial even in $H_2(X; \mathbb{Z}/2)$, since the only linear combination which may vanish in $H_2(X; \mathbb{Z}/2)$ is $[T_1] + [T_2]$. (This follows, e.g., from the Smith exact sequence.) These classes belong to the eigenspace of $t_*^{(1)}$ corresponding to the eigenvalue 1. But the latter space is hyperbolic, and in such a space two orthogonal isotropic classes cannot be linearly independent (see [6] for details). □

A.4. Proof of 3.5.2. Consider the quotient 4-manifold $X' = X/\operatorname{conj}$. Then

$$\sigma(X') = \frac{1}{2}\bigl(\sigma(X) + [X_{\mathbb{R}}] \circ_X [X_{\mathbb{R}}]\bigr) = \frac{1}{2}\sigma(X) = -8$$

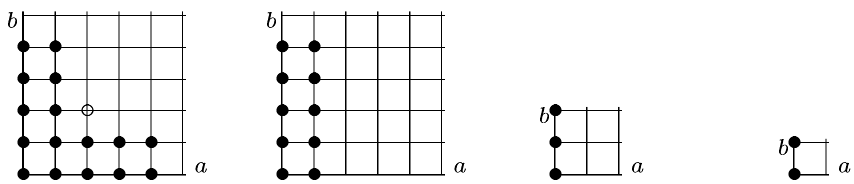
and

$$w_2(X) = [X_{\mathbb{R}}] + \pi^* w_2(X'),$$

where $\pi\colon X \to X'$ is the quotient map. As it is shown in [6], the kernel $\operatorname{Ker}\{\pi^*\colon H_2(X';\mathbb{Z}/2) \to H_2(X;\mathbb{Z}/2)\}$ is generated by the elements realized by the components of $X_{\mathbb{R}}$. Hence, if in the case under consideration $X_{\mathbb{R}}$ realized zero in $H_2(X';\mathbb{Z}/2)$, then $w_2(X')$ would be zero too, and this would contradict to the Rokhlin congruence $\sigma(X') \equiv 0 \pmod{16}$. □

A.5. Distribution of the components. In this section we briefly discuss the problem of distribution of the components of $E_{\mathbb{R}}$ between the two halves.

For all the topological types except those listed in Figures A.1, A.2 below, our construction shows that any distribution which satisfies the restriction of 3.4 can be realized. The exceptional types with at most one nonspherical component are listed in Figure A.1. (We use the notation $\{\text{half } E_{\mathbb{R}}^{(1)}\} \sqcup \{\text{half } E_{\mathbb{R}}^{(2)}\}$.)



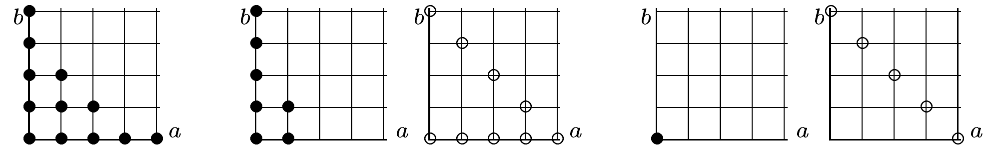
$\{aS\}\sqcup\{bS\}$, $\{V_2\sqcup aS\}\sqcup\{bS\}$ $\{V_4\sqcup aS\}\sqcup\{bS\}$ $\{V_6\sqcup aS\}\sqcup\{bS\}$ $\{V_8\sqcup aS\}\sqcup\{bS\}$, $\{V_{10}\sqcup aS\}\sqcup\{bS\}$

FIGURE A.1. Topological types with at most one nonspherical component

The distributions marked by black nodes are constructed in Section 5. The distributions $\{2S\}\sqcup\{2S\}$ and $\{V_2\sqcup 2S\}\sqcup\{2S\}$ (the white node in Figure A.1) are constructed by Nikulin [11], and there is a strong reason to believe that they cannot be obtained from a hyperboloid or ellipsoid using the approach of Section 4.

A.6. Theorem. *The distribution of the components of a real Enriques surface with at most one nonspherical component is one of those represented in Figure* A.1.

Proof will appear elsewhere.



$\{2V_1 \sqcup aS\} \sqcup \{bS\}$,
$\{V_1 \sqcup aS\} \sqcup \{V_1 \sqcup bS\}$

$\{V_3 \sqcup aS\} \sqcup \{V_1 \sqcup bS\}$,
$\{V_3 \sqcup V_1 \sqcup aS\} \sqcup \{bS\}$

$\{V_2 \sqcup aS\} \sqcup \{V_2 \sqcup bS\}$

FIGURE A.2. Topological types with two nonspherical components

The other exceptional types are given in Figure A.2, where, similar to Figure A.1, the black and white nodes correspond, respectively, to the distributions constructed in Section 5 and to those constructed by Nikulin [11].

At present, we can only conjecture that for these types any distribution is possible.

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