Rigid Isotopy Classification of Real Algebraic Curves of Bidegree (3,3) on Quadrics

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Abstract. A rigid isotopy of nonsingular real algebraic curves on a quadric is a path in the space of such curves of a given bidegree. We obtain the rigid isotopy classification of nonsingular real algebraic curves of bidegree (3,3) on a hyperboloid and on an ellipsoid. We also study the space of real algebraic curves of bidegree (3,3) with a single node or cusp. Bibliography: 11 items.

1. Introduction. The notion of rigid isotopy was introduced by Rokhlin [1]. On the projective plane the classification of nonsingular real algebraic curves up to rigid isotopy is known for degree $m \le 6$ (see [1], [2], [3]). On quadrics rigid isotopies of real algebraic curves of bidegrees $(m, 1)$, $(m, 2)$ were studied by the authors, see [4], [5], where the rigid isotopy classification of such nonsingular curves is obtained. For (nonsingular) curves of bidegree (3,3) on quadrics the classification of their real schemes (i.e., real isotopy classification) was obtained in [6] and [7] (see also [8]), and the classification of their complex schemes (i.e., real schemes enriched with a type and complex orientations, see below), in [8]. In the present paper we prove that *a nonsingular curve of bidegree* (3,3) *on a hyperboloid (on an ellipsoid) is determined up to rigid isotopy by its complex (respectively, real) scheme*, see Theorem 2. In the proof of Theorem 1 we enumerate all the connected components of the space of curves of bidegree (3,3) with a single node or cusp (see Figures 1 and 2). We use the approach to the rigid isotopy classification of plane real quartics suggested in [9].

2. Definitions and notation. Let X be a nonsingular quadric. The complex part CX of X is $CP^1 \times CP^1$, and up to biholomorfism X admits two antiholomorphic involutions with nonempty real part; the resulting real surfaces are the hyperboloid, with the real part RX homeomorphic to torus, and the ellipsoid, with $RX \cong S^2$.

Fix a pair P_1, P_2 of generatrices of X. The fundamental classes $[CP_1]$, $[CP_2]$ form a basis of $H_2(CX) \cong Z \oplus Z$. For any curve $A \subset X$ one has $[CA] = m_1[CP_1] + m_2[CP_2]$ for some nonnegative integers m_1, m_2. The pair (m_1, m_2) is called the *bidegree* of A. If $[x_0 : x_1]$, $[y_0 : y_1]$ are homogeneous coordinates in P_1, P_2 respectively, A is given by a polynomial

$$F(x_0, x_1; y_0, y_1) = \sum_{i,j=1}^{m_1,m_2} a_{ij} x_1^i x_0^{m_1-i} y_1^j y_0^{m_2-j},$$

which is homogeneous of degrees m_1 and m_2 in x_0, x_1 and y_0, y_1, respectively.

In the case of hyperboloid the antiholomorphic involution conj acts by conjugating all the four coordinates; hence A is real iff all a_{ij} are real. In the case of ellipsoid conj acts via $([x_0 : x_1], [y_0 : y_1]) \mapsto ([\bar{y}_0 : \bar{y}_1], [\bar{x}_0 : \bar{x}_1])$, and A is real iff $a_{ij} = \bar{a}_{ji}$ (in particular, $m_1 = m_2$).

In order to encode the topology of a real curve on a quadric we use a modification of the standard encoding scheme used for plane projective curves, see, e.g., [10]. First, let $(X, conj)$ be a hyperboloid and $A \in X$ be a nonsingular real curve. The real part RA may have components of two types: those contractible in RX and those noncontractible; they are called *ovals* and *nonovals*, respectively. The number of ovals (of nonovals) is denoted by l (by h). Each oval bounds a topological disk in RX, which is called the *interior* of this oval. The fundamental classes $[RP_1]$ and $[RP_2]$, endowed with some orientations (which are to be fixed), form a basis of $H_1(RX) \cong Z \oplus Z$. Let $N_1, ..., N_h$ be the nonovals of A. All of them realize the same nontrivial class (c_1, c_2) in $H_1(RX)$, with c_1, c_2 relatively prime. So the real scheme of $RA \subset RX$ can be encoded by

$$\langle (c_1, c_2), scheme_1, (c_1, c_2), scheme_2, ..., (c_1, c_2), scheme_h \rangle,$$

where $scheme_1, ..., scheme_h$ are the schemes of ovals in the connected components of $RX \setminus (N_1 \cup ... \cup N_h)$, cf. [10], [8].

If $(X, conj)$ is an ellipsoid, all the components of RA are ovals; their number is denoted by l. In this case we fix a point $\infty \in RX \setminus RA$, which is called the *exterior point*, and for an oval $C \subset RX$ define its *interior* to be the component of $RX \setminus C$ that does not contain ∞. This gives rise to a natural partial order on the set of ovals, and the scheme of ovals can be encoded as in [10].

Following F.Klein, see [11] or [1], we say that a real curve A is of *type* I or *type* II if RA divides or does not divide CA. If A is of type I, the natural orientations of the components U and V of $CA \setminus RA$ induce a pair of opposite orientations on $RA = \partial U = \partial V$ called the *complex orientations* of RA. The real scheme endowed with the type and, in the case of the type I, with the complex orientations is the *complex scheme* of A. The type of a curve with a real scheme $\langle B \rangle$ is encoded via $\langle B \rangle_I$ or $\langle B \rangle_{II}$.

All real curves of a given bidegree (m, n) form a space $C_{m,n} \cong RP^N$, $N = mn + m + n$. The set $\Delta \subset C_{m,n}$ of singular curves has dimension $N - 1$. Denote by $S \subset \Delta$ the subset of curves that have a singular point other than a node or a cusp, or have several singular points. Then $\Delta \setminus S$ is a manifold (although not a smooth submanifold of $C_{m,n}$). The *rigid isotopy class* of a curve $A \in C_{m,n} \setminus \Delta$ (or $A \in \Delta \setminus S$) is the component of $C_{m,n} \setminus \Delta$ (respectively, $\Delta \setminus S$) containing A. The components of $C_{m,n} \setminus \Delta$ (or $\Delta \setminus S$) are *chambers* (respectively, *walls*).

3. Curves with a node or a cusp. First we enumerate the walls in $C_{3,3}$.

THEOREM 1. *The number of walls in $C_{3,3}$ equals 20 in the case of hyperboloid and 6 in the case of ellipsoid.*

PROOF. Consider a curve $A \in \Delta \setminus S$, blow up its singular point, and blow down the two generatrices through this point. The result is a nonsingular quartic

$Q \subset P^2$. The inverse transformation is given by a pair (q_1, q_2) of distinct points in Q (the images of the generatrices), which are real in the case of hyperboloid or complex conjugate in the case of ellipsoid. In the case of hyperboloid these points are ordered and the real line through them is oriented. If A has a cusp, the line is tangent to Q.

Thus in the case of ellipsoid it is clear that the walls are enumerated by the rigid isotopy classes of real quartics; due to [11] their number equals 6.

In the case of hyperboloid the walls are enumerated by the connected components of the space Conf of configurations (Q, q_1, q_2, ϵ), where ϵ is an orientation of the real line through q_1 and q_2. Thus in addition to the rigid isotopy type of Q one needs to distinguish whether q_1 and q_2 belong to the same or distinct ovals of RQ. In the latter case denote the configuration (Q, q_1, q_2, ϵ) by b_α, $\alpha = 2, 3, 4$, if the real scheme of Q is $\langle \alpha \rangle$, and by b_{inn} (or b_{out}) if it is *a nest* $\langle 1\langle 1\rangle\rangle$ and q_1 lies in the inner (or outer) oval of the nest. (Here $\langle \alpha \rangle$ denotes α ovals lying outside of each other, and $\langle 1\langle 1\rangle\rangle$ denotes 2 ovals, one inside another.) In the former case denote the configuration by a_α, $\alpha = 1, ..., 4$, a_{inn} (or a_{out}), respectively.

Besides, all the configurations except b_2, b_3 can be given a sign + or − in the following way. The points q_1 and q_2 divide the real line through them into two segments oriented via ϵ; let $\overline{q_1 q_2}$ be the one having q_1 as the origin. Denote the configuration (Q, q_1, q_2, ϵ) by a_α^- (or b_{inn}^-, b_{out}^-) if a neighborhood of q_1 in $\overline{q_1 q_2}$ lies inside the oval containing the points q_1 and q_2 (or inside the outer oval of the nest), and by a_α^+ (or b_{inn}^+, b_{out}^+) otherwise. For b_4 consider a one-sided topological circle γ in RP^2 not intersecting $RQ \cup \overline{q_1 q_2}$ and denote the configuration by b_4^- (or b_4^+) if the complex orientations of the two ovals containing q_1, q_2 are (respectively, are not) coherent in $RP^2 \setminus \gamma$.

To complete the proof it remains to notice that *the set of connected components of Conf is in a natural one-to-one correspondence with the set* $\{a_\alpha^\pm (\alpha = 1, ..., 4),\ a_{inn}^\pm,\ a_{out}^\pm,\ b_2,\ b_3,\ b_4^\pm,\ b_{inn}^\pm,\ b_{out}^\pm\}$ *of 20 elements.* The assertion is an obvious consequence of the following lemma.

Lemma on Permutation of Ovals. *For any nestless real quartic Q there is a rigid isotopy that takes Q to itself and induces any given permutation of the ovals of Q. Besides, if Q consists of two or three ovals, there is a rigid isotopy which takes every oval to itself and reverses the orientation of a given line crossing two of the ovals.*

Proof. Since the rigid isotopy class of a real quartic is determined by its real scheme (see [11] or [1]) and since $PGL(3; R)$ is connected it suffices to realize each real scheme $\langle \alpha \rangle$, $\alpha = 2, 3, 4$, by a quartic Q and to find projective transformations of Q inducing necessary rigid isotopies. If $\alpha = 2$, one can take for Q the curve $(4x_1^2 + 4x_2^2 - x_0^2)(4x_0^2 + 4x_2^2 - x_1^2) = t(x_0^4 + x_1^4 + x_2^4)$, where $t \in R$ is small. Then the transformation $(x_0, x_1, x_2) \to (x_1, -x_0, x_2)$ permutes the ovals of Q. For $\alpha = 3$ and 4 consider the cube $\{(x_0, x_1, x_2) \mid |x_i| \leq 1\}$ in R^3. There are four planes through the origin which intersect the cube in regular hexagons. Let

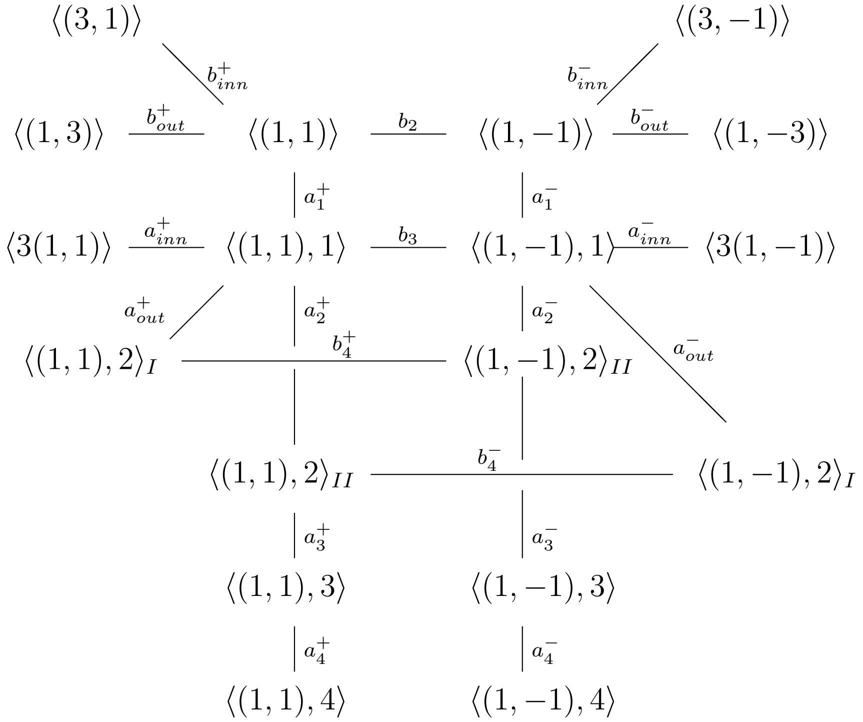

Figure 1: Chambers of curves of bidegree $(3, 3)$ on a hyperboloid

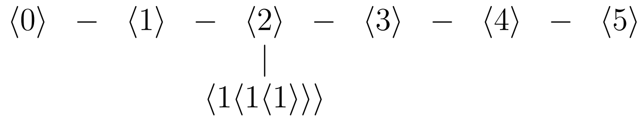

Figure 2: Chambers of curves of bidegree (3,3) on an ellipsoid

$u(x_0, x_1, x_2) = 0$ be the equation of the union of the planes. Then the equation $u(x_0, x_1, x_2) + t(x_0^4 + x_1^4 + x_2^4) = 0$ with small $t \in R$ defines a nonsingular quartic $Q \subset P^2$, which, depending on the sign of t, has 4 or 3 ovals; they correspond to pairs of opposite vertices or, respectively, faces of the cube and, hence, any permutation of the ovals can be realized by a symmetry of the cube. Besides, for $\alpha = 2$ and 3 the transformation $(x_0, x_1, x_2) \to (x_1, -x_0, x_2)$ takes every oval to itself and reverses the orientation of the line $x_2 = 0$. □

4. Main result.

Theorem 2. *The rigid isotopy classification of nonsingular real curves of bidegree* $(3, 3)$ *on a hyperboloid (on an ellipsoid) coincides with the classification of their complex (real) schemes. The adjacency graphs of the chambers in* $C_{3,3}$ *are shown in Figures 1 (for the hyperboloid) and 2 (for the ellipsoid).*

PROOF.

Consider the exact sequence of $(C_{3,3}, \Delta, S)$ (recall that $\dim C_{3,3} = 15$)

$$0 \to H_{15}(C_{3,3}, S) \to H_{15}(C_{3,3}, \Delta) \to H_{14}(\Delta, S) \xrightarrow{in} H_{14}(C_{3,3}, S). \qquad (1)$$

(Here and below all homology groups have $Z/2$-coefficients). It is clear that the number c of chambers equals $\dim_{Z/2} H_{15}(C_{3,3}, \Delta)$ and the number w of walls equals $\dim_{Z/2} H_{14}(\Delta, S)$. Since $H_{15}(C_{3,3}, S) = H_{15}(C_{3,3}) = Z/2$, from the exactness of (1) it follows that $c = 1 + w - codim_{Z/2} \ker in$.

In the case of ellipsoid due to Theorem 1 one has $w = 6$ and, hence, $w \leq 7$. On the other hand, perturbing the corresponding singular curves one obtains seven chambers, which differ by the real schemes, and the statement follows.

In the case of hyperboloid the number of complex schemes of curves of bidegree $(3, 3)$ equals 18 (see [8], §3.10).[1] Thus, $c \geq 18$. Since $w = 20$, see Theorem 1, in order to prove the opposite inequality it suffices to show that $codim_{Z/2} \ker in \geq 3$. Let x_j be the coordinates of a class $x \in H_{14}(\Delta, S)$ with respect to the basis of $H_{14}(\Delta, S)$ formed by the classes w_j realized by the walls. Then $codim_{Z/2} \ker in$ is the number of independent equations in x_j that determine $\ker in$, and it remains to find three such equations. Due to the Alexander-Pontryagin duality one has

$$H_{14}(C_{3,3}, S) \cong H^1(C_{3,3} \setminus S) = Hom(H_1(C_{3,3} \setminus S), Z/2). \qquad (2)$$

Hence, $x \in \ker in$ if and only if $in\, x \circ H_1(C_{3,3} \setminus S) = 0$, and the required equations are obtained by multiplying the relation $in\, x = \Sigma\, x_j\, in\, w_j$ by three linearly independent classes of $H_1(C_{3,3} \setminus S)$. The latter are represented by small circles about S centered at the points corresponding to the three curves shown in Figure 3. (These curves can easily be constructed by perturbing unions of lines. Note that the curves b and c differ by their complex orientations.) Each circle intersects only four walls, transversally at one point each. So from (2) it follows that the classes realized by the circles are independent and the corresponding equations are nontrivial. □

[1] The number given in [8] is 9, but one should take into account that the complex schemes are studied in [8] up to permutation of the factors of $P^1 \times P^1$.

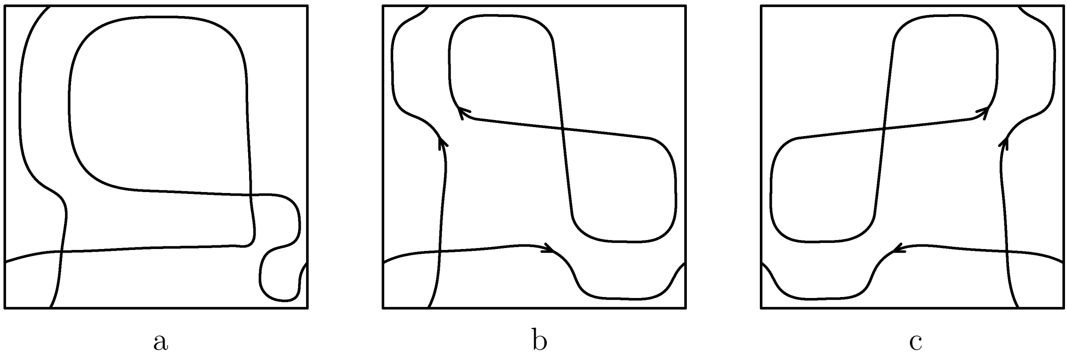

Figure 3:

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