THE ALEXANDER MODULE OF A TRIGONAL CURVE. II

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ABSTRACT. We complete the enumeration of the possible roots of the Alexander polynomial (both conventional and over finite fields) of a trigonal curve. The curves are not assumed proper or irreducible.

1. Introduction

This paper is a continuation of [4]: we complete the enumeration of the possible roots of the Alexander polynomial (both conventional and over finite fields) of a trigonal curve. Unlike [4], here we do not assume the curves irreducible, as this assumption does not lead to an essential simplification of the results.

An emphasis is given to the *exceptional* roots ξ, *i.e.*, those with the multiplicative order $N := \operatorname{ord}(-\xi) \geqslant 7$. Such roots are *not* controlled by congruence subgroups of the modular group.

Since the paper is a sequel, we only recall very briefly the necessary notions and preliminary results, concentrating on the explanation of the new approach that lets us improve the estimates found in [4]. For all details, further speculations, and references concerning the history of the subject and the previously known results on the Alexander module/polynomial of an algebraic curve in an algebraic surface, the reader is directed to [4] and [5].

1.1. **Principal results.** Let $C \subset \Sigma_d$ be a trigonal curve in a Hirzebruch surface, see §3.1 for the definitions, and consider the affine fundamental group

$$\pi^{\mathrm{aff}}(C) := \pi_1(\Sigma_d \smallsetminus (C \cup E \cup F_\infty)),$$

where F_∞ is a generic fiber of Σ_d. and E is the exceptional section. There is a natural epimorphism $\deg\colon \pi^{\mathrm{aff}}(C) \twoheadrightarrow \mathbb{Z}$ sending a meridian of a tubular neighborhood of C to $1 \in \mathbb{Z}$. The abelianization A_C of the kernel $\operatorname{Ker}\deg$ is called the *Alexander module* (or *Alexander invariant*) of C. This group is indeed a module over the ring $\Lambda := \mathbb{Z}[t, t^{-1}]$ of Laurent polynomials; the action of t is given by $[h] \mapsto [aha^{-1}]$, where $h \in \operatorname{Ker}\deg$ and $a \in \pi^{\mathrm{aff}}(C)$ is any element of degree 1. Alternatively, $\mathrm{A}_C = H_1(X)$ is the homology of the infinite cyclic covering $X \to \Sigma_d \smallsetminus (C \cup E \cup F_\infty)$ corresponding to deg, and the action of t is induced by the deck translation of the covering.

Denote $\Bbbk_0 := \mathbb{Q}$ and $\Bbbk_p := \mathbb{F}_p$ (the field with p elements) if p is a prime. Unless C is isotrivial, the product $\mathrm{A}_C \otimes \Bbbk_p$ is a torsion module over the principal ideal domain $\Lambda \otimes \Bbbk_p$; its order $\Delta_{C,p} \in \Lambda \otimes \Bbbk_p$ is called the $(\operatorname{mod} p)$-*Alexander polynomial* of C. We are interested in the roots of $\Delta_{C,p}$. More precisely, let ξ be an algebraic number over $\Bbbk_p$, denote by $\psi_\xi \in \Lambda \otimes \Bbbk_p$ its minimal polynomial, and consider the minimal

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Table 1. Exceptional factors of Δ, $N \geqslant 7$ (see Remark 1.2)

#	p	N	Factors $\psi_\xi \in \mathbb{F}_p[t]$ of $\Delta_{C,p}$	$\bar{G} \subset \Gamma$
*1	2	7	$t^3+t+1,\ \ t^3+t^2+1$	$(9;1,0;1^2 7^1)$
*2		15	$t^4+t+1,\ \ t^4+t^3+1$	$(17;1,2;1^2 15^1)$
*3	3	8	$t^2+2t+2,\ \ t^2+t+2$	$(10;0,1;1^2 8^1)$
*4	5	8	$t^2+2,\ \ t^2+3$	$(78;0,0;1^6 8^9)$
5		12	$t^2+2t+4,\ \ t^2+3t+4$	$(52;0,4;1^4 12^4)$
*6	11	10	$t+2;\ \ t+6;\ \ t+7;\ \ t+8$	$(24;2,0;1^2 2^1 10^2)$
*7	13	12	$t+2,\ \ t+7;\ \ t+6,\ \ t+11$	$(14;0,2;1^2 12^1)$
*8	17	8	$t+2,\ \ t+9;\ \ t+8,\ \ t+15$	$(36;0,0;1^4 8^4)$
9	19	9	$t+4,\ \ t+5;\ \ t+6,\ \ t+16;\ \ t+9,\ \ t+17$	$(20;0,2;1^2 9^2)$
10		18	$t+2;\ \ t+3;\ \ t+10;\ \ t+13;\ \ t+14;\ \ t+15$	$(40;2,4;1^2 2^1 18^2)$
*11	29	7	$t+7,\ \ t+25;\ \ t+16,\ \ t+20;\ \ t+23,\ \ t+24$	$(60;0,0;1^4 7^8)$
12	37	9	$t+7,\ \ t+16;\ \ t+9,\ \ t+33;\ \ t+12,\ \ t+34$	$(76;0,4;1^4 9^8)$
*13	43	7	$t+4,\ \ t+11;\ \ t+16,\ \ t+35;\ \ t+21,\ \ t+41$	$(132;0,0;1^6 7^{18})$

field $\bar{\Bbbk} := \Lambda(\xi) := (\Lambda \otimes \Bbbk_p)/\psi_\xi$ containing ξ. Then $\mathrm{A}_C(\xi) := (\mathrm{A}_C \otimes \Bbbk_p)/\psi_\xi$ is a $\bar{\Bbbk}$-vector space, and we are interested in the pairs (p, ξ) for which this space may have positive dimension, *i.e.*, ψ_ξ may appear as a factor of the Alexander polynomial $\Delta_{C,p}$ of a non-isotrivial trigonal curve. According to [4], the multiplicative order $N := \operatorname{ord}(-\xi)$ must be finite, and the principal result of the present paper is the following theorem, which is proved in §4, see §4.7.

Theorem 1.1. *Let p be a prime or zero, and assume that the* $(\bmod\, p)$*-Alexander polynomial of a non-isotrivial (see §3.1) trigonal curve C has a root $\xi \in \bar{\Bbbk} \supset \Bbbk_p$ such that $N := \operatorname{ord}(-\xi) \geqslant 7$. Then the pair (p, ψ_ξ) is one of those listed in Table 1. The pairs (p, ψ_ξ) marked with a * in the table do appear in the Alexander polynomials of* proper *trigonal curves; the others do not.*

Remark 1.2 (comments to Table 1)**.** Listed in the table are triples (p, N, ψ_ξ) and, for each triple, certain information about the skeleton (see §2.3) of the corresponding universal subgroup (see §3.5): $(e; v_\circ, v_\bullet; r)$, where e is the number of edges, $v_\circ$ and $v_\bullet$ are the numbers of monovalent $\circ$- and $\bullet$-vertices, respectively, and r is the set of the region widths in the partition notation. Note that these data do not determine the skeleton uniquely: in fact, the polynomials ψ_ξ with isomorphic skeletons are separated by commas rather than semicolons in the table.

Remark 1.3. The case $N > 10$ is settled in [4], where the rest of Theorem 1.1 (the range $7 \leqslant N \leqslant 10$) was conjectured. All triples (p, N, ψ_ξ) with $N \leqslant 6$ are also enumerated in [4], see §3.6 for a few further details.

Addendum 1.4. *In all cases listed in Table 1, the module $\mathrm{A}_C(\xi)$ has a geometric presentation of the form $\mathrm{A}(\xi)/\bar{\Bbbk}\mathbf{e}_1$, see §3.4 and §2.2 for the definition of A and $\mathbf{e}_i$; in particular, $\dim_{\bar{\Bbbk}} \mathrm{A}_C(\xi) = 1$. Furthermore, at most one factor as in the table may appear in the Alexander polynomial of any particular curve.*

This statement is proved in §4.8.

1.2. **Open problems.** The approach chosen here and in [4], namely, the study of the specializations $\mathrm{A}_C(\xi)$, simplifies the original question about the structure of the Λ-module A_C. In particular, we ignore the higher torsion of the form $\mathbb{Z}_{p^r}$ and $(\Lambda \otimes \mathbb{F}_p)/\psi^r$, $p > 0$. In general, it is clear that A_C is annihilated by a polynomial of the form $t^n - 1$, and it appears reasonable to study the quotients $A_C/\Phi_N(-t)$, where Φ_N is the cyclotomic polynomial of order N. (Still, since Λ is not a principal ideal domain, extra work is needed to recover A_C from these quotients.) At present, the structure of the $\Lambda/\Phi_N(-t)$-module $A_C/\Phi_N(-t)$ is known for $N = 1$, see [5], and for $2 \leqslant N \leqslant 6$, see §3.6. If $N \geqslant 7$, using Theorem 1.1 and Addendum 1.4, one can only state that $A_C/\Phi_N(-t)$ is generated over Λ by a single element and that it is a finite abelian p-group for some value of prime p (compatible with that of N) as in Table 1.

Examples of modules A_C with higher torsion $\mathbb{Z}_{p^r}$ or $(\Lambda \otimes \mathbb{F}_p)/\psi^r$ are known, see [5] and §3.6. However, in all these examples one has $N \leqslant 6$; the case $N \geqslant 7$ has never been studied from this point of view.

Another open question is the realizability of all modules discovered by trigonal curves. Our proof of Theorem 1.1 is in the group-theoretical settings, and each pair (p, ψ_ξ) listed in Table 1 *is* realized by a certain genus zero subgroup G of the Burau group Bu_3. If $G \subset \mathbb{B}_3$, it is in turn realized as the monodromy group of a proper trigonal curve due to Theorem 3.5; the other cases, leading to improper curves, are to be the subject of a further investigation.

1.3. **Contents of the paper.** The first two sections contain preliminary material: we site a few notions and results needed in the proof of the main theorem. In §2, we introduce the braid group $\mathbb{B}_3$ and its extension $\mathbb{B}_3 \cdot \operatorname{Inn} \mathfrak{F}$ and recall the reduced Burau representation (see §2.2) and its further specialization to the modular group $\Gamma := PSL(2, \mathbb{Z})$. Crucial for the sequel is the description of subgroups of Γ in terms of *skeletons* (certain bipartite ribbon graphs, see §2.3) and the description of their lifts to $\mathbb{B}_3$ or Bu_3 by means of the *type specification* (see §2.4). In §3, we discuss trigonal curves in Hirzebruch surfaces (see §3.1), the braid monodromy and the monodromy group of such a curve (see §3.2), and the Zariski–van Kampen theorem (see §3.3). Then we introduce the conventional and extended Alexander module of a trigonal curve, expressing them in terms of the monodromy group (see §3.4) and introduce the universal subgroup corresponding to a given Alexander module (see §3.5). For completeness, we extend to all, not necessarily proper or irreducible, trigonal curves the results of [4] concerning the roots ξ of the Alexander polynomial with $N := \operatorname{ord}(-\xi) \leqslant 6$ (see §3.6).

Theorem 1.1 is proved in §4. We recall the computation of the local modules and the estimate $N \leqslant 26$ found in [4] (see §4.2), show that $\dim_{\bar{\Bbbk}} A_C(\xi) \leqslant 1$ whenever $N \geqslant 7$ (see §4.3), and engage into improving the above estimate to $N \leqslant 6$ with the exception of finitely many explicitly listed cases (see §4.5): the strategy consists in showing, by computing a number of resultants, that the assumption $N \geqslant 7$ implies that a certain ribbon graph has too many too large regions and thus cannot be planar. The exceptional cases are eliminated by the explicit computation of the corresponding universal groups, which are all finite (see §4.6), and this concludes the proof of Theorem 3.5, which restates Theorem 1.1 in group-theoretical terms. The computation in §4.5 and §4.6 is heavily computer aided; it was done using `Maple`, and we only outline the approach. The formal reduction of Theorem 1.1 to Theorem 3.5 is explained in §4.7, and Addendum 1.4 is proved in §4.8.

As usual, ends of proofs are marked with $\square$. Statements whose proofs are omitted are marked with either $\lhd$ or $\rhd$. In the former case, the proof is either trivial or already explained; in the latter case, the reader is directed to the literature, which is usually referred to in the header of the statement.

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2. The groups $\mathbb{B}_3$, Bu_3, and Γ

In this section, se recall briefly a few necessary fact concerning the braid group $\mathbb{B}_3$, the modular group $\Gamma := PSL(2,\mathbb{Z})$, and their relation to bipartite ribbon graphs. A detailed treatment of the latter subject is found in [9]; here, we merely recall the basic definitions and explain briefly the geometric insight. For all proofs, the reader is also referred to [9].

2.1. **The braid group $\mathbb{B}_3$.** Artin's *braid group* $\mathbb{B}_3$ on three strands is the group

$$\mathbb{B}_3 := \langle \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

The generators σ_1, σ_2 above are called *Artin generators*. There is an epimorphism $\mathrm{dg}\colon \mathbb{B}_3 \twoheadrightarrow \mathbb{Z}$, $\sigma_1, \sigma_2 \mapsto 1 \in \mathbb{Z}$, called the *degree*. Furthermore, there is a canonical faithful representation $\mathbb{B}_3 \to \operatorname{Aut}\langle \alpha_1, \alpha_2, \alpha_3 \rangle$, the Artin generators acting *via*

$$\sigma_1\colon \alpha_1 \mapsto \alpha_1\alpha_2\alpha_1^{-1}, \quad \alpha_2 \mapsto \alpha_1; \qquad \sigma_2\colon \alpha_2 \mapsto \alpha_2\alpha_3\alpha_2^{-1}, \quad \alpha_3 \mapsto \alpha_2. \tag{2.1}$$

According to Artin [1], $\mathbb{B}_3$ can be identified with the subgroup of $\operatorname{Aut}\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ consisting of the automorphisms taking each generator to a conjugate of a generator and preserving the product $\rho := \alpha_1\alpha_2\alpha_3$.

Throughout the paper, we reserve the notation $\mathfrak{F}$ for the free group $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ equipped with a distinguished $\mathbb{B}_3$-orbit of bases, called *geometric*. (Note that each particular geometric basis gives rise to its own pair σ_1, σ_2 of Artin generators of $\mathbb{B}_3$.) The element $\rho := \alpha_1\alpha_2\alpha_3 \in \mathfrak{F}$ does not depend on the choice of a geometric basis. The center of $\mathbb{B}_3$ is the infinite cyclic group generated by Δ^2, where $\Delta := \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ is the *Garside element*; one has $\Delta^2(\alpha) = \rho\alpha\rho^{-1}$ for any $\alpha \in \mathfrak{F}$. The Garside element Δ depends on the choice of a geometric basis, whereas Δ^2 does not.

There is a well defined *degree epimorphism* $\deg\colon \mathfrak{F} \twoheadrightarrow \mathbb{Z}$ taking each geometric generator to $1 \in \mathbb{Z}$; it is preserved by $\mathbb{B}_3$.

With improper trigonal curves in mind, we will consider a larger group $\mathbb{B}_3 \cdot \operatorname{Inn}\mathfrak{F}$, where $\operatorname{Inn}\mathfrak{F} \subset \operatorname{Aut}\mathfrak{F}$ is the subgroup of inner automorphisms. Since $\operatorname{Inn}\mathfrak{F}$ is normal in $\operatorname{Aut}\mathfrak{F}$, the product is indeed a subgroup. One has $\mathbb{B}_3 \cap \operatorname{Inn}\mathfrak{F} = \langle \Delta^2 \rangle$; hence, the map $\mathrm{dg}\colon \mathbb{B}_3 \twoheadrightarrow \mathbb{Z}$ extends to $\mathbb{B}_3 \cdot \operatorname{Inn}\mathfrak{F}$ *via* $\mathrm{dg}(\beta\alpha) = \mathrm{dg}\,\beta + 2\deg\alpha$, where $\beta \in \mathbb{B}_3$ and $\alpha \in \operatorname{Inn}\mathfrak{F} \cong \mathfrak{F}$. (We consider the left adjoint action $\alpha(\alpha') = \alpha\alpha'\alpha^{-1}$; observe that $\mathrm{dg}\,\Delta^2 = 6 = 2\deg\rho$.)

2.2. **The Burau representation.** The *universal Alexander module* is the abelianization A of the kernel $\operatorname{Ker}[\deg\colon \mathfrak{F} \twoheadrightarrow \mathbb{Z}]$. It is a module over the ring $\Lambda := \mathbb{Z}[t, t^{-1}]$ of integral Laurent polynomials, t acting *via* $t[h] = [\alpha h \alpha^{-1}]$, where $h \in \operatorname{Ker}\deg$ and $\alpha \in \mathfrak{F}$ is any element of degree 1. A simple computation using the Reidemeister–Schreier method shows that $\mathrm{A} = \Lambda\mathbf{e}_1 \oplus \Lambda\mathbf{e}_2$, where $\mathbf{e}_i := [\alpha_{i+1}\alpha_i^{-1}]$, $i = 1, 2$; these generators depend on the choice of a geometric basis.

Since the action of $\mathbb{B}_3 \cdot \operatorname{Inn}\mathfrak{F}$ preserves the degree, it descends to an action on A, giving rise to a representation $\mathbb{B}_3 \cdot \operatorname{Inn}\mathfrak{F} \to GL(2,\Lambda)$. The Artin generators σ_1, σ_2 corresponding to the chosen geometric basis (the one used to define $\mathbf{e}_1$, $\mathbf{e}_2$) act *via*

$$\sigma_1 = \begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 1 & 0 \\ t & -t \end{bmatrix},$$

and an element $\alpha \in \operatorname{Inn}\mathfrak{F} \cong \mathfrak{F}$ maps to $t^{\deg\alpha}\operatorname{id}$. For the image $\bar\beta$ of $\beta \in \mathbb{B}_3 \cdot \operatorname{Inn}\mathfrak{F}$, one has $\det\bar\beta = t^{\operatorname{dg}\beta}$. The restriction $\mathbb{B}_3 \to GL(2,\Lambda)$ is called the (*reduced*) *Burau representation*, see [2]; it is faithful, and for this reason we identify braids and their images in $GL(2,\Lambda)$. The image in $GL(2,\Lambda)$ of the whole group $\mathbb{B}_3 \cdot \operatorname{Inn}\mathfrak{F}$ is called the *Burau group* Bu_3; it is the central product

$$\mathrm{Bu}_3 = \mathbb{B}_3 \odot \langle t\operatorname{id}\rangle = \mathbb{B}_3 \times \langle t\operatorname{id}\rangle / \{\Delta^2 = t^3\operatorname{id}\}.$$

The center of Bu_3 is the infinite cyclic subgroup generated by the scalar matrix $t\operatorname{id}$.

Given two submodules $\mathcal{U}, \mathcal{V} \subset \mathrm{A}$, we say that $\mathcal{U}$ is *conjugate* to $\mathcal{V}$, $\mathcal{U} \sim \mathcal{V}$ (*subconjugate* to $\mathcal{V}$, $\mathcal{U} \prec \mathcal{V}$) is $\mathcal{U} = \beta\mathcal{V}$ (respectively, $\mathcal{U} \subset \beta\mathcal{V}$) for some $\beta \in \mathbb{B}_3$. In this definition, $\mathbb{B}_3$ can be replaced with the Burau group Bu_3. Similar terminology is used for subgroups of $\mathbb{B}_3$ and Bu_3: a subgroup $H \subset \mathrm{Bu}_3$ is said to be *conjugate* (*subconjugate*) to a subgroup $G \subset \mathrm{Bu}_3$ if $H = \beta G \beta^{-1}$ (respectively, $H \subset \beta G \beta^{-1}$) for some element $\beta \in \mathbb{B}_3$.

Specializing all matrices at $t = -1$, we obtain epimorphisms $\mathbb{B}_3 \subset \mathrm{Bu}_3 \twoheadrightarrow \tilde\Gamma := SL(2,\mathbb{Z})$, which factor further to the *modular representation*

$$\operatorname{pr}_\Gamma \colon \mathbb{B}_3, \mathrm{Bu}_3 \twoheadrightarrow \Gamma := PSL(2,\mathbb{Z}) = \tilde\Gamma/\pm\operatorname{id}.$$

We abbreviate $\bar H := \operatorname{pr}_\Gamma H$ and $\bar\beta := \operatorname{pr}_\Gamma \beta$ for a subgroup $H \subset \mathrm{Bu}_3$ and an element $\beta \in \mathrm{Bu}_3$. The degree homomorphisms $\operatorname{dg}\colon \mathbb{B}_3 \to \mathbb{Z}$ and $\operatorname{dg}\colon \mathrm{Bu}_3 \to \mathbb{Z}$ descend to epimorphisms $\operatorname{dg}\colon \Gamma \twoheadrightarrow \mathbb{Z}_6$ and $\operatorname{dg} \bmod 2\colon \Gamma \twoheadrightarrow \mathbb{Z}_2$, respectively; the former coincides with the abelianization epimorphism $\Gamma \twoheadrightarrow \Gamma/[\Gamma,\Gamma] \cong \mathbb{Z}_6$. The kernels of the projections $\operatorname{pr}_\Gamma \colon \mathbb{B}_3, \mathrm{Bu}_3 \twoheadrightarrow \Gamma$ are the centers of the corresponding groups.

2.3. **Skeletons.** Recall that a *bipartite graph* is a graph whose vertices are divided into two kinds, $\bullet$- and $\circ$-, so that the two ends of each edge are of the opposite kinds. A *ribbon graph* is a graph equipped with a distinguished cyclic order (*i.q.* transitive $\mathbb{Z}$-action) on the star of each vertex. Any graph embedded into an oriented surface S is a ribbon graph, with the cyclic order induced from the orientation of S. Conversely, any finite ribbon graph defines a unique, up to homeomorphism, closed oriented surface S into which it is embedded: the star of each vertex is embedded into a small oriented disk (it is this step where the cyclic order is used), these disks are connected by oriented ribbons along edges producing a tubular neighborhood of the graph, and finally each boundary component of the resulting compact surface is patched with a disk. (Intuitively, the boundary components patched at the last step are the *regions* reintroduced combinatorially in Definition 2.2 below, where the construction of S is discussed in more details.) The surface S thus constructed is called the *minimal supporting surface* of the ribbon graph.

Below, we redefine a certain class of bipartite ribbon graphs in purely combinatorial terms, relating them to the modular group and its subgroups. In spite of this combinatorial approach, we will freely use the topological language applicable to the geometric realizations of the graphs.

As is well known, the *modular group* $\Gamma := PSL(2,\mathbb{Z})$ is generated by two elements $\mathbb{X} := (\bar\sigma_2\bar\sigma_1)^{-1}$ and $\mathbb{Y} := \bar\sigma_2\bar\sigma_1^2$, the defining relations being $\mathbb{X}^3 = \mathbb{Y}^2 = 1$; thus,

$\Gamma \cong \mathbb{Z}_3 * \mathbb{Z}_2$. According to [8] (see also [10], where this construction appeared first), a subgroup $G \subset \Gamma$ can be described by its *skeleton* $\operatorname{Sk} := \operatorname{Sk}_G$, which is the bipartite ribbon graph, possibly infinite, defined as follows: the set of edges of Sk is the left Γ-set Γ/G, its $\bullet$- and $\circ$-vertices are the orbits of $\mathbb{X}$ and $\mathbb{Y}$, respectively, and the cyclic order (the ribbon graph structure) at each trivalent $\bullet$-vertex is given by the action of $\mathbb{X}^{-1}$. The incidence map assigns to an edge e its $\bullet$- and $\circ$-ends by sending e to, respectively, the $\mathbb{X}$- and $\mathbb{Y}$-orbits containing e. Note that a cyclic order at each mono- or bivalent vertex is unique and hence redundant; however, it is convenient to agree that the cyclic order at *all* $\bullet$-vertices is given by $\mathbb{X}^{-1}$, and that at $\circ$-vertices is given by $\mathbb{Y}$. The graph Sk_G is finite if and only if G is a subgroup of finite index.

By definition, Sk is a connected bipartite ribbon graph, possibly infinite, with the following properties:

- the valency of each $\bullet$-vertex is 3 or 1;
- the valency of each $\circ$-vertex is 2 or 1.

Such a graph is called an (*abstract*) *skeleton*; its set of edges can be regarded as a transitive left Γ-set, with the action of $\mathbb{X}^{-1}$ and $\mathbb{Y}$ given by the cyclic order at the $\bullet$- and $\circ$-vertices, respectively. A skeleton is *regular* if it has no monovalent vertices.

The skeleton Sk_G of a subgroup $G \subset \Gamma$ is equipped with a distinguished edge e, namely the coset G/G. Conversely, any pair (Sk, e), where Sk is a skeleton and e is a distinguished edge, gives rise to a subgroup $G := \operatorname{stab} e \subset \Gamma$. If no edge is distinguished, Sk defines a conjugacy class of subgroups of Γ, which is denoted by Stab Sk.

Topologically, we regard a skeleton Sk as an orbifold, assigning to each monovalent $\bullet$- or $\circ$-vertex the ramification index 3 or 2, respectively. Under this convention, the homotopy classes of paths in Sk (starting and ending inside an edge) can be identified with pairs (e_0, g), where the *initial point* e_0 is an edge of Sk and $g \in \Gamma$; then the *terminal point* is the edge $e_1 := g^{-1}e_0$. Hence, we have an isomorphism

$$G = \operatorname{stab} e = \pi_1^{\mathrm{orb}}(\operatorname{Sk}_G, e),$$

where the basepoint for the fundamental group is chosen inside the edge e.

Definition 2.2. A *region* in a skeleton Sk is an orbit of $\mathbb{X}\mathbb{Y}$. The cardinality of a region R is called its *width* and denoted by $\operatorname{wd} R$. A region R of width n is also referred to as an *n-gon* or *n-gonal region*. (The $\bullet$-ends of the edges constituting R can be regarded as its corners.) The region containing an edge e is denoted by $(\!(e)\!)$.

Let Sk be a finite skeleton. Patching each region of Sk with an oriented disk, one obtains the *minimal supporting surface* Supp Sk: it is an oriented closed surface containing Sk and inducing its ribbon graph structure. (More precisely, the boundary of the disk patching a region $R = \{e_0, e_1, \ldots\}$ is composed by $e_0, e_0', e_1, e_1', \ldots$, where $e_i' := \mathbb{Y}e_i$; the edges e_i' appear in the boundary with the opposite orientation.) The genus g of Supp Sk is called the *genus* of Sk. If $\operatorname{Sk} = \operatorname{Sk}_G$ for a finite index subgroup $G \subset \Gamma$, then g is also called the *genus* of G; this definition is equivalent to the conventional one in terms of modular curves, see [8].

The skeleton Sk_G and genus of a finite index subgroup $G \subset \mathrm{Bu}_3$ (or $G \subset \mathbb{B}_3$) are defined as those of the image $\bar{G} \subset \Gamma$. Since an inclusion of subgroups gives rise to a ramified covering of the minimal supporting surfaces of their skeletons, one has

$$\operatorname{genus}(H) \geqslant \operatorname{genus}(G) \quad \text{whenever} \quad H \prec G. \tag{2.3}$$

2.4. **The type specification.** Define the *depth* $\operatorname{dp} G$ of a subgroup $G \subset \mathrm{Bu}_3$ as the degree of the minimal positive generator of the cyclic subgroup $G \cap \operatorname{Ker} \operatorname{pr}_\Gamma$, or zero if the latter intersection is trivial. Clearly, $\operatorname{dp} G = 0 \bmod 2$ for any subgroup $G \subset \mathrm{Bu}_3$, and $\operatorname{dp} G = 0 \bmod 6$ if $G \subset \mathbb{B}_3$.

Our primary concern are subgroups of genus zero. Let $G \subset \mathrm{Bu}_3$ be such a subgroup, and denote by $S^\sharp := \operatorname{Supp}^\sharp G$ the punctured surface obtained from the sphere $\operatorname{Supp} \operatorname{Sk}_G$ by removing the center of each region of Sk_G and each monovalent vertex of Sk_G. Then there is an epimorphism

$$\pi_1(S^\sharp, e) \twoheadrightarrow G/\langle t^k \operatorname{id}\rangle, \quad 2k := \operatorname{dp} G, \tag{2.4}$$

which is included into the commutative diagram

$$\begin{array}{ccc} \pi_1(S^\sharp, e) & \longrightarrow & G/\langle t^k \operatorname{id}\rangle \\ \downarrow & & \cong \downarrow \operatorname{pr}_\Gamma \\ \pi_1^{\mathrm{orb}}(\operatorname{Sk}_G, e) & \xrightarrow{\cong} & \bar{G}. \end{array}$$

As above, the basepoint for all fundamental groups is chosen inside the distinguished edge e of Sk_G.

Since $S^\sharp$ is a punctured sphere, the group $\pi_1(S^\sharp, e)$, and hence also the quotient $G/\langle t^k \operatorname{id}\rangle$, is generated by (the images of) a system of lassoes in $S^\sharp$ about the centers of the regions of Sk_G and its monovalent vertices. It follows that the subgroup $G/\langle t^k \operatorname{id}\rangle \subset \mathrm{Bu}_3/\langle t^k \operatorname{id}\rangle$ can be described by means of its *type specification* tp, which is a function on the set of regions and monovalent vertices of Sk_G, taking values in $\mathbb{Z}_{\operatorname{dp} G}$ (with the convention that $\mathbb{Z}_0 = \mathbb{Z}$) and defined as follows: the value of tp on a monovalent vertex or a region is the degree of the lift to $G/\langle t^k \operatorname{id}\rangle$ of the corresponding lasso about the vertex or the center of the region, respectively. This function is well defined and has the following properties.

Proposition 2.5 (see [4])**.** *Let $d = 6$ if $G \subset \mathbb{B}_3$ and $d = 2$ otherwise. Then*:

1. $\operatorname{dp} G = 0 \bmod d$;
2. $\operatorname{tp}(R) = \operatorname{wd} R \bmod d$ *for any region* R;
3. $\operatorname{tp}(\bullet) = 2 \bmod d$ *and* $3\operatorname{tp}(\bullet) = 0$;
4. $\operatorname{tp}(\circ) = 3 \bmod d$ *and* $2\operatorname{tp}(\circ) = 0$;
5. *the sum of all values of* tp *equals zero.*

Given a skeleton Sk*, a pair* (dp, tp) *satisfying conditions (1)–(5) above defines a unique subgroup* $G \subset \mathrm{Bu}_3$; *one has* $G \subset \mathbb{B}_3$ *if and only if the pair* (dp, tp) *satisfies conditions (1)–(4) with* $d = 6$. ▷

3. Trigonal curves

3.1. **Trigonal curves in Hirzebruch surfaces.** A *Hirzebruch surface* is a geometrically ruled rational surface $\pi\colon \Sigma_d \to B \cong \mathbb{P}^1$ with an exceptional section E of self-intersection $-d \leqslant 0$. If $d > 0$, such a section is unique. A (*generalized*) *trigonal curve* is a reduced curve $C \subset \Sigma_d$, not containing E or a fiber of π as a component, and such that the restriction $\pi\colon C \to B$ is a map of degree three. A trigonal curve is *genuine* or *proper* if it is disjoint from the exceptional section E. A *singular fiber* of a trigonal curve C is a fiber of π intersecting $C \cup E$ at fewer that four points.

A *positive* (*negative*) *Nagata transformation* is a birational map $\Sigma_d \dashrightarrow \Sigma_{d\pm 1}$ consisting in blowing up a point P in (respectively, not in) the exceptional section E and blowing down the proper transform of the fiber through P. A *d-fold Nagata transformation* is a sequence of d Nagata transformations in the same fiber and *of the same sign*. Two trigonal curves are *Nagata equivalent* (*d-Nagata equivalent*) if they can be related by a sequence of Nagata transformations (respectively, d-fold Nagata transformations).

By a sequence of positive Nagata transformations, any trigonal curve C can be made proper; the result is called a *proper model* of C.

In appropriate affine coordinates (x, y) in Σ_d such that $E = \{y = \infty\}$, a proper trigonal curve C can be given by its *Weierstraß equation*

$$y^3 + g_2(x)y + g_3(x) = 0, \tag{3.1}$$

where g_2, g_3 are certain polynomials in x. The (*functional*) *j-invariant* of C is the meromorphic function $j_C \colon B \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ given by

$$j_C(x) = -\frac{4g_2^3}{\Delta}, \qquad \text{where} \quad \Delta := -4g_2^3 - 27g_3^2$$

is the discriminant of (3.1) with respect to y. (We use Kodaira's normalization, with respect to which the 'special' values of the j-invariant are 0, 1, and ∞.) By definition, j_C is preserved by Nagata transformations, and the j-invariant of an improper trigonal curve is defined as that of any of its proper models. A curve C is called *isotrivial* if $j_C = \text{const}$.

3.2. **The monodromy group.** In this subsection, we outline the construction and basic properties of the braid monodromy of a trigonal curve. For more details and all proofs, which are omitted here, we refer to [6] and [7].

Let $C \subset \Sigma_d \to B$ be a proper trigonal curve. A *monodromy domain* is a closed topological disk $\Omega \subset B$ containing in its interior all singular fibers of C. A continuous section $s \colon \Omega \to \Sigma_d$ of π is called *proper* if its image is disjoint from both E and the fiberwise convex hull of C (with respect to the canonical affine structure in the *affine fibers* $F_b^\circ := \pi^{-1}(b) \smallsetminus E$, $b \in B$, which are affine spaces over $\mathbb{C}$). Since Ω is contractible, a proper section exists and is unique up to homotopy in the class of such sections.

Fix a monodromy domain Ω and a proper section s over Ω. Let $b_1, \ldots, b_r \in \Omega$ be the singular fibers of C, and denote $\Omega^\sharp := \Omega \smallsetminus \{b_1, \ldots, b_r\}$. Then, s is a section of the locally trivial fibration $\pi \colon \pi^{-1}(\Omega^\sharp) \smallsetminus (C \cup E) \to \Omega^\sharp$, and the monodromy of the associated bundle with the discrete fibers $\operatorname{Aut} \pi_1(F_b^\circ \smallsetminus C, s(b))$, $b \in \Omega^\sharp$, gives rise to an anti-homomorphism $\mathfrak{m} \colon \pi_1(\Omega^\sharp, b) \to \operatorname{Aut} \pi_F$, where $\pi_F := \pi_1(F_b^\circ \smallsetminus C, s(b))$, $b \in \Omega^\sharp$, is the fundamental group of a fixed nonsingular affine fiber punctured at C. The latter anti-homomorphism is called the *braid monodromy* of C, and its image $\mathfrak{Im}_C := \operatorname{Im} \mathfrak{m} \subset \operatorname{Aut} \pi_F$ is called the *monodromy group* of C.

The free group π_F has a distinguished class of geometric bases; a choice of one of these bases identifies π_F with $\mathfrak{F}$. (In fact, if $j_C(b) \neq 0, 1$, then π_F has a *canonical basis* $\{\alpha_1, \alpha_2, \alpha_3\}$, which is well defined up to conjugation by $\rho := \alpha_1\alpha_2\alpha_3$.) Under this identification, the monodromy $\mathfrak{m}$ takes values in the braid group $\mathbb{B}_3 \subset \operatorname{Aut} \mathfrak{F}$ and, up to conjugation in $\mathbb{B}_3$, the monodromy group $\mathfrak{Im}_C$ is independent of the choices made in the construction.

The following statement is crucial for Theorem 1.1.

Theorem 3.2 (see [5]). *The monodromy group of a non-isotrivial proper trigonal curve is of genus zero. Conversely, given a subgroup $G \subset \mathbb{B}_3$ of genus zero and depth $6d > 0$, there is a unique, up to isomorphism and d-Nagata equivalence, proper trigonal curve C_G such that, for another non-isotrivial proper trigonal curve C, one has $\mathfrak{Im}_C \prec G$ if and only if C is d-Nagata equivalent to a curve induced from C_G. This curve C_G is called the* universal curve *corresponding to G.* ▷

Now, let C be an improper trigonal curve. Consider a proper model C' of C and, after making the necessary choices, its braid monodromy $\mathfrak{m}' \colon \pi_1(\Omega^\sharp, b) \to \operatorname{Aut}\pi_F$. Let $\{\gamma_1, \ldots, \gamma_r\}$ be a geometric basis for the free group $\pi_1(\Omega^\sharp, b)$. To each basis element γ_j one can assign the *slope* $\varkappa_j \in \pi_F$, which depends on both curves C, C' and the generator γ_j. In this notation, the *braid monodromy* of C is defined as the anti-homomorphism $\mathfrak{m} \colon \gamma_j \mapsto \mathfrak{m}_j$, $j = 1, \ldots, r$, where $\mathfrak{m}_j$ is the automorphism $\alpha \mapsto \varkappa_j^{-1}\mathfrak{m}(\gamma_j)\varkappa_j$, $\alpha \in \pi_F$. The image $\mathfrak{Im}_C := \operatorname{Im}\mathfrak{m}$ is called the *monodromy group* of C; under the identification $\pi_F = \mathfrak{F}$ it is a subgroup of $\mathbb{B}_3 \cdot \operatorname{Inn}\mathfrak{F}$.

3.3. **The Zariski–van Kampen theorem.** The following theorem is the most well-known means of computing the fundamental group of the complement of an algebraic curve. It is essentially contained in [11]. There is a great deal of modifications and generalizations of this theorem making use of various pencils; the particular case of improper trigonal curves is treated in details in [7].

Theorem 3.3 (see [7]). *Let $C \subset \Sigma_d$ be a trigonal curve, and let $\mathfrak{Im}_C \subset \mathbb{B}_3 \cdot \operatorname{Inn}\mathfrak{F}$ be its monodromy group. Then one has a presentation*

$$\pi^{\mathrm{aff}}(C) = \mathfrak{F}/\{\alpha = \beta(\alpha),\ \alpha \in \mathfrak{F},\ \beta \in \mathfrak{Im}_C\}. \qquad \triangleright$$

A presentation of the group $\pi^{\mathrm{aff}}(C)$ as in Theorem 3.3 is called *geometric.*

3.4. **The Alexander modules.** Given a subgroup $G \subset \mathbb{B}_3 \cdot \operatorname{Inn}\mathfrak{F}$, let

$$\bar{\mathcal{V}}_G := \sum_{\beta \in G} \operatorname{Im}(\beta - \mathrm{id}) \subset \mathrm{A}, \qquad \mathcal{V}_G := \sum_{\beta \in G,\ \alpha \in \mathfrak{F}} \Lambda[\beta(\alpha) \cdot \alpha^{-1}] \subset \mathrm{A}$$

and define the *Alexander module* $\mathrm{A}_G := \mathrm{A}/\mathcal{V}_G$ and the *extended Alexander module* $\bar{\mathrm{A}}_G := \mathrm{A}/\bar{\mathcal{V}}_G$. As in the case of curves, pick an algebraic number $\xi \in \bar{\Bbbk}$ over $\Bbbk_p$, consider the specializations

$$\bar{\mathrm{A}}_G(\xi) := (\bar{\mathrm{A}}_G \otimes \Bbbk_p)/\psi_\xi, \qquad \mathrm{A}_G(\xi) := (\mathrm{A}_G \otimes \Bbbk_p)/\psi_\xi,$$

and define the subspaces

$$\bar{\mathcal{V}}_G(\xi) := \operatorname{Ker}[\mathrm{A}(\xi) \twoheadrightarrow \bar{\mathrm{A}}_G(\xi)], \qquad \mathcal{V}_G(\xi) := \operatorname{Ker}[\mathrm{A}(\xi) \twoheadrightarrow \mathrm{A}_G(\xi)].$$

Clearly,

$$\bar{\mathcal{V}}_G(\xi) = \sum_{\beta \in G} \operatorname{Im}(\beta(\xi) - \mathrm{id}) \subset \mathrm{A}(\xi),$$

where $\beta \mapsto \beta(\xi)$ is the composition of the Burau representation and specialization homomorphism $GL(2, \Lambda) \to GL(2, \bar{\Bbbk})$. In particular, both $\bar{\mathcal{V}}_G$ and $\bar{\mathcal{V}}_G(\xi)$ depend on the image of G in Bu_3 only and thus can be defined for subgroups of Bu_3.

Lemma 3.4 (see [4]). *For any subgroup $G \subset \mathbb{B}_3 \cdot \operatorname{Inn}\mathfrak{F}$ and any algebraic number ξ, one has $\bar{\mathcal{V}}_G(\xi) \subset \mathcal{V}_G(\xi)$; hence, there is an epimorphism $\bar{\mathrm{A}}_G(\xi) \twoheadrightarrow \mathrm{A}_G(\xi)$. If $G \subset \mathbb{B}_3$ and $\xi^2 + \xi + 1 \neq 0$, then $\bar{\mathcal{V}}_G(\xi) = \mathcal{V}_G(\xi)$ and $\bar{\mathrm{A}}_G(\xi) = \mathrm{A}_G(\xi)$.* ▷

According to Theorem 3.3, for a trigonal curve C and algebraic number ξ one has $\mathrm{A}_C(\xi) = \mathrm{A}_G(\xi)$, where $G := \mathfrak{Im}_C$; the corresponding epimorphism $\mathrm{A}(\xi) \twoheadrightarrow \mathrm{A}_C(\xi)$ is called a *geometric presentation* of the Alexander module of C. Hence, there is an epimorphism $\bar{\mathrm{A}}_G(\xi) \twoheadrightarrow \mathrm{A}_C(\xi)$, and Theorem 1.1 is essentially a consequence of the following restatement in terms of the monodromy groups.

Theorem 3.5. *Let $G \subset \mathrm{Bu}_3$ be a subgroup of genus zero and let $\xi \in \bar{\Bbbk} \supset \Bbbk_p$ be an algebraic number such that $\bar{\mathrm{A}}_G(\xi) \neq 0$. Then $N := \operatorname{ord}(-\xi) < \infty$. Furthermore, one has $N \leqslant 6$ unless (p, ψ_ξ) is one of the pairs listed in Table 1. Each pair listed in the table is realized by a certain subgroup $G \subset \mathrm{Bu}_3$ of genus zero; the pairs marked with a * are also realized by subgroups $G \subset \mathbb{B}_3$ of genus zero.*

This theorem is proved in §4, see §4.6.

3.5. **The universal subgroups.** The existence part of Theorem 3.5 is based on the concept of universal subgroup. Fix an algebraic number ξ and consider a subspace $\mathcal{V} \subset \mathrm{A}(\xi)$. Then the subset

$$G_{\mathcal{V}} := \{\beta \in \mathrm{Bu}_3 \mid \operatorname{Im}(\beta(\xi) - \mathrm{id}) \subset \mathcal{V}\}$$

is a subgroup of Bu_3; it is called the *universal subgroup* corresponding to $\mathcal{V}$.

Remark 3.6. Clearly, one has $G_0 = \operatorname{Ker}[\beta \mapsto \beta(\xi)]$ and $G_{\mathrm{A}} = \mathrm{Bu}_3$. In all other cases, $\mathcal{V} = \bar{\Bbbk}\mathbf{v}$ for a certain vector $\mathbf{v} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 \in \mathrm{A}(\xi)$ and the universal subgroup $G_{\mathcal{V}}$ is given by linear equations: $\beta \in G_{\mathcal{V}}$ if and only if $\mathbf{v}^\perp\beta(\xi) = \mathbf{v}^\perp$, where $\mathbf{v}^\perp := [a_2, -a_1]$ generates the annihilator $\mathcal{V}^\perp \subset \mathrm{A}(\xi)^*$.

The following statements are obvious:

(1) if $G := G_{\mathcal{V}}$, then $\bar{\mathcal{V}}_G \subset \mathcal{V}$;
(2) one has $\bar{\mathcal{V}}_G \prec \mathcal{V}$ if and only if $G \prec G_{\mathcal{V}}$.

Here, in Statement (1), the inclusion may be proper; in fact, very few subspaces of dimension one result in nontrivial universal subgroups, *cf.* Corollary 4.5.

Lemma 3.7. *Let $N := \operatorname{ord}(-\xi)$, and assume that $2 \leqslant N < \infty$ and that $G := G_{\mathcal{V}}$ is the universal subgroup corresponding to a subspace $\mathcal{V} \subset \mathrm{A}(\xi)$. Then, the width of each region of the skeletons Sk_G and $\mathrm{Sk}_{G \cap \mathbb{B}_3}$ divides N.*

Proof. Observe that

$$\sigma_1^N = \begin{bmatrix} (-t)^N & \tilde{\varphi}_N(-t) \\ 0 & 1 \end{bmatrix},$$

where $\tilde{\varphi}_N(t) := (t^N - 1)/(t-1)$. Hence, $\sigma_1^N(\xi) = \mathrm{id}$ and $(\mathbb{X}\mathbb{Y})^N \in G_{\mathcal{V}} \cap \mathbb{B}_3$. □

3.6. **Digression: the case $N \leqslant 6$.** For completeness, we discuss a few extensions of the results of [4] concerning the specializations of the Alexander modules at algebraic numbers ξ with $N := \operatorname{ord}(-\xi) \leqslant 6$.

Strictly speaking, only irreducible curves (equivalently, subgroups of Bu_3 with transitive image in $\mathbb{S}_3$) are considered in [4]. However, the preliminary results of [4] hold in the general case. Thus, if $2 \leqslant N \leqslant 5$ and $G_{\mathcal{V}}$ is the universal subgroup corresponding to a submodule $\mathcal{V} \subset \mathrm{A}/\Phi_N(-t)$ (where Φ_N is the cyclotomic polynomial of order N), then $\bar{G}_{\mathcal{V}} \subset \Gamma$ is a congruence subgroup of level N. (In fact, this statement is contained in Lemma 3.7, as the principal congruence subgroup of level $N \leqslant 5$ is of genus zero and is normally generated by $\bar{\sigma}_1^N$.) The number of such subgroups is finite and, using, *e.g.*, the tables found in [3] and trying various type

TABLE 2. Alexander modules $A_C(\xi)$ with $N := \operatorname{ord}(-\xi) \leqslant 5$

#	p	N	$\psi_\xi \in \mathbb{F}_p[t]$	$\bar{\mathcal{V}}_G$	$\bar{G} \subset \Gamma$	Remarks
1	0	2	$t-1$	I	$2B^0 = \Gamma_1(2)$	$\Leftrightarrow$ 9
2				0	$2C^0 = \Gamma(2)$	$\Leftrightarrow$ 10
3		3	t^2-t+1	I	$3B^0 = \Gamma_1(3)$	$\Leftrightarrow$ 13
4				0	$3D^0 = \Gamma(3)$	$\Leftrightarrow$ 14
5		4	t^2+1	I	$4B^0 = \Gamma_1(4)$	$\Rightarrow$ 1, 9
6				0	$4G^0 = \Gamma(4)$	$\Rightarrow$ 2, 10
7		5	$\Phi_5(-t)$	I	$5D^0 = \Gamma_1(5)$	$\Rightarrow$ 18
8				0	$5H^0 = \Gamma(5)$	$\Rightarrow$ 19
9	2	1	$t+1$	I	$2B^0 = \Gamma_1(2)$	$\Leftrightarrow$ 1
10				0	$2C^0 = \Gamma(2)$	$\Leftrightarrow$ 2
11		3	t^2-t+1	II	$3A^0 = \Gamma^3$	
12		5	$\Phi_5(-t)$	IV	$5E^0$	
13	3	1	$t+1$	I	$3B^0 = \Gamma_1(3)$	$\Leftrightarrow$ 3
14				0	$3D^0 = \Gamma(3)$	$\Leftrightarrow$ 4
15		2	$t-1$	II	$2A^0 = \Gamma^2$	
16		4	t^2+1	III	$4D^0$	$\Rightarrow$ 15
17		5	$\Phi_5(-t)$	III	$5F^0$	
18	5	1	$t+1$	I	$5D^0 = \Gamma_1(5)$	
19				0	$5H^0 = \Gamma(5)$	
20	7	1	$t+1$	I	$7E^0 = \Gamma_1(7)$	

specifications, one arrives at a finite list of submodules of the form $\bar{\mathcal{V}}_G \subset A/\Phi_N(-t)$. Details are left to the reader, and the final result, in terms of the specializations $\bar{A}_G(\xi)$, is represented in Table 2. Listed in the table are:

- the values of p, N, and ψ_ξ,
- the corresponding subspace $\bar{\mathcal{V}}_G(\xi) \subset A(\xi)$ (see below),
- the projection $\bar{G} \subset \Gamma$ of the corresponding universal subgroup G, in the notation of [3] and, whenever available, in the conventional notation, and
- a list of dependencies, *i.e.*, whether the non-vanishing of the module $A_G(\xi)$ implies the non-vanishing of another module $A_G(\xi')$ for the same group G.

The subspace $\bar{\mathcal{V}}_G(\xi) \subset A(\xi)$ is either 0 or conjugate to $\bar{\Bbbk}\mathbf{v}_T$, where T is the *type* I, II, III, or IV listed in the table and $\mathbf{v}_T := a_T(\xi)\mathbf{e}_1 + \mathbf{e}_2$, see (4.4). The implications in the last column are given by the inclusions of the universal subgroups, see [3].

The case $N = 1$ (the maximal dihedral quotients of the fundamental group) is settled in [5]: in this case, the universal subgroups are also congruence subgroups of $\tilde{\Gamma} := SL(2, \mathbb{Z})$ (but not necessarily of level 1).

Finally, if $N = 6$, the $\mathbb{B}_3$-action on the module $A' := A/(t^2+t+1)$ has invariant vector $\mathbf{v} := -t\mathbf{e}_1 + \mathbf{e}_2$. Hence, in the basis $\{\mathbf{v}, \mathbf{e}_2\}$, the Bu_3-action is given by upper triangular matrices and can easily be studied. Assume that $G \subset \mathrm{Bu}_3$ is a subgroup of genus zero and the submodule $\bar{\mathcal{V}}_G \subset A'$ is distinct from A'. If $G \subset \mathbb{B}_3$ (proper trigonal curves), then $\bar{\mathcal{V}}_G \sim \Lambda'\mathbf{u} + J\mathbf{v}$, where $\Lambda' := \Lambda/(t^2+t+1)$, $\mathbf{u}$ is one of the following five vectors

$$\mathbf{u}_1 := \mathbf{e}_2, \quad \mathbf{u}_2 := (t+2)\mathbf{e}_2, \quad \mathbf{u}_3 := 2\mathbf{e}_2, \quad \mathbf{u}_\circ := t\mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{u}_\bullet := \mathbf{e}_1 - \mathbf{e}_2,$$

and $J \subset \Lambda'$ is an ideal *of finite index*. If $G \not\subset \mathbb{B}_3$ (improper curves), then $\bar{\mathcal{V}}_G$ is conjugate to the submodule generated by one of the following seven (pairs of) vectors:

$$2\mathbf{e}_2, \mathbf{v}; \quad \mathbf{v}; \quad (t-1)\mathbf{e}_2, \mathbf{v}; \quad 2\mathbf{e}_2, (t-1)\mathbf{v}; \quad \mathbf{e}_2, (t-1)\mathbf{v};$$
$$(t-1)\mathbf{e}_2, (t-1)\mathbf{v}; \quad (t-1)\mathbf{e}_2 - \mathbf{v}, (t-1)\mathbf{v}.$$

Details will appear elsewhere.

4. Proof of Theorems 1.1 and 3.5

4.1. **The set-up.** Fix a subgroup $G \subset \mathrm{Bu}_3$ *of genus zero* and let $\mathrm{Sk} := \mathrm{Sk}_G = \Gamma/\bar{G}$ be its skeleton, $e := \bar{G}/\bar{G}$ the distinguished edge of Sk, and tp the type specification of G. Fix, further, a value p, prime or zero, and an algebraic number $\xi \in \bar{\Bbbk}_p$. We assume that $N := \mathrm{ord}(-\xi) \geqslant 7$; in particular, $\xi \neq \pm 1$ and $\xi^2 + \xi + 1 \neq 0$.

We will also make use of the multiplicative order $M := \mathrm{ord}\,\xi$. One obviously has $M = \mathrm{e}_p(N)$ and $N = \mathrm{e}_p(M)$, where $\mathrm{e}_2(N) := N$ and

$$\mathrm{e}_p(N) := \begin{cases} 2N, & \text{if } N = 1 \bmod 2, \\ \frac{1}{2}N, & \text{if } N = 2 \bmod 4, \\ N, & \text{if } N = 0 \bmod 4 \end{cases}$$

for $p \neq 2$ prime or zero. The Bu_3-action on $\mathrm{A}(\xi)$ factors through $\mathrm{Bu}_3/\langle t^M \,\mathrm{id}\rangle$. In particular, we can assume that $\mathrm{dp}\, G = 2M$ and pass to the group $G/\langle t^M \,\mathrm{id}\rangle$.

We are interested in a subgroup G such that $\bar{\mathrm{A}}_G(\xi) \neq 0$. Since $\dim_{\bar{\Bbbk}} \mathrm{A}(\xi) = 2$, the latter condition is equivalent to $\dim_{\bar{\Bbbk}} \bar{\mathcal{V}}_G(\xi) \leqslant 1$ and, according to [4], one has $N < \infty$. A region R of Sk is called *trivial* (*essential*) if $N \mid \mathrm{wd}\, R$ (respectively, $N \nmid \mathrm{wd}\, R$). Since genus is monotonous, see (2.3), we can assume that G is the universal subgroup corresponding to the subspace $\bar{\mathcal{V}}_G(\xi) \subset \mathrm{A}(\xi)$. Then the width of each region divides N, see Lemma 3.7; hence, trivial are the regions R with $\mathrm{wd}\, R = N$, and essential are those with $\mathrm{wd}\, R < N$.

Consider a copy of $\mathfrak{F}$ and a geometric basis $\alpha_1, \alpha_2, \alpha_3$ with respect to which the action of $\mathbb{B}_3$ is given by (2.1). Given another edge e' of Sk, we fix a path (e, g), $g \in \Gamma$, from e to e', lift g to an element $\tilde{g} \in \mathbb{B}_3$, and consider a new geometric basis $\alpha'_i := \tilde{g}(\alpha_i)$, $i = 1, 2, 3$; it is called a *canonical basis* over e'. Using these canonical bases for $\mathfrak{F}$, we define the (local) canonical bases $\mathbf{e}_1, \mathbf{e}_2$ (over e) and $\mathbf{e}'_1, \mathbf{e}'_2$ (over e') for the universal Alexander module A, see §2.2.

4.2. **The local modules.** Consider a region R or a monovalent vertex v of Sk and denote $\bar{\mathcal{V}}_*(\xi) := \mathrm{Im}(\mathfrak{m}_* - \mathrm{id})$, where $\mathfrak{m}_*$ is the monodromy about the boundary ∂R if $* = R$ or the monodromy about v if $* = v$. (More precisely, $\mathfrak{m}_*$ is the image under (2.4) of a lasso about the center of R or v, respectively.) In view of (2.4), one has $\bar{\mathcal{V}}_G(\xi) = \sum \bar{\mathcal{V}}_*(\xi)$, where $*$ runs over all regions and monovalent vertices of Sk. Hence, a necessary condition for the non-vanishing of $\bar{\mathrm{A}}_G(\xi)$ is $\dim_{\bar{\Bbbk}} \bar{\mathcal{V}}_*(\xi) \leqslant 1$ for each region and each monovalent vertex.

The submodules $\bar{\mathcal{V}}_*(\xi)$ are easily computed in terms of a local canonical basis over an edge e' 'close' to the region or vertex in question. More precisely, if $* = R$ is a region, we let $e' := \mathbb{Y}e''$, where e'' is any edge contained in R; if $* = v$ is a monovalent $\bullet$-vertex, we take for e' the only edge incident to v; finally, if $* = v$ is a monovalent $\circ$-vertex, we let $e' = \mathbb{X}e''$, where e'' is the only edge incident to v.

The following two statements are contained in [4].

Lemma 4.1 (see [4])**.** *In the notation above, assume that* $\dim_{\bar{\Bbbk}} \bar{\mathcal{V}}_*(\xi) \leqslant 1$*, where* $*$ *is a region* R *or a monovalent vertex* v*. Let* $M := \operatorname{ord} \xi = \mathrm{e}_p(N)$*.*

(1) *If* R *is a trivial region, then* $\operatorname{tp}(R) = \operatorname{wd} R \bmod 2M$ *and* $\bar{\mathcal{V}}_R(\xi) = 0$*.*
(2) *Essential regions are subdivided into two types,* I *and* II*, as explained below.*
(3) *If* R *is a region of type* I*, then* $\operatorname{tp}(R) = \operatorname{wd} R \bmod 2M$ *and* $\bar{\mathcal{V}}_R(\xi) = \bar{\Bbbk}\mathbf{e}'_2$*.*
(4) *If* R *is a region of type* II *and* $n := \operatorname{wd} R$*, one has: if* n *is even or* $p = 2$*, then* $\operatorname{tp}(R) = -n \bmod 2M$*; otherwise,* $\operatorname{tp}(R) = M - n \bmod 2M$ *and* M *is even; in both cases,* $\bar{\mathcal{V}}_R(\xi) = \bar{\Bbbk}(\xi^{-1}(\xi+1)\mathbf{e}'_1 + \mathbf{e}'_2)$*.*
(5) *If* v *is a monovalent* $\bullet$*-vertex, one has: if* $p \neq 3$*, then* $M = 0 \bmod 3$ *and* $\operatorname{tp}(v) = \pm\frac{2}{3}M \bmod 2M$*; otherwise,* $M \neq 0 \bmod 3$ *and* $\operatorname{tp}(v) = 0 \bmod 2M$*; in both cases,* $\bar{\mathcal{V}}_v(\xi) = \bar{\Bbbk}(-\xi^s\mathbf{e}'_1 + \mathbf{e}'_2)$*, where* $s := \frac{1}{2}\operatorname{tp}(v) - 1$*.*
(6) *If* v *is a monovalent* $\circ$*-vertex, then* M *is odd,* $\operatorname{tp}(v) = M \bmod 2M$*, and* $\bar{\mathcal{V}}_v(\xi) = \bar{\Bbbk}(\xi^s\mathbf{e}'_1 + \mathbf{e}'_2)$*, where* $s := \frac{1}{2}(M-1)$*.* ▷

Lemma 4.2 (see [4])**.** *Assume, in addition, that* $\bar{\mathrm{A}}_G(\xi) \neq 0$*. Then:*

(1) *at most one of the three regions incident to a trivalent* $\bullet$*-vertex is essential;*
(2) *the region incident to a monovalent vertex is trivial;*
(3) *two monovalent vertices cannot be incident to a common edge.* ▷

Note that in Lemma 4.2(1) we do *not* assume that the three regions are pairwise distinct. In particular, it follows that a $\bullet$-vertex may appear at most once as a corner of an essential region.

4.3. **The case $\bar{\mathcal{V}}_G(\xi) = 0$.** If $\bar{\mathcal{V}}_G(\xi) = 0$, then, due to Lemma 4.1, Sk is a regular skeleton and the widths of all regions of Sk are multiples of N. Hence, by Euler's formula (see, *e.g.*, (4.7) below), one has $N \leqslant 5$. This case is settled in [4], where it is shown that $\bar{G} = \Gamma(N)$ is the principal congruence subgroup of level N.

4.4. **The case $0 \subsetneq \bar{\mathcal{V}}_G(\xi) \subsetneq \mathrm{A}(\xi)$.** From now on, we assume that $\dim_{\bar{\Bbbk}} \bar{\mathcal{V}}_G(\xi) = 1$, *i.e.*, the skeleton Sk has at least one essential region or monovalent vertex.

Consider an edge e' of Sk. If e' is the support of a canonical basis used in the computation of a local module $\bar{\mathcal{V}}_*(\xi)$, see the explanation prior to Lemma 4.1, we assign to e' a *type* $T(e')$ as follows:

- if $* = R$ is an essential region of type I or II, see Lemma 4.1(3) and (4), then e' is of type I or II, respectively;
- if $* = v$ is a monovalent $\bullet$-vertex and $p \neq 3$, then e' is of type $\mathrm{III}_\pm$, where $\operatorname{tp}(v) = \pm\frac{2}{3}M \bmod 2M$, see Lemma 4.1(5);
- if $* = v$ is a monovalent $\bullet$-vertex and $p = 3$, then e' is of type III;
- if $* = v$ is a monovalent $\circ$-vertex, then e' is of type IV, see Lemma 4.1(6);
- otherwise (e' is not related to a 'special' fragment of Sk), e' is of type 0.

An edge of type $T \neq 0$ is called *special.* According to Lemmas 4.1 and 4.2, the type is well defined, *i.e.*, an edge cannot be related to two distinct 'special' fragments. (Indeed, otherwise the subspace $\bar{\mathcal{V}}_G(\xi)$ would contain a pair of linearly independent vectors and one would have $\bar{\mathrm{A}}_G(\xi) = 0$.) In other words, there is a well defined *surjective* map

$$\psi\colon \mathbf{S} \twoheadrightarrow \{\text{monovalent vertices}\} \cup \{\text{essential regions}\}, \tag{4.3}$$

where $\mathbf{S}$ is the set of the special edges of Sk. It follows also that to each special edge e' one can assign the *local subspace* $\bar{\mathcal{V}}_{e'}(\xi) := \bar{\mathcal{V}}_{\psi(e')} \subset \mathrm{A}$. If $e' = \bar{\beta}^{-1}e$, $\beta \in \mathbb{B}_3$, Lemma 4.1 implies that $\bar{\mathcal{V}}_{e'}(\xi) = \bar{\Bbbk}\big(\beta\mathbf{v}_{T(e')}\big)$, where $\mathbf{v}_T := a_T(\xi)\mathbf{e}_1 + \mathbf{e}_2$ and the Laurent polynomial $a_T(t)$, $T \neq 0$, is given by

$$a_{\mathrm{I}} = 0, \quad a_{\mathrm{II}} = t^{-1}(t+1), \quad a_{\mathrm{III}} = -t^s, \quad a_{\mathrm{IV}} = t^{(M-1)/2}. \tag{4.4}$$

Here, $s := \pm(M/3) - 1$ for $T = \mathrm{III}_\pm$ $(p \neq 3)$ and $s := -1$ for $T = \mathrm{III}$ $(p = 3)$.

Corollary 4.5 (see [4])**.** *If $G \subset \mathrm{Bu}_3$ is a subgroup of genus zero, $N := \operatorname{ord}(-\xi) \geqslant 7$, and $\bar{\mathcal{V}}_G(\xi) \subset \mathrm{A}$ is a subspace of dimension one, then one has $\bar{\mathcal{V}}_G(\xi) \sim \bar{\Bbbk}\mathbf{v}_T$ for some type $T \neq 0$ (in fact, for any type $T \neq 0$ present in the skeleton Sk_G).* ◁

(According to [4], the conclusion of Corollary 4.5 also holds for $N \leqslant 5$; the only exception is the case $N = 6$, *i.e.*, $\xi^2 + \xi + 1 = 0$.)

Let $\mathbf{R}$ be the set of the *trivial* regions of Sk. Let, further, $k_N := \lceil 5/(N-6) \rceil$, *i.e.*, $k_7 = 5$, $k_8 = 3$, $k_9 = k_{10} = 2$, and $k_N = 1$ for $N \geqslant 11$.

Lemma 4.6. *Assume that there is a map $\varphi\colon \mathbf{S} \to 2^{\mathbf{R}}$ with the following properties:*

- *$|\varphi(e')| \geqslant k_N$ for each special edge $e' \in \mathbf{S}$;*
- *$\varphi(e') \cap \varphi(e'') = \varnothing$ whenever $e' \neq e''$.*

Then G is not *a subgroup of genus zero.*

Proof. Let $n_\bullet$ and $n_\circ$ be the numbers of *monovalent* $\bullet$- and $\circ$-vertices of Sk, and let n_i be the number of its regions of width $i \geqslant 1$. As a simple consequence of Euler's formula, Sk is of genus zero if and only if

$$3n_\circ + 4n_\bullet + \sum_{i=1}^{N}(6-i)n_i = 12. \tag{4.7}$$

Recall that a region R is trivial if and only if $\operatorname{wd} R = N$, *i.e.*, $|\mathbf{R}| = n_N$. Replacing in (4.7) all coefficients except $(6-N)$ with their maximum $5 = \max_{i\geqslant 1}\{3, 4, 6-i\}$, in view of (4.3) we obtain the inequality $5|\mathbf{S}| > (N-6)|\mathbf{R}|$. On the other hand, under the hypotheses of the lemma, we have $|\mathbf{R}| \geqslant k_N|\mathbf{S}| \geqslant 5|\mathbf{S}|/(N-6)$. □

4.5. **Reduction to a finite number of cases.** We still assume that $G \subset \mathrm{Bu}_3$ is the universal subgroup corresponding to a subspace $\bar{\mathcal{V}}_G(\xi) \subset \mathrm{A}(\xi)$ of dimension 1.

In order to construct a 'universal' map φ as in Lemma 4.6, we fix a value of N and consider a finite set $B = \{\beta_1, \ldots, \beta_k\} \subset \mathrm{Bu}_3$, $k \geqslant k_N$, with all projections $\bar\beta_i \in \Gamma$ pairwise distinct. For a type $T \neq 0$, denote $\mathbf{v}_T(t) := a_T(t)\mathbf{e}_1 + \mathbf{e}_2 \in \mathrm{A}$, so that $\mathbf{v}_T = \mathbf{v}_T(\xi)$, and consider the Laurent polynomials

$$D_{ij,l}(T', T'')(t) := \det\big[\sigma_1^l\beta_i\mathbf{v}_{T'}(t) \bigm| \beta_j\mathbf{v}_{T''}(t)\big] \in \Lambda,$$

where

$$\begin{gathered} T', T'' \neq 0, \quad i, j = 1, \ldots, k, \quad l = 0, \ldots, N-1, \\ \text{and} \quad T' \neq T'' \quad \text{or} \quad i \neq j \quad \text{or} \quad l \neq 0. \end{gathered} \tag{4.8}$$

Note that excluded in (4.8) are precisely those sequences (T', T'', i, j, l) for which the determinant is identically zero.

Lemma 4.9. *Let $e', e'' \in \mathbf{S}$ be two special edges, not necessarily distinct, and let $\beta', \beta'' \in \mathrm{Bu}_3$. Then, if $\bar\beta' e' = \bar\beta'' e''$, one must have $\det\big[\beta'\mathbf{v}_{T(e')} \bigm| \beta''\mathbf{v}_{T(e'')}\big] = 0$.*

Proof. Replacing G with a conjugate subgroup, we can assume that $\bar\beta' e' = \bar\beta'' e''$ is the distinguished edge e. Then the vectors $\beta'\mathbf{v}_{T(e')}$ and $\beta''\mathbf{v}_{T(e'')}$ span $\bar{\mathcal{V}}_{e'}(\xi)$ and $\bar{\mathcal{V}}_{e''}(\xi)$, respectively, and, unless these vectors are linearly dependent, we have $\dim_{\bar{\Bbbk}} \bar{\mathcal{V}}_G(\xi) \geqslant 2$, *i.e.*, $\bar{\mathcal{V}}_G(\xi) = \mathrm{A}(\xi)$. □

Lemma 4.10. *Assume that G is a subgroup of genus zero and that $N \geqslant 7$. Then, for any subset $B \subset \mathrm{Bu}_3$ of size $k \geqslant k_N$, there is a sequence (T', T'', i, j, l) as in (4.8) such that $D_{ij,l}(T', T'')(\xi) = 0$. Furthermore, for at least one of such sequences one has $\bar{\mathcal{V}}_G(\xi) \sim \bar{\Bbbk}\mathbf{v}_{T'} \sim \bar{\Bbbk}\mathbf{v}_{T''}$.*

Proof. Assume that the conclusion does *not* hold, *i.e.*, that all determinants are non-zero. Then, by Lemma 4.9, for any pair of special edges $e', e'' \in \mathbf{S}$ one has $\bar{\sigma}_1^l \bar{\beta}_i e' \neq \bar{\beta}_j e''$ whenever $e' \neq e''$ or $i \neq j$ or $l \neq 0 \bmod N$. In particular (from the special case $e' = e''$ and $i = j$), each region $(\!(\bar{\beta}_i e')\!)$ is trivial and, letting

$$\varphi(e') := \bigl\{ (\!(\bar{\beta}_i e')\!) \bigm| i = 1, \ldots, k \bigr\},$$

we obtain a well defined map $\varphi\colon \mathbf{S} \to 2^{\mathbf{R}}$ satisfying the hypotheses of Lemma 4.6. Hence, G is not of genus zero.

For the last statement, observe that, if $D_{ij,l}(T', T'')(\xi) \neq 0$ for all types T', T'' present in Sk_G, then the map φ in this particular skeleton is still well defined and satisfies the hypotheses of Lemma 4.6. Hence, again, G is not of genus zero. □

Fix a value $N \geqslant 7$, consider a subset $B := \{\beta_1, \ldots, \beta_k\} \subset \mathrm{Bu}_3$, and compute the resultants $\mathcal{R}_{ij,l}(T', T'') \in \mathbb{Z}$ of the determinants $D_{ij,l}(T', T'')(t)$ and the cyclotomic polynomial $\Phi_N(-t)$, where (T', T'', i, j, l) is an index sequence as in (4.8). The set B is called *informative* if $k \geqslant k_N$ and all $\mathcal{R}_{ij,l}(T', T'') \neq 0$ in $\mathbb{Z}$. Due to Lemma 4.10, the existence of an informative set, see below, rules out the case $p = 0$. (In [4], this case was prohibited for irreducible curves only.) Furthermore, each informative set B gives rise to a finite collection $\mathcal{E}(B)$ of 'exceptional' triples (p, ψ_ξ, T) such that there *may* exist a subgroup $G \subset \mathrm{Bu}_3$ of genus zero with $\bar{\mathcal{V}}_G(\xi) \sim \bar{\Bbbk}\mathbf{v}_T(\xi)$. This list is obtained as follows: for each resultant $\mathcal{R}_{ij,l}(T', T'') \neq \pm 1$, we let $T = T'$ and record all prime divisors p of $\mathcal{R}_{ij,l}(T', T'')$ (so that $\mathcal{R}_{ij,l}(T', T'') = 0 \bmod p$) and, for each such divisor p, all irreducible common factors ψ_ξ of $D_{ij,l}(T', T'')(t)$ and $\Phi_N(-t)$ over $\Bbbk_p$.

It is shown in [4] that $N \leqslant 26$ and, furthermore, $N \leqslant 10$ unless (p, ψ_ξ) is one of the pairs listed in Table 1. (Note that the latter statement can also be proved using the approach outlined in this subsection: for most values $N \geqslant 11$ the subset $B = \{\mathrm{id}\}$ is informative.) Let $\beta_1 = t\sigma_1\sigma_1^{-1}$ and $\beta_2 = t\sigma_2^{-1}\sigma_1$. (We multiply the matrices by t in order to clear the denominators.) Using `Maple`, one can show that each of the following subsets

$$\begin{aligned}
N = 7&: \quad \{\mathrm{id}, \beta_1^2, \beta_1^3, \beta_1\beta_2, \beta_2\beta_1\} \text{ and } \{\mathrm{id}, \beta_1^2, \beta_1\beta_2, (\beta_1\beta_2)^2, \beta_2\beta_1\};\\
N = 8&: \quad \{\mathrm{id}, \beta_1^2, \beta_1\beta_2\} \text{ and } \{\mathrm{id}, \beta_1^2, \beta_1\beta_2\beta_1\};\\
N = 9, 10&: \quad \{\mathrm{id}, \beta_2\},\ \{\mathrm{id}, \beta_1\beta_2\},\ \text{and } \{\mathrm{id}, \beta_2\beta_1\}
\end{aligned}$$

is informative and, for each subset B, compile the list $\mathcal{E}(B)$ of exceptional triples. (To shorten the further computation, for each N we consider several subsets B_i and take the intersection $\bigcap_i \mathcal{E}(B_i)$ of the corresponding lists.) As a result, we obtain a finite list (too long to be reproduced here) of exceptional triples (p, ψ_ξ, T) that might appear in the extended Alexander module of a subgroup of genus zero.

4.6. **End of the proof of Theorem 3.5.** The rest of the proof proceeds as in [4]: for each exceptional triple (p, ψ_ξ, T) found in the previous subsection, we use `Maple` to compute the universal subgroup G of Bu_3 or $\mathbb{B}_3$ corresponding to the subspace $\bar{\Bbbk}\mathbf{v}_T(\xi) \subset \mathrm{A}(\xi)$ and select those triples for which this subgroup is of genus zero. The result is Table 1.

For the computation, we specialize the Burau representation at $t = \xi$ and map $\mathbb{B}_3 \subset \mathrm{Bu}_3$ to the finite group $GL(2, \bar{\Bbbk})$. (Recall that $p \ne 0$ and $\bar{\Bbbk}$ is a finite field. In fact, in most cases $\deg \psi_\xi = 1$ and hence $\bar{\Bbbk} = \Bbbk_p$. In the few exceptional cases, we are working with (2×2)-matrices over $\Bbbk_p[t]$ considering them modulo ψ_ξ.) Denote the resulting specialization homomorphism by $\kappa\colon \mathrm{Bu}_3 \to GL(2, \bar{\Bbbk})$. Then $G \supset \operatorname{Ker} \kappa$ and the set of edges of the skeleton Sk_G is the quotient of $\operatorname{Im} \kappa / \kappa(G)$ (or $\kappa(\mathbb{B}_3)/\kappa(G)$ if the universal subgroup of $\mathbb{B}_3$ is to be found) by the further identification $m \sim \xi^s m$, where $s \in \mathbb{Z}$ (respectively, $s = 0 \bmod 3$). The $\bullet$- and $\circ$-vertices of Sk_G are the orbits of $\kappa(\sigma_2\sigma_1)$ and $\kappa(\sigma_2\sigma_1^2)$, respectively, and its regions are the orbits of $\kappa(\sigma_1)$.

Technically, since the image $\operatorname{Im} \kappa$ is not known *a priori*, the coset enumeration proceeds as follows. We start with $m = \mathrm{id}$ and keep multiplying matrices by $\kappa(\sigma_2\sigma_1)$ and $\kappa(\sigma_2\sigma_1^2)$, comparing each matrix against those already recorded. Each new matrix m is added to the list together with all products $\xi^s m$, $s = 0, \ldots, M-1$, where $M := \mathrm{e}_p(N)$. (If $M = 0 \bmod 3$ and a subgroup $G \subset \mathbb{B}_3$ is to be found, only the values $s = 0 \bmod 3$ are used.) Note that the equivalence relation is, in fact, linear, *cf.* Remark 3.6: for two matrices $m_1, m_2 \in GL(2, \bar{\Bbbk})$ one has $m_1 m_2^{-1} \in \kappa(G)$ if and only if $\mathbf{v}_T^\perp(m_1 - m_2) = 0$, where $\mathbf{v}_T^\perp := [-1, a_T(\xi)]$ generates the annihilator of the subspace $\bar{\Bbbk}\mathbf{v}_T \subset \mathrm{A}(\xi)$. This observation simplifies the coset enumeration. □

4.7. **End of the proof of Theorem 1.1.** In view of the epimorphism $\bar{\mathrm{A}}_G(\xi) \twoheadrightarrow \mathrm{A}_C(\xi)$, $G := \mathfrak{Im}_C$, the restrictions on the pairs (p, ψ_ξ) that may result in a nontrivial Alexander module follow from Theorem 3.2 (the monodromy group is a subgroup of genus zero) and Theorem 3.5. If a pair (p, ψ_ξ) can be realized by a subgroup $G \subset \mathbb{B}_3$ of genus zero (the lines marked with a * in Table 1), then $\mathrm{A}_G(\xi) = \bar{\mathrm{A}}_G(\xi) \ne 0$, see Lemma 3.4, and, due to Theorem 3.2 again, G is the monodromy group of a certain proper trigonal curve C, so that one has $\mathrm{A}_C(\xi) = \mathrm{A}_G(\xi) \ne 0$. □

4.8. **Proof of Addendum 1.4.** The first statement follows from the computation in §4.6: in each case resulting in a universal subgroup G of genus zero, we either start with a triple (p, ψ_ξ, T) with $T = \mathrm{I}$ (and hence $\bar{\mathcal{V}}_G = \bar{\Bbbk}\mathbf{e}_2$) or, using the coset enumeration, can show that the subspaces $\bar{\mathcal{V}}_G = \bar{\Bbbk}\mathbf{v}_T$ and $\bar{\Bbbk}\mathbf{e}_2 = \bar{\Bbbk}\mathbf{v}_{\mathrm{I}}$ are conjugate.

The second statement is also proved by a computer aided computation. One needs to show that, given two universal subgroups $G_1, G_2 \subset \mathrm{Bu}_3$ corresponding to two distinct pairs (p, ψ_ξ) and (q, ψ_η), the intersection $G_1 \cap G_2'$, where $G_2' \sim G_2$, cannot be of genus zero. The skeletons $\mathrm{Sk}_i := \mathrm{Sk}_{G_i}$, $i = 1, 2$, have already been computed and, using the double coset formula, one can see that the skeletons of the intersections of the form $G_1 \cap G_2'$, $G_2' \sim G_2$, are the connected components of the fibered product $\mathrm{Sk}_1 \times_{\bullet\!-\!\circ} \mathrm{Sk}_2$, where $\bullet\!-\!\circ$ is the skeleton of Γ itself. Considering all products/components one by one, one concludes that they all have positive genus. Details will appear elsewhere. □

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