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A DIVISIBILITY THEOREM FOR THE ALEXANDER POLYNOMIAL OF A PLANE ALGEBRAIC CURVE

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Abstract. An upper estimate for the Alexander polynomial of an algebraic curve is obtained, which sharpens Libgober's estimate in terms of the local polynomials at the singular points of the curve: only those singular points may contribute to the Alexander polynomial of the curve which are in excess of the hypothesis of Nori's vanishing theorem.

Introduction

Consider an irreducible algebraic curve $B \subset \mathbb{C}\mathrm{p}^2$ of degree n, and denote by π the fundamental group $\pi_1(\mathbb{C}\mathrm{p}^2 \smallsetminus B)$ of its complement. In general, studying this group is a difficult problem (see, e.g., Moishezon [Mo], Nori [N], Oka [O], and Zariski [Z] for a survey of the few known results and examples in this direction; some newer results can be found in Bartolo [B], Bartolo, Tokunaga [BT], Degtyarev [D3], Dimca [Di], Tokunaga [T], and Tono [To]). That is why O. Zariski [Z] suggested, as the first approximation to π, to study its Alexander polynomial $\Delta_B(t)$, which can be defined as follows: Denote by $K = [\pi, \pi]$ the commutant of π, and by $K' = [K, K]$, its second commutant, i.e., the subgroup generated by the commutators $[x_1, x_2]$ for all $x_1, x_2 \in K$. Then $\pi/K = \mathbb{Z}_n$ acts on the quotient K/K' (which is an abelian group of finite rank), and one can define $\Delta_B(t)$ to be the characteristic polynomial of the automorphism of K/K' corresponding to any generator of π/K.

It is well known that, if B is a non-singular curve, then $\pi = \mathbb{Z}_n$, and, hence, $\Delta_B(t) = 1$. Hence, one can expect that there is a relationship between the complexity of $\Delta_B(t)$ and singularities of B. In order to describe this influence, A. Libgober [L] introduced the notion of local Alexander polynomial $\Delta_{B|O}(t)$ of B at a singular point O, which, by definition, is just the ordinary Alexander polynomial of the link cut by B on the boundary of a small ball about O. The result of Libgober is the following:

Theorem (see Libgober [L]). $\Delta_B(t)$ *divides the product* $\prod \Delta_{B|O_i}(t)$ *over all the singular points* O_i *of* B.

Another result in this direction is due to Nori [N], who proved the following generalized version of the famous Zariski conjecture on nodal curves:

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Theorem (see Nori [N]). *Let B be a reduced ample divisor on a projective algebraic surface X. Consider an embedded resolution $\sigma\colon Y \to X$ of all the singular points of B other than nodes, and for each irreducible component B_i of B denote by $\widetilde{B}_i$ its proper transform in Y and by $\delta(\widetilde{B}_i)$ the number of nodes of $\widetilde{B}_i$. If σ can be chosen so that $\widetilde{B}_i^2 > 2\delta(\widetilde{B}_i)$ for all i, then the kernel of the inclusion homomorphism $\pi_1(X \smallsetminus B) \to \pi_1(X)$ is an abelian subgroup whose centralizer has finite index.*

In the classical case $X = \mathbb{C}\mathrm{p}^2$ Nori's theorem implies that, under the hypothesis, $\pi_1(\mathbb{C}\mathrm{p}^2 \smallsetminus B)$ is abelian and, in particular, $\Delta_B(t) = 1$.

The main result of this paper occupies an intermediate position between the above two theorems:

Main theorem. *Suppose that the set of the singular points of B is split into two subsets, S_+ and S_{exc}, so that there exists a resolution of the points in S_+ such that the proper transform of B has positive self-intersection. Then $\Delta_B(t)$ divides the product $\prod \Delta_{B|O_i}(t)$ over all the points $O_i \in S_{\mathrm{exc}}$*

Roughly speaking, this result means that only those singular points of B may affect $\Delta_B(t)$ (in the way of Libgober's theorem), which are in excess of the hypothesis of Nori's theorem. Moreover, one can estimate the multiplicities of different roots of $\Delta_B(t)$ separately, each time trying to gather in S_+ the singular points whose local Alexander polynomials vanish at the root in question.

Remark. Note that the local Alexander polynomial of a node is $t - 1$. Hence, nodes never contribute to the Alexander polynomial of an irreducible curve (see [Z]) and one can always keep them in S_{exc}.

Remark. In fact, we prove a slightly stronger result, dealing with an arbitrary ample divisor on an algebraic surface; see Theorem 4.1 for details.

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1. Alexander polynomial of an algebraic curve

The definition of Δ given in Introduction applies to any affine algebraic surface (like $\mathbb{C}\mathrm{p}^2 \smallsetminus B$) or, more generally, to any topological space Y with $H_1(Y) = \mathbb{Z}_n$ (or with a preferred homomorphism $H_1(Y) \to \mathbb{Z}_n$, which in the algebraic case can be given by a line bundle $E \in H^1(Y; \mathcal{O}_Y^*)$ with $E^{\otimes n} \cong \mathcal{O}_Y$). Then the (rational) Alexander polynomial is defined to be the characteristic polynomial of the deck translation action on $H_1(Y_n; \mathbb{Q})$, where Y_n is the cyclic covering of Y determined by the above $H_1(Y) \to \mathbb{Z}_n$. However, it is traditional for the subject, like in knot theory, to include into the definition the original curve B (which plays the rôle of the pereferical structure). Among other advantages, this assures the correct branching at infinity. (Another, more practical, reason for treating the Alexander polynomial as an invariant of the curve rather than its complement is that it is its relation to the singularities of the curve that is studied.) Another advantage of considering a pair (X, B) instead of the complement $Y = X \smallsetminus B$ is that it provides for a canonical compactification X' of Y_n, and, as shown in Libgober [L], if B is irreducible an reduced, the first cohomology of the desingularization of X' differ from that of Y_n by an easily controllable part with trivial deck translation action which comes from X. (In [L] one can also find a survey of various definition of the Alexander polynomial and relation between them.)

Thus, let $B \subset X$ be an algebraic curve in a projective algebraic surface X, and let $E \in H^1(X; \mathcal{O}_X^*)$ be a class such that $nE = [B]$ for some positive integer n. Consider some linear bundles L_B and L_E corresponding to $[B]$ and E respectively, fix an isomorphism $L_E^{\otimes n} \cong L_B$ and a section $s\colon X \to L_B$ with the zero-set B, and denote by $\widetilde{X}' \subset L_E$ the set of the n-th roots of s (i.e., locally $\widetilde{X}'$ consists of the points $(x, t) \in L_E$ such that $t^n = s(x)$), and by p', the restriction to $\widetilde{X}'$ of the bundle projection $L_E \to X$. The pair $(\widetilde{X}', p')$ is called an *n-fold covering of X branched over B,* and E is called the *class* of $\widetilde{X}'$.

Denote by tr the restriction to $\widetilde{X}'$ of the bundle automorphism $L_E \to L_E$, $(x, t) \mapsto (x, t \cdot \exp(2\pi i/n))$, and let $\rho\colon \widetilde{X} \to \widetilde{X}'$ be some fixed tr-invariant resolution of singularities of $\widetilde{X}'$. Then $H^1(\widetilde{X}; \mathbb{C})$ is a $\mathbb{C}[t]$-module, t acting via the induced automorphism tr^*. Let $p = p' \circ \rho$.

1.1. Definition. $H^1(\widetilde{X}; \mathbb{C})$ is called the (*reduced*) *Alexander module* of B, and the characteristic polynomial of tr^* on $H^1(\widetilde{X}; \mathbb{C})$ is denoted by $\widetilde{\Delta}_B(t)$ and is called the (*reduced*) *Alexander polynomial* of B.

Remarks. (1) It is shown in Hirzebruch [H] (see also [D1]) that any topological covering of X branched over B (with the given multiplicities of the components) can be given by the above construction, and there is a natural one-to-one correspondence between the isomorphism classes of such coverings and classes $E \in H^1(X; \mathcal{O}_X^*)$ such that $nE = [B]$.

(2) In Libgober [L] it is shown that $H^1(\widetilde{X}; \mathbb{C})$ does not depend on the choice of $\widetilde{X}$. If $X = \mathbb{C}\mathrm{p}^2$ and B is irreducible an reduced, then $\widetilde{\Delta}$ defined above coincides with the classical Alexander polynomial of $\pi_1(X \smallsetminus B)$.

Since tr is a finite order analytical automorphism, $H^1(\widetilde{X}; \mathbb{C})$ splits into direct sum of the eigenspaces of tr, and this splitting is compatible with the Hodge decomposition. Thus, one has

$$H^1(\widetilde{X}; \mathbb{C}) = \bigoplus \bigl(H_r^{1,0}(\widetilde{X}) \oplus H_r^{0,1}(\widetilde{X})\bigr),$$

where $H_r^{p,q}(\widetilde{X})$ is the eigenspace of $H^p(\widetilde{X}; \Omega^q(\widetilde{X}))$ corresponding to the eigenvalue $\exp(2\pi i r)$ of tr^*, $r \in \bigl\{-\frac{n-1}{n}, \dots, -\frac{1}{n}, 0\bigr\}$. Denote $h_r^{p,q} = h_r^{p,q}(\widetilde{X}) = \dim_{\mathbb{C}} H_r^{p,q}(\widetilde{X})$. Then from the Serre duality it follows that $h_r^{1,0} = h_r^{1,2}$ and

$h^{0,1}_r = h^{2,1}_r$. Put

$$h_r(\widetilde X) = h_r = \begin{cases} h^{1,2}_r, & r \in \bigl\{-\frac{n-1}{n},\dots,-\frac{1}{n},0\bigr\},\\ h^{1,2}_0 + h^{2,1}_0, & r = 0,\\ h^{2,1}_{-r}, & r \in \bigl\{0,\frac{1}{n},\dots,\frac{n-1}{n}\bigr\}.\end{cases}$$

(We assume $h_r = 0$ if $r \notin (-1,1)$ or $nr \notin \mathbb{Z}$.)

1.2. Definition. The set of rationals $\bigl\{r \in \mathbb{Q} \bigm| h_r \ne 0\bigr\}$, each counted with the multiplicity h_r, is called the *spectrum* of the pair (B,E) and is denoted $\operatorname{Spec}(B,E)$.

1.3. Proposition.

(1) $\operatorname{Spec}(B,E)$ *is symmetric in respect to 0;*
(2) $h_0 = \dim_{\mathbb{C}} H^1(X;\mathbb{C})$*;*
(3) *if* $r<0$*, then* $h_r = \dim_{\mathbb{C}}\bigl[H^1(X;p_*\omega_{\widetilde X})\bigr]_r$*, where* $[\,\cdot\,]_r$ *stands for the eigenspace of* tr^* *corresponding to the eigenvalue* $\exp(2\pi i r)$*;*
(4) *one has*

$$\widetilde\Delta_B(t) = \prod_{r\in\operatorname{Spec}(B,E)} (t-\exp(2\pi i r))^{h_r}.$$

Proof. (1) This statement follows from the fact that $H^{p,q}(\widetilde X)$ and $H^{q,p}(\widetilde X)$ are complex conjugate linear spaces.

(2) h_0 is the dimension of the tr-invariant part of $H^1(\widetilde X;\mathbb{C})$, which is isomorphic to $H^1(X;\mathbb{C})$.

(3) For $r<0$ one has $h_r = \dim_{\mathbb{C}}\bigl[H^1(\widetilde X;\omega_{\widetilde X})\bigr]_r$, and the statement follows from the fact that $R^i p_*\omega_{\widetilde X} = 0$ for $i>0$ (see Kollár [K]).

(4) This statement is obvious. $\square$

2. Local Alexander polynomials

Let C be a (germ of a) plain curve at an isolated singular point O. Pick a small ball $\mathcal{B}$ centered at O. It is well known (see, e.g., Milnor [M]), that the intersection $C\cap\partial\mathcal{B}$ is a link in $\partial\mathcal{B} = S^3$, and the isotopy type of $C\cap\partial\mathcal{B}$ does not depend of the choice of $\mathcal{B}$, provided that the latter is sufficiently small.

2.1. Definition. The Alexander polynomial of the link $C\cap\partial\mathcal{B}$ is called the *local Alexander polynomial* of C at O and is denoted $\Delta_{C|O}(t)$.

Remark. $\Delta_{C|O}(t)$ admits another description (see Milnor [M]): it is the characteristic polynomial of the monodromy action on the vanishing cohomology group.

Let $\sigma\colon \widetilde Y \to \mathbb{C}^2$ be a resolution of O such that $\sigma^{-1}C$ is a divisor with normal crossings. Denote by E_i, $i=1,\dots,k$, the reduced components of $\sigma^{-1}O$, and by m_i, their multiplicities in $\sigma^{-1}C$. Pick a rational $r\in[-1,0)$, and consider the sheaves $\mathcal{J}_r = \mathcal{J}_r(C|O) = \sigma_*\mathcal{K}(-\sum\lfloor(r+1)m_i\rfloor E_i)$ and

$\overline{\mathcal{J}}_r = \bigcap_{-1\leqslant r'<r} \mathcal{J}_{r'}$, where $\mathcal{K} = \omega_{\widetilde{Y}} \otimes (\sigma^*\omega_{\mathbb{C}^2})^{\otimes -1}$, and $\lfloor x \rfloor$ stands for the integral part of a rational x. Obviously, $\overline{\mathcal{J}}_r$ and $\mathcal{J}_r$ are sheaves of ideals on $\mathbb{C}^2$, and $\mathcal{J}_{r'} \subset \overline{\mathcal{J}}_{r'} \subset \mathcal{J}_r \subset \overline{\mathcal{J}}_r$ whenever $-1 \leqslant r < r' < 0$. Besides, $\mathcal{J}_{-1} = \mathcal{O}_{\mathbb{C}^2}$, and all the quotients $\overline{\mathcal{J}}_r/\mathcal{J}_r$ and $\mathcal{J}_{r'}/\mathcal{J}_r$, $r' \leqslant r$, are concentrated at O.

2.2. Proposition (see Varchenko [Var]). *For any* $r \in (-1, 0)$ *the integer* $h_r = h_r(C|O) = \dim_{\mathbb{C}}(\overline{\mathcal{J}}_r/\mathcal{J}_r)\big|_O$ *coincides with the multiplicity of* r *in the spectrum* $\operatorname{Spec}_O C$ *of* O. (*See Varchenko* [Var] *for the definition of* $\operatorname{Spec}_O C$.)

Remark. In [Var], this result is stated and proved in somewhat different terms. A "translation" to the language of this paper can be found in [D2].

Remark. In fact, the sheaves $\overline{\mathcal{J}}_r$ are only introduced to simplify the notation. Since the function $r \mapsto \mathcal{J}_r$ is, obviously, piecewise constant, one can replace $\overline{\mathcal{J}}_r$ with $\mathcal{J}_{r-\varepsilon}$, ε being sufficiently small. (If N is an integer divisible by all the m_i's, one can take any $\varepsilon < 1/N$.) Note also that $\mathcal{J}_r$ can as well be defined as $\mathcal{J}_r(C|O) = \sigma_*\mathcal{K}(\sum \lceil -(r+1)m_i \rceil E_i)$, where $\lceil x \rceil$ is the upper integral approximation of x.

2.3. Corollary. *The non-invariant cyclotomic part of* $\Delta_{C|O}(t)$ *is given by*

$$\prod (t - \exp(2\pi i r))^{h_r + h_{-(r+1)}},$$

r *running over all the rationals in* $(-1, 0)$ *with* $h_r \neq 0$. *In other words, the multiplicity in* $\Delta_{C|O}(t)$ *of a non-trivial root of unity* $\exp(2\pi i r)$, $r \in (-1, 0)$, *equals* $h_r + h_{-(r+1)}$.

Proof. The statement immediately follows from the symmetry of $\operatorname{Spec}_O C$ in respect to 0 and the following result of Varchenko [Var]: the cyclotomic part of $\Delta_{C|O}(t)$ is given by

$$\prod_{r \in \operatorname{Spec}_O B} (t - \exp(2\pi i r))$$

(where the elements of $\operatorname{Spec}_O C$ are counted with their multiplicities). □

3. The sheaves $p_*\omega_{\widetilde{X}}$.

Notation. Given a formal divisor $D = \sum r_i D_i$ with *rational* coefficients r_i, let $\lfloor D \rfloor = \sum \lfloor r_i \rfloor D_i$, $\lceil D \rceil = \sum \lceil r_i \rceil D_i$, and $D_{\text{red}} = \sum D_i$, where $\lfloor x \rfloor$ and $\lceil x \rceil$ stand, respectively, for the lower and upper integral approximations of a rational x.

Let $B \subset X$ be a divisor with isolated singularities, $E \in H^1(X; \mathcal{O}_X^*)$ a class such that $nE = [B]$, and $p'\colon \widetilde{X}' \to X$ an n-fold covering branched over B with the class E. In order to construct a desingularization $\widetilde{X}$ of $\widetilde{X}'$, consider an embedded resolution $\sigma\colon Y \to X$ of the singularities of B, let $q' = \sigma^* p'\colon \widetilde{Y}' \to Y$, pick an equivariant desingularization $\rho\colon \widetilde{Y} \to \widetilde{Y}'$, and put $\widetilde{X} = \widetilde{Y}$. Let $q = q' \circ \rho\colon \widetilde{Y} \to Y$ and $p = \sigma \circ q\colon \widetilde{Y} \to X$ be the projections.

From Proposition 1.3 it follows that $\widetilde{\Delta}_B(t)$ can be expressed in terms of the cohomology of the sheaves $\overline{\mathcal{S}}_r = \bigl[q_*\omega_{\widetilde{Y}}\bigr]_r$ or $\mathcal{S}_r = \bigl[p_*\omega_{\widetilde{Y}}\bigr]_r = \sigma_*\overline{\mathcal{S}}_r$. (As in Section 2, we denote by $[\,\cdot\,]_r$, $r = -1, -\frac{n-1}{n}, \ldots, -\frac{1}{n}$, the eigensheaf corresponding to the eigenvalue $\exp(2\pi i r)$ of tr^*.) The following result is proved in [D2]:

3.1. Proposition. *Let $\overline{B} = \sigma^* B$ be the full inverse image of B in Y. Then for any $r = -1, -\frac{n-1}{n}, \ldots, -\frac{1}{n}$ one has:*

1. $\overline{\mathcal{S}}_r = \omega_Y\bigl((r+1)nE - \lfloor (r+1)\overline{B}\rfloor\bigr)$*;*
2. $\mathcal{S}_r = \omega_X\bigl((r+1)nE\bigr) \otimes \bigotimes \mathcal{J}_r(B|O_i)$*, the product over all the singular points O_i of B;*
3. $R^i\sigma_*\overline{\mathcal{S}}_r = 0$ *for $i > 0$.*

Remark. Note that Statement (1) of Proposition 3.1 applies, in fact, to any divisor $\overline{B}$ in Y with normal crossings.

4. Proof of the main theorem

4.1. Theorem (generalization of the Main Theorem). *Let X be a projective algebraic surface, and B an ample irreducible divisor on X with only isolated singularities. Suppose that the set $\operatorname{Sing} B$ of the singular points of B is split into two disjoint subsets, S_+ and S_{exc}, and there is a resolution of all the points in S_+ such that the proper transform of B has positive self-intersection. Then $\widetilde{\Delta}_B(t)$ divides the product*

$$(t-1)^{b_1(X)} \prod_{O_i \in S_{\mathrm{exc}}} \Delta_{B|O_i}(t)$$

(*no matter what class $E \in H^1(X; \mathcal{O}_X^*)$ is used to define $\widetilde{\Delta}_B(t)$*)

Remark. As it was mentioned in Introduction, the nodes of B never contribute to the Alexander polynomial and, hence, one can always keep them in S_{exc}.

Proof. Let the spaces $Y, \widetilde{Y}', \widetilde{Y} = \widetilde{X}$ and maps σ, ρ, p, q be as in Section 3. (We suppose that the resolutions of the points in S_+ are the ones mentioned in the theorem.) First of all, note that the trivial part of the Alexander module $H^1(\widetilde{X}; \mathbb{C})$ is isomorphic to $H^1(X; \mathbb{C})$. This corresponds to the factor $(t-1)^{b_1(X)}$ in the above formula. Hence, one can confine oneself to considering the non-trivial roots of $\widetilde{\Delta}_B(t)$, i.e., those of the form $\exp(2\pi i r)$, $r = -\frac{n-1}{n}, \ldots, -\frac{1}{n}$, and, according to Proposition 1.3 and Corollary 2.3, the result would follow from the inequalities

$$h_r(\widetilde{Y}) = \dim_{\mathbb{C}}\bigl[H^1(X; p_*\omega_{\widetilde{X}})\bigr]_r \leqslant \sum_{O_i \in S_{\mathrm{exc}}} h_r(B|O_i).$$

We will use Viehweg's vanishing theorem, which, in the two-dimensional case, can be stated as follows:

4.2. Theorem (see Viehweg [V] or Miyaoka [Mi]). *Let Y be a projective surface (over $\mathbb{C}$), D a formal divisor on Y with rational coefficients, and $\mathcal{L}$ an invertible sheaf on Y. Suppose that the support of D is a divisor with normal crossings, $(c_1\mathcal{L} - [D])^2 > 0$, and $(c_1\mathcal{L} - [D]) \circ C \geqslant 0$ for any curve $C \subset Y$. Then $H^p(X; \mathcal{L} \otimes \omega_Y(-\lfloor D \rfloor)) = 0$ for any $p > 0$.*

Denote by $\overline{B}$ the proper transform of B in Y, and by F_i, the part of its full inverse image $\sigma^* B$ which lies over O_i. Let $\overline{B}' = \overline{B} + \sum_{O_i \in S_{\text{exc}}} F_i$. According to our assumption, $(\overline{B}')^2 > 0$. Prove that $\overline{B}' \circ C \geqslant 0$ for any curve $C \subset Y$. This is obviously true for any curve which is not a component of $\overline{B}'$ (since $\overline{B}'$ is an effective divisor). If C is a component of F_i (with $O_i \in S_{\text{exc}}$), then $\overline{B}' \circ C = 0$, since, at least, homologically, $\overline{B}'$ can be pushed out of the fiber $\sigma^{-1} O_i$. Finally, if $C = \overline{B}$, then $\overline{B}' \circ \overline{B} = \overline{B}' \circ (\overline{B}' - \sum F_i) = (\overline{B}')^2 > 0$.

Pick some $r \in \left\{-\frac{n-1}{n}, \dots, -\frac{1}{n}\right\}$. Since $nE - \sigma^* B$ is numerically equivalent to zero, the pair $(\mathcal{L}, D) = ((r+1)nE, (r+1)\sigma^* B - \varepsilon \overline{B}')$, $\varepsilon > 0$, satisfies the hypothesis of Theorem 4.2, and, hence, $H^1\big(Y; \omega_Y((r+1)nE - \lfloor D \rfloor)\big) = 0$. Now note that, if ε is sufficiently small, the sheaf $\overline{\mathcal{R}}_r = \omega_Y((r+1)nE - \lfloor D \rfloor)$ contains $\overline{\mathcal{S}}_r = \left[q_* \omega \widetilde{X}\right]_r$, and in the pull-back of a neighborhood of a singular point O_i of B it coincides with either $\overline{\mathcal{S}}_r$ (if $O_i \in S_+$), or $\overline{\mathcal{S}}_{r'}$ for some $r' < r$ (if $O_i \in S_{\text{exc}}$). In particular, this implies that

1. (1) $R^i \sigma_* \overline{\mathcal{R}}_r = 0$ for $i > 0$ (The statement is local in X, and Proposition 3.1(3) applies),
2. (2) $H^1(X; \sigma_* \overline{\mathcal{R}}_r) = 0$ (This follows from (1)), and
3. (3) the following sequence is exact

$$0 \to \left[p_* \omega_{\widetilde{Y}}\right]_r \to \sigma_* \overline{\mathcal{R}}_r \to \bigoplus_{O_i \in S_{\text{exc}}} \overline{\mathcal{J}}_r(B|O_i)/\mathcal{J}_r(B|O_i) \to 0.$$

Hence, the cohomology exact sequence

$$\bigoplus_{O_i \in S_{\text{exc}}} H^0\big(X; \overline{\mathcal{J}}_r(B|O_i)/\mathcal{J}_r(B|O_i)\big) \to H^1\big(X; \left[p_* \omega_{\widetilde{Y}}\right]_r\big) \to H^1\big(X; \sigma_* \overline{\mathcal{R}}_r\big)$$

yields

$$h_r(\widetilde{Y}) = \dim_{\mathbb{C}} H^1\big(X; \left[p_* \omega_{\widetilde{X}}\right]_r\big) \leqslant \sum_{O_i \in S_{\text{exc}}} \dim_{\mathbb{C}} H^0\big(X; \overline{\mathcal{J}}_r(B|O_i)/\mathcal{J}_r(B|O_i)\big) = \sum_{O_i \in S_{\text{exc}}} h_r(B|O_i),$$

and the theorem follows. □

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