LINES ON QUARTIC SURFACES

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Abstract. We show that the maximal number of (real) lines in a (real) nonsingular spatial quartic surface is 64 (respectively, 56). We also give a complete projective classification of all quartics containing more than 52 lines: all such quartics are projectively rigid. Any value not exceeding 52 can appear as the number of lines of an appropriate quartic.

1. Introduction

1.1. **Principal results.** Throughout the paper, all algebraic varieties are defined over $\mathbb{C}$. Given an algebraic surface $X \subset \mathbb{P}^3$, we denote by $\operatorname{Fn}(X)$ the set of projective lines contained in X. If X is real (see definition below), $\operatorname{Fn}_{\mathbb{R}}(X)$ stands for the set of real lines contained in X.

Theorem 1.1 (see §8.3)**.** *Let $X \subset \mathbb{P}^3$ be a nonsingular quartic, and assume that $|\operatorname{Fn}(X)| > 52$. Then X is projectively equivalent to either*

- *Schur's quartic X_{64}, see §9.1, or*
- *one of the three quartics X'_{60}, X''_{60}, $\bar{X}''_{60}$ described in §9.4.1, or*
- *the quartic Y_{56}, see §9.2, or quartics X_{56}, $\bar{X}_{56}$, Q_{56} described in §9.4.1, or*
- *one of the two quartics X_{54}, Q_{54} described in §9.4.*

In particular, one has $|\operatorname{Fn}(X)| = 64$, 60, 56, or 54, respectively.

Corollary 1.2 (see Segre [24] and Rams, Schütt [19])**.** *Any nonsingular quartic in $\mathbb{P}^3$ contains at most 64 lines.* ◁

Note that the field of definition $\mathbb{C}$ is essential for all statements. For example, over $\mathbb{F}_9$, the quartic given by the equation $z_0z_3^3 + z_1z_2^3 + z_1^3z_2 + z_0^3z_3 = 0$ contains 112 lines. According to Rams, Schütt [19], the bound $|\operatorname{Fn}(X)| \leqslant 64$ holds over any field of characteristic other than 2 or 3.

As was observed by T. Shioda, X_{56} and $\bar{X}_{56}$ are alternative projective models of the Fermat quartic: this fact follows from the description of their transcendental lattice, see Lemma 6.19. I. Shimada has recently found an explicit defining equation of these surfaces. Other similar examples are discussed in Remark 9.14.

Recall that a *real variety* is a complex algebraic variety X equipped with a *real structure*, *i.e.*, an anti-holomorphic involution $\operatorname{conj}\colon X \to X$. The *real part* of X is the fixed point set $X_{\mathbb{R}} := \operatorname{Fix}\operatorname{conj}$. A subvariety (*e.g.*, a line) $Y \subset X$ is called *real* if it is conj-invariant. When speaking about a *real quartic* $X \subset \mathbb{P}^3$, we assume that

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the real structure on X is the restriction of the standard coordinatewise complex conjugation $z \mapsto \bar{z}$ on $\mathbb{P}^3$.

Corollary 1.3 (see §8.4)**.** *Let $X \subset \mathbb{P}^3$ be a nonsingular (over $\mathbb{C}$) real quartic, and assume that $|\mathrm{Fn}_{\mathbb{R}}(X)| > 52$. Then X is projectively equivalent (over $\mathbb{R}$) to the quartic Y_{56} given by (9.4). In particular, one has $|\mathrm{Fn}_{\mathbb{R}}(X)| = 56$, and this is the maximal number of real lines that can be contained in a nonsingular real quartic.*

Addendum 1.4 (see §8.5)**.** *For any number*

$$n \in \{0, 1, \ldots, 51, 52, 54, 56, 60, 64\},$$

there exists a nonsingular quartic $X \subset \mathbb{P}^3$ such that $|\mathrm{Fn}(X)| = n$. For any number

$$m \in \{0, 1, \ldots, 47, 48, 52, 56\},$$

there exists a nonsingular real quartic $X \subset \mathbb{P}^3$ such that $|\mathrm{Fn}_{\mathbb{R}}(X)| = m$.

Thus, for the moment we are not certain about the values $|\mathrm{Fn}_{\mathbb{R}}(X)| = 49, 50, 51$. We know three families of real quartics with 52 real lines; for a list of currently known large configurations of lines, see Table 1 in §6.2.

The quartic Y_{56} can be defined over $\mathbb{Q}$; however, some of the lines are still defined only over $\mathbb{Q}(\sqrt{2})$ (see Remark 9.11). At present, we do not know how many lines defined over $\mathbb{Q}$ a quartic defined over $\mathbb{Q}$ may have; since $\mathbb{Q} \subset \mathbb{R}$ and Y_{56} has been ruled out, Corollary 1.3 implies that this maximal number is at most 52, the first candidates being the configurations $\mathbf{Y}'_{52}$, $\mathbf{Y}''_{52}$, $\mathbf{Z}_{52}$. Though, see Remark 9.13.

Another open question is the maximal number of lines contained in a triangle free configuration, see Theorem 7.9 and Remark 7.10.

1.2. **Contents of the paper.** In §2, we start with a brief introduction to the history of the subject. In §3, we recall basic notions and facts related to integral lattices and $K3$-surfaces and use the theory of $K3$-surfaces to reduce the original geometric problem to a purely arithmetical question about *configurations*; the main results of this section are stated in §3.4. The simplest properties of configurations, not related directly to quartic surfaces, are treated in §4, whereas §5 deals with the more subtle arithmetic properties of the main technical tool of the paper, the so-called *pencils*. The technical part is §6: we outline the algorithm used for counting lines in a pair of obverse pencils and state the counts obtained in the output. Table 1 lists most known large configurations of lines. In §7, we digress to the so-called *triangle free* configurations, for which one can obtain a stronger bound on the number of lines, see Theorem 7.9. The principal results of the paper stated in §1.1 are proved in §8. Finally, in §9, we discuss the properties of quartics with many lines (in particular, §9.2 contains an explicit equation of Y_{56}) and make a few concluding remarks.

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2. History of the subject

The fact that there always exist exactly twenty seven lines on every smooth cubic surface in the complex projective space $\mathbb{P}^3$ naturally leads to inquiries about higher degree surfaces in $\mathbb{P}^3$. The situation however seems to be more involved for higher degree surfaces since it follows immediately from a standard dimension count that a generic surface in $\mathbb{P}^3$ of degree four or higher does not contain any lines at all, whereas each Fermat surface of the form

$$z_0^d + z_1^d + z_2^d + z_3^d = 0,$$

where $[z_0 : z_1 : z_2 : z_3]$ are homogeneous coordinates of $\mathbb{P}^3$, contains exactly $3d^2$ lines, for all $d \geqslant 3$. This then prompts the more relevant question about how many lines a surface of degree $d \geqslant 4$ can have. In particular, for a fixed $d \geqslant 4$, is there an upper bound for the number of lines that a surface of degree d can contain?

At this point it is appropriate to call attention to the difference between the existence of rational curves on a surface and the existence of lines. A line in $\mathbb{P}^3$ is defined as the intersection of two hyperplanes but a rational curve is an isomorphic image of $\mathbb{P}^1$, which need not be a line. Whereas we are expecting a finite number of lines on a surface the situation is drastically different for the existence of rational curves. While a generic quartic does not contain a line, it is shown by Mori and Mukai [17] that every projective $K3$-surface, in particular every smooth quartic in $\mathbb{P}^3$, contains at least one rational curve. Moreover Bogomolov, Hassett and Tschinkel showed in [5] that a generic $K3$-surface, including a generic quartic surface in $\mathbb{P}^3$, contains infinitely many rational curves. Going away from generic case to specific examples, Bogomolov and Tschinkel showed in [4] that if a $K3$-surface admitting an elliptic fibration has Picard number at most 19, then it contains infinitely many rational curves.

Xi Chen showed in [9] that for a generic quartic in $\mathbb{P}^3$, every linear system $\mathcal{O}(n)$, for any $n > 0$, contains a nodal rational curve. In fact Yau and Zaslow in [25], inspired by string theory, counted those rational curves for the $n = 1$ case.

Existence of smooth curves on quartic surfaces in $\mathbb{P}^3$ is also relatively well understood. Mori showed in [16] that a quartic surface in $\mathbb{P}^3$ contains a smooth curve of degree $n > 0$ and genus $g \geqslant 0$ if and only if either

- $g = (n^2/8) + 1$, or
- $g < (n^2/8)$ and $(n, g) \neq (5, 3)$.

The problem of counting lines on smooth surfaces in $\mathbb{P}^3$ is on the other hand a totally different game.

The first work which we can trace about this problem is Schur's article [22] where he exhibits a certain quartic surface which contains 64 lines. This surface is now known as Schur's quartic and is given by the equation

$$z_0(z_0^3 - z_1^3) = z_2(z_2^3 - z_3^3).$$

In §9.1 we give an account of the 64 lines on this quartic.

Apparently no progress was made on this result for about half a century until 1943 when Segre published some articles on the arithmetic and geometry of surfaces in $\mathbb{P}^3$. In one of these articles, in [24], he claimed that the number of lines which can lie on a quartic surface cannot exceed 64. Since Schur's quartic already contains 64 lines, this result of Segre would close the question for quartics were it not for a flaw in his arguments which was only recently detected and corrected by Rams and Schütt in [19]. Rams and Schütt showed that the theorem is correct but the proof needs some modifications using techniques which were not available to Segre at that time.

Segre article [24] contains an upper bound for the number of lines which can lie on a surface of degree $d \geqslant 4$. His upper bound, which is not affected by his erroneous argument about quartics, is $(d-2)(11d-6)$. This bound is not expected to be sharp. For quartics it predicts 76, larger than the actual 64.

There is one curious fact about Segre's work of 1943. Most of the techniques he uses were already in Salmon's book [21] which was originally published in 1862. It would be reasonable to expect that a work similar to Segre's be published much earlier than 1943. We learn from a footnote in [24] that the problem was mentioned by Meyer in an encyclopedia article [14] as early as 1908 but even that was not enough to spur interest in the subject at the time.

After Segre's work there was again a period of long silence on the problem of lines on surfaces. In 1983 Barth mentioned this problem in [2] which turned out to be an influential manuscript on the subject. There he also noted that since a smooth quartic in $\mathbb{P}^3$ is a $K3$-surface and since by Torelli theorems a $K3$-surface is nothing but its Picard lattice, all results of Segre on quartics could possibly be reproduced in the lattice language. This teaser was one of the challenges which prompted us to work on this problem thirty years later.

In 1995, Caporaso, Harris and Mazur, in [8], while investigating the number of rational points on a curve over an algebraic number field, attacked the problem of finding a lower bound for the maximal number N_d of lines lying on a surface of the form $\varphi(z_0, z_1) = \varphi(z_2, z_3)$, where φ is a homogeneous form of degree d. Their arguments being purely geometric, their findings made sense in the complex domain. They found that in general for all $d \geqslant 4$,

$$N_d \geqslant 3d^2, \text{ but } N_4 \geqslant 64, \quad N_6 \geqslant 180, \quad N_8 \geqslant 256, \quad N_{12} \geqslant 864, \quad N_{20} \geqslant 1600.$$

Here the equality $N_4 = 64$ follows from Segre's work [24].

In 2006 Boissiére and Sarti attacked this problem in [6] using group actions. They studied the maximal number of lines on symmetric surfaces in $\mathbb{P}^3$, where we called a surface symmetric if its equation is of the form

$$\varphi(z_0, z_1) = \psi(z_2, z_3),$$

where φ and ψ are homogeneous forms of degree d, as studied by Caporaso, Harris and Mazur. This approach may seem restrictive at first; nonetheless, it is reasonable since Schur's surface which contains the maximal possible number of lines a quartic surface can contain is itself of this form. Boissiére and Sarti first showed that for symmetric surfaces, the inequalities about N_d which Caporaso, Harris and Mazur obtained are actually equalities. This increased the hope that the symmetric surfaces are candidates to carry the most number of lines among other surfaces of the same degree. However, Boissiére and Sarti showed in the same work that this expectation fails. They showed that the non-symmetric surface given by

$$z_0^8 + z_1^8 + z_2^8 + z_3^8 + 168 z_0^2 z_1^2 z_2^2 z_3^2 \\ + 14(z_0^4 z_1^4 + z_0^4 z_2^4 + z_0^4 z_3^4 + z_1^4 z_2^4 + z_1^4 z_3^4 + z_2^4 z_3^4) = 0$$

contains 352 lines, which is far greater than the upper bound of 256 for the symmetric surfaces of the same degree. Notice that the number 352 is within the limits allowed by Segre's upper bound, which gives 492 in this case.

Finally, almost thirty years after Barth's teaser, two teams started to work on this problem, unaware of each other, from two different points of approach. While we concentrated on understanding the "lines on surfaces" problem for $K3$-surfaces in $\mathbb{P}^3$ and aimed at transliterating Segre's results into the lattice language, Rams and Schütt decided to re-attack the problem by using elliptic fibration techniques in [19]. They discovered a flow in Segre's arguments which rendered his proof void; nonetheless, his theorem proved to be correctly stated. Moreover, Rams and Schütt's proof works on any algebraically closed field of any characteristic $p \neq 2, 3$. Schur's quartic becomes singular when $p = 2$ (still containing 64 lines); when $p = 3$, it is shown in [19] that the surface contains 112 lines.

It is interesting to note that the concept of an elliptic fibration is inevitable in studying the lines on a quartic. If X is a smooth quartic in $\mathbb{P}^3$ and L is a line lying on X, one can parametrize the space of planes Λ_t in $\mathbb{P}^3$ passing through L by $t \in \mathbb{P}^1$. Then any point $p \in X$ determines a unique plane Λ_t, and the map sending p to t is an elliptic fibration. If $p \in L$, we take Λ_t as the plane tangent to X at p. Segre starts with this observation but, using intuitive geometric arguments, he erroneously claims that the maximal number of lines in the fibers of the pencil is 18. The true bound is 20, see [19] or (5.7), which calls for more work to establish the ultimate bound 64 for the total number of lines in X.

3. The reduction

Throughout the paper, we consider various abelian groups A equipped with bilinear and/or quadratic forms. Whenever the form is fixed, we use the abbreviation $x \cdot y$ (respectively, x^2) for the value of the bilinear form on $x \otimes y$ (respectively, the quadratic form on x). Given a subset $B \subset A$, its *orthogonal complement* is $B^\perp = \{x \in A \mid x \cdot y = 0 \text{ for all } y \in B\}$.

3.1. **Integral lattices.** An (*integral*) *lattice* is a finitely generated free abelian group S supplied with a symmetric bilinear form $b\colon S \otimes S \to \mathbb{Z}$. A lattice S is *even* if $x^2 = 0 \bmod 2$ for all $x \in S$. As the transition matrix between two integral bases has determinant ± 1, the determinant $\det S \in \mathbb{Z}$ (*i.e.*, the determinant of the Gram matrix of b in any basis of S) is well defined. A lattice S is called *nondegenerate* if $\det S \neq 0$; it is called *unimodular* if $\det S = \pm 1$. Alternatively, S is nondegenerate if and only if its *kernel* $\ker S := S^\perp$ is trivial. An *isometry* $\psi\colon S \to S'$ between two lattices is a group homomorphism respecting the bilinear forms; obviously, one always has $\operatorname{Ker} \psi \subset \ker S$. The group of auto-isometries of a nondegenerate lattice S is denoted by $O(S)$. Given a collection of subsets/elements $A_1, \ldots$ in S, we use the notation $O(S, A_1, \ldots)$ for the subgroup of $O(S)$ preserving each A_i as a set.

Given a lattice S, the bilinear form extends to $S \otimes \mathbb{Q}$ by linearity. The inertia indices $\sigma_\pm S$, $\sigma_0 S$ and the signature σS of S are defined as those of $S \otimes \mathbb{Q}$. The orthogonal projection establishes a linear isomorphism between any two maximal positive definite subspaces of $S \otimes Q$, thus providing a way for comparing their orientations. A coherent choice of orientations of all maximal positive definite subspaces is called a *positive sign structure*. Assuming S nondegenerate, we denote by $O^+(S) \subset O(S)$ the subgroup formed by the auto-isometries preserving a positive sign structure.

A *d-polarized lattice* is a lattice S with a distinguished vector $h \in S$, referred to as the *polarization*, such that $h^2 = d$. We use the abbreviation $O_h(A_1, \ldots)$ for $O(h, A_1, \ldots)$; a similar convention applies for O^+.

If S is nondegenerate, the dual group $S^\vee = \operatorname{Hom}(S, \mathbb{Z})$ can be identified with the subgroup

$$\bigl\{x \in S \otimes \mathbb{Q} \bigm| x \cdot y \in \mathbb{Z} \text{ for all } y \in S\bigr\}.$$

In particular, $S \subset S^\vee$ and the quotient $S^\vee/S$ is a finite group; it is called the *discriminant group* of S and is denoted by $\operatorname{discr} S$ or $\mathcal{S}$. The discriminant group $\mathcal{S}$ inherits from $S \otimes \mathbb{Q}$ a symmetric bilinear form $\mathcal{S} \otimes \mathcal{S} \to \mathbb{Q}/\mathbb{Z}$, called the *discriminant form*, and, if S is even, a quadratic extension $\mathcal{S} \to \mathbb{Q}/2\mathbb{Z}$ of this form. When speaking about the discriminant groups, their (anti-)isomorphisms, *etc.*, we always assume that the discriminant form (and its quadratic extension if the lattice is even) is taken into account. The number of elements in $\mathcal{S}$ is equal to $|\det S|$; in particular, $\mathcal{S} = 0$ if and only if S is unimodular.

Given a prime number p, we denote by $\mathcal{S}_p$ or $\operatorname{discr}_p S$ the p-primary part of $\mathcal{S} = \operatorname{discr} S$. The form $\mathcal{S}$ is called *even* if there is no order 2 element $\alpha \in \mathcal{S}_2$ with $\alpha^2 = \pm\frac{1}{2} \bmod 2\mathbb{Z}$. We use the notation $\ell(\mathcal{S})$ for the minimal number of generators of $\mathcal{S}$, and we put $\ell_p(\mathcal{S}) = \ell(\mathcal{S}_p)$. The quadratic form on $\mathcal{S}$ can be described by means of an analog (ε_{ij}) of the Gram matrix: assuming that $d_1 \mid d_2 \mid \ldots \mid d_\ell$ are the invariant factors of $\mathcal{S}$, we pick a basis $\alpha_1, \alpha_2, \ldots, \alpha_\ell \in \mathcal{S}$ so that the order of α_i is d_i, and let $\varepsilon_{ij} = \alpha_i \cdot \alpha_j \bmod \mathbb{Z}$ for $i \neq j$ and $\varepsilon_{ii} = \alpha_i^2 \bmod 2\mathbb{Z}$. A similar construction applies to $\mathcal{S}_p$. Furthermore, according to R. Miranda and D. Morrison [15], unless $p = 2$ and $\mathcal{S}_2$ is odd, the determinant of the resulting matrix is a unit in $\mathbb{Z}_p$ well defined modulo $(\mathbb{Z}_p^*)^2$; this determinant is denoted by $\det_p \mathcal{S} \in \mathbb{Z}_p^*/(\mathbb{Z}_p^*)^2$.

Two nondegenerate lattices are said to have the same *genus* if their localizations at all primes and at infinity are pairwise isomorphic. The genus of an even lattice is determined by its signature and the isomorphism class of the quadratic extension of the discriminant form, see [18].

In what follows, we denote by $[s]$ the rank one lattice $\mathbb{Z}w$, $w^2 = s$. The notation $\mathbf{U}$ stands for the *hyperbolic plane*, *i.e.*, the lattice generated by a pair of vectors u, v (referred to as a *standard basis* for $\mathbf{U}$) with $u^2 = v^2 = 0$ and $u \cdot v = 1$. Furthermore, given a lattice S, we denote by nS, $n \in \mathbb{N}$, the orthogonal direct sum of n copies of S, and by $S(q)$, $q \in \mathbb{Q}$, the lattice obtained from S by multiplying the form by q (assuming that the result is still an integral lattice). The notation $n\mathcal{S}$ is also used for the orthogonal sum of n copies of a discriminant group $\mathcal{S}$.

A *root* in an even lattice S is a vector $r \in S$ of square -2. A *root system* is an even negative definite lattice generated by its roots. Recall that each root system splits (uniquely up to order of the summands) into orthogonal sum of indecomposable root systems, the latter being those of types $\mathbf{A}_n$, $n \geqslant 1$, $\mathbf{D}_n$, $n \geqslant 4$, $\mathbf{E}_6$, $\mathbf{E}_7$, or $\mathbf{E}_8$, see [7].

From now on, we fix an even unimodular lattice $\mathbf{L}$ of rank 22 and signature -16. All such lattices are isomorphic to $2\mathbf{E}_8 \oplus 3\mathbf{U}$. It can easily be shown that, up to the action $O^+(S)$, this lattice has a unique 4-polarization h; thus, $\mathbf{L}$ is always considered equipped with a distinguished 4-polarization h and a positive sign structure.

We also fix the notation for certain discriminant forms. Given coprime integers m, n such that one of them is even, $\langle\frac{m}{n}\rangle$ is the quadratic form $1 \mapsto \frac{m}{n} \bmod 2\mathbb{Z}$ on $\mathbb{Z}/n$. Given a positive integer k, consider the group $\mathbb{Z}/2^k \times \mathbb{Z}/2^k$ generated by $\alpha = (1, 0)$ and $\beta = (0, 1)$; denote by $\mathcal{U}_{2^k}$ (respectively, $\mathcal{V}_{2^k}$) the quadratic form on the above group such that $\alpha \cdot \beta = \frac{1}{2^k} \bmod \mathbb{Z}$ and $\alpha^2 = \beta^2 = 0 \bmod 2\mathbb{Z}$ (respectively, $\alpha^2 = \beta^2 = \frac{1}{2^{k-1}} \bmod 2\mathbb{Z}$).

An *extension* of a nondegenerate lattice S is another lattice M containing S. An *isomorphism* between two extensions $M', M'' \supset S$ is a bijective isometry $M' \to M''$ identical on S. More generally, given a subgroup $G \subset O(S)$, a G-isomorphism is a bijective isometry $M' \to M''$ whose restriction to S is an element of G.

The two extreme cases are those of *finite index* extensions (where S has finite index in M) and *primitive* ones (where M/S is torsion free). The general case $M \supset S$ splits into the finite index extension $\tilde{S} \supset S$ and primitive extension $M \supset \tilde{S}$, where

$$\tilde{S} = \bigl\{x \in M \bigm| nx \in S \text{ for some } n \in \mathbb{Z}\bigr\}$$

is the *primitive hull* of S in M.

If S is nondegenerate and $M \supset S$ is a finite index extension, we have a chain of inclusions

$$S \subset M \subset M^\vee \subset S^\vee,$$

and, hence, a subgroup $\mathcal{K} = M/S \subset \mathcal{S}$; this subgroup is called the *pivot* of $M \supset S$. The pivot $\mathcal{K}$ is *b-isotropic*, that is, the restriction to $\mathcal{K}$ of the discriminant form $\mathcal{S} \otimes \mathcal{S} \to \mathbb{Q}/\mathbb{Z}$ is trivial. Furthermore, the lattice M is even if and only if S is even and $\mathcal{K}$ is *isotropic*, that is, the restriction to $\mathcal{K}$ of the quadratic extension $\mathcal{S} \to \mathbb{Q}/2\mathbb{Z}$ of the discriminant form is trivial.

Theorem 3.1 (V. Nikulin [18]). *Given a nondegenerate lattice S, the map sending $M \supset S$ to the pivot $\mathcal{K} = M/S \subset \mathcal{S}$ establishes a one-to-one correspondence between the set of isomorphism classes of finite index extensions of S and the set of b-isotropic subgroups of $\mathcal{S}$. Under this correspondence, one has* $\operatorname{discr} M = \mathcal{K}^\perp/\mathcal{K}$ *and* $M = \bigl\{x \in S^\vee \bigm| x \bmod S \in \mathcal{K}\bigr\}$.

In the other extreme case, we confine ourselves to primitive extensions $M \supset S$ to an even unimodular lattice M. Assuming S nondegenerate, these are equivalent to appropriate finite index extensions of $S \oplus S^\perp$, the pivot of the latter giving rise to an anti-isomorphism $\mathcal{S} \to \operatorname{discr} S^\perp$ and thus determining the genus of $S^\perp$. It follows that, given a subgroup $G \subset O(S)$ and the signature of M, a G-isomorphism class of even unimodular primitive extensions $M \supset S$ is determined by a choice of

- an even lattice T such that $\operatorname{discr} T \cong -\mathcal{S}$ and $\sigma_\pm T = \sigma_\pm M - \sigma_\pm S$, and
- a bi-coset in $G\backslash \operatorname{Aut} \operatorname{discr} T/O(T)$.

For details see [18]. The following theorem is a combination of the above observation and Nikulin's existence theorem [18] applied to the genus of $S^\perp$.

Theorem 3.2 (V. Nikulin [18]). *A nondegenerate even lattice S admits a primitive extension to the lattice* $\mathbf{L}$ *if and only if the following conditions are satisfied*:

1. $\sigma_+ S \leqslant 3$, $\sigma_- S \leqslant 19$, *and* $\operatorname{rk} S + \ell(\mathcal{S}) \leqslant 22$;
2. $(-1)^{\sigma_+ S - 1}|\mathcal{S}| = \det_p \mathcal{S} \bmod (\mathbb{Z}_p^*)^2$ *for all odd prime numbers p such that* $\operatorname{rk} S + \ell_p(\mathcal{S}) = 22$;
3. *either* $\operatorname{rk} S + \ell_2(\mathcal{S}) < 22$, *or $\mathcal{S}_2$ is odd, or* $|\mathcal{S}| = \pm \det_2 \mathcal{S} \bmod (\mathbb{Z}_2^*)^2$.

3.2. $K3$**-surfaces.** Let $X \subset \mathbb{P}^3$ be a nonsingular quartic. It is a minimal $K3$-surface. Introduce the following objects:

- $L_X = H_2(X) = H^2(X)$, regarded as a lattice *via* the intersection form (we always identify homology and cohomology *via* Poincaré duality);
- $h_X \in L_X$, the class of a generic plane section of X;
- $\mathcal{F}(X) \subset H_2(X; \mathbb{Z})$, the primitive sublattice spanned over $\mathbb{Q}$ by h_X and the classes of lines $l \subset X$ (the *Fano configuration* of X);
- $\omega_X \subset L_X \otimes \mathbb{R}$, the oriented 2-subspace spanned by the real and imaginary parts of the class of a holomorphic 2-form on X (the *period* of X).

Note that ω_X is positive definite and orthogonal to h_X; furthermore, the Picard group $\operatorname{Pic} X$ equals $\omega_X^\perp \cap L_X$.

The following statement is an immediate consequence of the above description of $\operatorname{Pic} X$ and the Riemann–Roch theorem.

Lemma 3.3. *A vector $a \in L_X$ is realized by a line $l \subset X$ if and only if $a \cdot \omega_X = 0$, $a^2 = -2$, and $a \cdot h_X = 1$. Distinct lines represent distinct classes in L_X.* ◁

In view of the uniqueness part of this statement, we identify lines in X and their classes in L_X.

As is well known, the lattice L_X is isomorphic to $\mathbf{L}$; a *marking* of X is a choice of a particular isomorphism $\psi\colon L_X \to \mathbf{L}$ such that $\psi(h_X) = h \in \mathbf{L}$ and the maximal positive definite subspace $\psi(\mathbb{R}h_X \oplus \omega_X)$ is positively oriented. Consider a period ω, *i.e.*, an oriented positive definite 2-subspace $\omega \subset \mathbf{L} \otimes \mathbb{R}$ orthogonal to h. The following statement provides a criterion for the realizability of the triple $(\mathbf{L}, h, \omega)$ by a quartic, *i.e.*, the existence of a marked nonsingular quartic (X, ψ) such that ψ takes ω_X to ω. It is a combination of the surjectivity of the period map for $K3$-surfaces (see Vik. Kulikov [13]) and Saint-Donat's description [20] of projective models of $K3$-surfaces.

Proposition 3.4. *A triple $(\mathbf{L}, h, \omega)$ is realizable by a quartic $X \subset \mathbb{P}^3$ if and only if $\mathbf{L}$ contains no vector e such that $e \cdot \omega = 0$ and either*

(1) *$e^2 = -2$ and $e \cdot h = 0$, or*
(2) *$e^2 = 0$ and $e \cdot h = 2$.* ▷

Denote by Ω the space of oriented positive definite 2-subspaces $\omega \subset \mathbf{L} \otimes \mathbb{R}$ orthogonal to h and such that $\mathbb{R}h \oplus \omega$ is positively oriented. By Proposition 3.4, the image of the *period map* $(X, \psi) \mapsto \psi(\omega_X)$ is the subset $\Omega^\circ \subset \Omega$ obtained by removing the locally finite collection of codimension two subspaces

$$\Omega_e = \{\omega \in \Omega \mid \omega \cdot e = 0\},$$

where $e \in \mathbf{L}$ runs over all vectors as in Proposition 3.4(1) or (2). Restricting to Ω° Beauville's universal family [3] of marked polarized $K3$-surfaces, we obtain the following statement on marked quartics.

Proposition 3.5. *The subset $\Omega^\circ \subset \Omega$ is a fine moduli space of marked nonsingular quartics in $\mathbb{P}^3$.* ▷

Now, let $X \subset \mathbb{P}^3$ be a real nonsingular quartic. The complex conjugation induces an involutive isometry $c_X\colon L_X \to L_X$ taking h_X to $-h_X$, preserving ω_X as a subspace and reversing its orientation. In particular, it follows that the positive inertia index of the skew-invariant eigenlattice of c_X equals 2.

Consider an involutive isometry $c\colon \mathbf{L} \to \mathbf{L}$ and denote by $L_{\pm c}$ its (± 1)-eigenlattices. The involution c is called *geometric* if $h \in L_{-c}$ and $\sigma_+ L_{-c} = 2$. As explained above, a marking of a nonsingular real quartic $X \subset \mathbb{P}^3$ takes c_X to a geometric involution on $\mathbf{L}$. This involution is called the *homological type* of X; it is determined by X up to the action of $O_h^+(\mathbf{L})$. Conversely, according to Nikulin [18], any geometric involution $c\colon \mathbf{L} \to \mathbf{L}$ is the homological type of a marked nonsingular real quartic, and the periods of such quartics constitute the whole space

$$\Omega^\circ \cap \{\mathbb{R}\omega_+ \oplus \mathbb{R}\omega_- \mid \omega_\pm \in L_{\pm c} \otimes \mathbb{R}\}. \tag{3.6}$$

3.3. **Configurations.** Motivated by Lemma 3.3, we define a *line* in a 4-polarized lattice S as a vector $a \in S$ such that $a^2 = -2$ and $a \cdot h = 1$. The set of all lines in S is denoted by $\operatorname{Fn}(S)$.

Definition 3.7. A *pre-configuration* is a 4-polarized lattice S generated over $\mathbb{Q}$ by its polarization h and all lines $a \in S$. A pre-configuration S is called *hyperbolic* if $\sigma_+(S) = 1$. A *configuration* is a nondegenerate hyperbolic pre-configuration S that contains no vector e such that either

(1) $e^2 = -2$ and $e \cdot h = 0$, or
(2) $e^2 = 0$ and $e \cdot h = 2$

(*cf.* Proposition 3.4). For a pre-configuration (S, h) and a subset $A \subset \operatorname{Fn}(S)$, the notation $\operatorname{span}_h(A)$ stands for the pre-configuration $S' \subset S$ generated (over $\mathbb{Z}$) by A and h.

Remark 3.8. Let S be a nondegenerate hyperbolic pre-configuration. Then

- S contains finitely many lines, and
- any pre-configuration $S' \subset S$ is also nondegenerate and hyperbolic.

In particular, if S is a configuration, then so is S'.

Let $L \subset \mathbf{L}$ be a nondegenerate primitive polarized sublattice. An *L-configuration* is a configuration $S \subset L$ *primitive* in L. Two L-configurations $S', S'' \subset L$ are said to be *isomorphic*, or *strictly isomorphic*, if there exists an element of the group $O_h^+(\mathbf{L}, L)$ sending S' to S''. An *L-realization* of a pre-configuration S is a polarized isometry $\psi\colon S \to L$ such that the image $\operatorname{Im}\psi$ is non-degenerate, *i.e.*, $\operatorname{Ker}\psi = \ker S$. If the primitive hull $(\operatorname{Im}(\psi) \otimes \mathbb{Q}) \cap L$ is an L-configuration, the realization ψ is called *geometric*. A configuration admitting a primitive geometric L-realization is called *L-geometric* (or just *geometric* if $L = \mathbf{L}$).

Note that there is a subtle difference between $\mathbf{L}$-configurations and geometric ones: typically, the former are considered up to the action of $O_h^+(\mathbf{L})$, whereas the latter, up to abstract automorphisms of polarized lattices (*cf.* Lemma 6.19).

To simplify the classification of configurations, we introduce also the notion of weak isomorphism. Namely, two $\mathbf{L}$-configurations are said to be *weakly isomorphic* if they are taken to each other by an element of the group $O_h(\mathbf{L})$; in other words, we disregard the positive sign structure on $\mathbf{L}$. Respectively, an $\mathbf{L}$-configuration $S \subset \mathbf{L}$ is called *symmetric* if it is preserved by an element $a \in O_h(\mathbf{L}) \smallsetminus O_h^+(\mathbf{L})$; if such an element a can be chosen involutive (respectively, involutive and identical on S), the configuration S is called *reflexive* (respectively, *totally reflexive*). Putting $c = -a$, one concludes that S is totally reflexive if and only if $S \subset L_{-c}$ for some geometric involution c. It is also clear that each weak isomorphism class consists of one or two strict isomorphism classes, depending on whether the configurations are symmetric or not, respectively.

Lemma 3.9. *An* $\mathbf{L}$*-configuration* S *is totally reflexive if and only if the orthogonal complement* $S^\perp$ *contains either* $[2]$ *or* $\mathbf{U}(2)$*.*

Proof. We use the classification of geometric involutions found in [18]. On the one hand, any sublattice isomorphic to $[2]$ or $\mathbf{U}(2)$ in $h^\perp \subset \mathbf{L}$ is of the form L_{+c} for some geometric involution c. On the other hand, for any geometric involution c the sublattice L_{-c} is totally reflexive. □

3.4. **The arithmetical reduction.** Let $X \subset \mathbb{P}^3$ be a nonsingular quartic surface. Choosing a marking $\psi\colon L_X \to \mathbf{L}$, we obtain an $\mathbf{L}$-configuration $\psi(\mathcal{F}(X))$ (see Proposition 3.4). Since any two markings differ by an element of $O_h^+(\mathbf{L})$, the surface X gives rise to a well-defined isomorphism class $[\mathcal{F}(X)]$ of $\mathbf{L}$-configurations.

Two nonsingular quartics X_0 and X_1 in $\mathbb{P}^3$ are said to be *equilinear deformation equivalent* if there exists a path X_t, $t \in [0,1]$, in the space of nonsingular quartics such that the number of lines in X_t remains constant.

Theorem 3.10. *The map* $X \mapsto [\mathcal{F}(X)]$ *establishes a bijection between the set of equilinear deformation classes of nonsingular quartics in* $\mathbb{P}^3$ *and that of strict isomorphism classes of* $\mathbf{L}$*-configurations.*

Proof. For the surjectivity, we choose a period $\omega \in \Omega^\circ$ so that $\omega^\perp \cap \mathbf{L}$ represents the chosen class of $\mathbf{L}$-configurations and apply Proposition 3.4 and Lemma 3.3. For the injectivity, we prove a stronger statement, *viz.* the connectedness of the space $\Omega'(S)$ of marked nonsingular quartics whose lines are taken by the marking to the lines of a fixed $\mathbf{L}$-configuration $S \subset \mathbf{L}$. To this end, consider the spaces

$$\Omega(S) = \{\omega \in \Omega \,|\, S \subset \omega^\perp\}, \qquad \Omega^\circ(S) = \Omega(S) \cap \Omega^\circ.$$

By Proposition 3.5, the latter is a fine moduli space of marked nonsingular quartics (X, ψ) such that $\psi(\operatorname{Pic} X) \supset S$; hence, by Lemma 3.3, the space $\Omega'(S)$ is obtained from $\Omega^\circ(S)$ by removing the union of the subspaces Ω_e, where

(3) $e \in \mathbf{L} \smallsetminus S$ is such that $e^2 = -2$ and $e \cdot h = 1$.

In other words, $\Omega'(S)$ is obtained from a connected (in a sense, convex) manifold $\Omega(S)$ by removing the codimension 2 subspaces Ω_e with e as in Proposition 3.4(1), (2) or as in (3) above. This family of subspaces is obviously locally finite, and this fact implies the connectedness of the complement. □

Proposition 3.11. *Let* S *be an* $\mathbf{L}$*-configuration, and denote by* $\mathcal{X}$ *the equilinear deformation class corresponding to* S *under the bijection of Theorem 3.10. Then*:

- $\mathcal{X}$ *is invariant under the complex conjugation if and only if* S *is symmetric*;
- $\mathcal{X}$ *contains a real quartic if and only if* S *is reflexive.*

Proof. Since $\omega_{\bar{X}}$ is ω_X with the orientation reversed, the statement follows from the description of the moduli space $\Omega'(S)$ given in the proof of Theorem 3.10. □

A nonsingular quartic $X \subset \mathbb{P}^3$ is called $\mathcal{F}$*-maximal* if $\operatorname{rk} \mathcal{F}(X) = 20$.

Addendum 3.12. *The map* $X \mapsto [\mathcal{F}(X)]$ *establishes a bijection between the set of projective equivalence classes of* $\mathcal{F}$*-maximal quartics in* $\mathbb{P}^3$ *and that of isomorphism classes of* $\mathbf{L}$*-configurations of rank* 20*.*

Proof. Such quartics have maximal Picard rank, and for $S \subset \mathbf{L}$ of rank 20, the moduli space $\Omega'(S)/PGL(4,\mathbb{C})$ (*cf.* the proof of Theorem 3.10) is discrete. □

Now, consider a nonsingular real quartic $X \subset \mathbb{P}^3$ of a certain homological type $c\colon \mathbf{L} \to \mathbf{L}$. The real structure on X reverses the orientation of any real algebraic curve $C \subset X$, thus reversing the class $[C] \in L_X$. Hence, as above, considering real lines only, we can define the *real Fano configuration* $\mathcal{F}_\mathbb{R}(X)$ and the isomorphism class $[\mathcal{F}_\mathbb{R}(X)]$ of L_{-c}-configurations.

The following statements are straightforward, *cf.* (3.6).

Theorem 3.13. *The real Fano configuration of a nonsingular real quartic $X \subset \mathbb{P}^3$ of homological type $c\colon \mathbf{L} \to \mathbf{L}$ is L_{-c}-geometric. Conversely, any isomorphism class of L_{-c}-configurations is of the form $[\mathcal{F}_\mathbb{R}(X)]$ for some nonsingular real quartic $X \subset \mathbb{P}^3$ of homological type c.* ◁

Corollary 3.14. *An $\mathbf{L}$-configuration S is in the class $[\mathcal{F}_\mathbb{R}(X)]$ for some nonsingular real quartic $X \subset \mathbb{P}^3$ if and only if S is totally reflexive.* ◁

A nonsingular real quartic X is called *$\mathcal{F}_\mathbb{R}$-maximal* if $\operatorname{rk} \mathcal{F}_\mathbb{R}(X) = 20$. Even though we do not study equivariant equilinear deformations of real quartics, in the case of the maximal Picard rank, where the moduli spaces are discrete, we still have projective equivalence; the precise statement is as follows.

Addendum 3.15. *The map $X \mapsto [\mathcal{F}_\mathbb{R}(X)]$ establishes a bijection between the set of real projective equivalence classes of $\mathcal{F}_\mathbb{R}$-maximal real quartics in $\mathbb{P}^3$ of a given homological type $c\colon \mathbf{L} \to \mathbf{L}$ and that of isomorphism classes of L_{-c}-configurations of rank* 20. ◁

4. Geometry of configurations

In this section, we study the simplest properties of configurations, *viz.* those with a simple geometric interpretation. Most statements hold without the assumption that the configuration should be geometric.

4.1. **Planes.** Fix a configuration S and denote by $h \in S$ its polarization.

Lemma 4.1. *For any two distinct lines $a_1, a_2 \in S$ one has $a_1 \cdot a_2 = 0$ or 1.*

Proof. Let $a_1 \cdot a_2 = x$, and consider the subconfiguration $S' := \operatorname{span}_h(a_1, a_2)$ (see Remark 3.8). From $\det S' > 0$, one has $-1 \leqslant x \leqslant 2$. If $x = -1$, then $a_1 - a_2$ is as in Definition 3.7(1); if $x = 2$, then $a_1 + a_2$ is as in Definition 3.7(2). □

Two distinct lines $a_1, a_2 \in S$ are said to *intersect* (respectively, to be *disjoint*, or *skew*) if $a_1 \cdot a_2 = 1$ (respectively, $a_1 \cdot a_2 = 0$). We regard the set of lines $\operatorname{Fn}(S)$ as a graph, with a pair of lines (regarded as vertices) connected by an edge if and only if the lines intersect. A *subgraph* of $\operatorname{Fn}(S)$ is always assumed induced.

A *plane* in a configuration S is a collection $\{a_1, a_2, a_3, a_4\} \subset S$ of four pairwise intersecting lines.

Lemma 4.2. *For any plane $\{a_1, a_2, a_3, a_4\} \subset S$ one has $a_1 + a_2 + a_3 + a_4 = h$.*

Proof. The difference $h - (a_1 + a_2 + a_3 + a_4)$ is in the kernel of $\operatorname{span}_h(a_1, a_2, a_3, a_4)$; hence, this difference is zero, see Remark 3.8. □

Corollary 4.3 (of Lemmas 4.1 and 4.2)**.** *Let $\alpha = \{a_1, a_2, a_3, a_4\} \subset S$ be a plane and $b \in S$ a line not contained in α. Then b intersects exactly one line of α.* ◁

The *valency* $\operatorname{val} l$ of a line $l \in S$ is the number of lines in S that intersect l.

Corollary 4.4 (of Corollary 4.3)**.** *For any plane $\alpha = \{a_1, a_2, a_3, a_4\} \subset S$, one has*

$$|\mathrm{Fn}(S)| = \operatorname{val} a_1 + \operatorname{val} a_2 + \operatorname{val} a_3 + \operatorname{val} a_4 - 8.$$

Lemma 4.5. *Let a_1, $a_2 \in S$ be two intersecting lines, and assume that there is a line $b_1 \in S$ that intersects both a_1 and a_2. Then, there exists exactly one other line $b_2 \in S$ intersecting a_1 and a_2. Furthermore, the lines a_1, a_2, b_1, b_2 form a plane.*

As a consequence, if two planes α_1, $\alpha_2 \subset S$ share two lines, then $\alpha_1 = \alpha_2$.

Proof. For the existence, let $b_2 = h - (a_1 + a_2 + b_1)$ (*cf.* Lemma 4.2). For the uniqueness, consider a line c as in the statement. If $b_1 \cdot c = 0$, then the difference $h - (a_1 + a_2 + b_1 + c)$ is as in Definition 3.7(1). Otherwise, one has $b_1 \cdot c = 1$ by Lemma 4.1, and $\{a_1, a_2, b_1, c\}$ is a plane. Hence, $c = b_2$ by Lemma 4.2. □

If two distinct lines lie in a (unique) plane $\alpha \subset S$, they are said to *span* α.

4.2. **Skew lines.** We keep the notation (S, h) from the previous section. The next lemma states some properties of skew lines.

Lemma 4.6. *Consider a number of lines $a_1, \ldots, a_m, b_1, \ldots, b_n \in S$ such that all a_i are pairwise disjoint, all b_j are pairwise distinct, and $a_i \cdot b_j = 1$ for all $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then the following holds*:

1. *if $m \geqslant 2$, then all lines b_j are pairwise disjoint*;
2. *if $m = 2$, then $n \leqslant 10$*; *if $n = 9$, then there exists a unique other line b_{10} such that $a_i \cdot b_{10} = 1$ for $i = 1, 2$*; cf. *also Corollary 5.43 below*;
3. *if $m = 4$, then $n \leqslant 4$*; *if $n = 3$, then there exists a unique other line b_4 such that $a_i \cdot b_4 = 1$ for $i = 1, 2, 3$*; *for this line, also $a_4 \cdot b_4 = 1$*;
4. *if $m = n = 4$, then any other line $c \in S$ intersects exactly two of the given lines $a_1, \ldots, a_4$, $b_1, \ldots, b_4$*;
5. *if $m \geqslant 3$, then $n \leqslant 4$*; *if $m \geqslant 5$, then $n \leqslant 2$.*

Proof. Item (1) is a partial restatement of Lemma 4.5. The next two statements are proved similarly, with

$$b_{10} = 4h - 3(a_1 + a_2) - (b_1 + \ldots + b_9)$$

in item (2) and

$$b_4 = 2h - (a_1 + \ldots + a_4 + b_1 + b_2 + b_3)$$

in item (3). In the latter case, if $a_4 \cdot b_4$ were 0, the vector $a_1 + \ldots + b_4 - 2h$ would be as in Definition 3.7(1). The expression for b_4 proves also item (4), and item (5) is a simple consequence of item (3). □

Recall that our ultimate goal is the study of the configuration S of lines in a nonsingular quartic surface X. From this perspective, as the name suggests, a plane is the subconfiguration cut on X by a plane in $\mathbb{P}^3$, provided that the intersection splits completely into components of degree one. A collection $a_1, \ldots, a_4, b_1, \ldots, b_4$ as in Lemma 4.6(3) and (4) can similarly be interpreted as the intersection of X with a quadric (the lines a_i and b_j lying in the two distinct families of generatrices), and a subconfiguration as in Lemma 4.6(2) is (probably, a special case of) the intersection of X with another quartic. The following lemma, not used in the paper, is in the same spirit: it describes the intersection of X with a cubic. For the statement, define a *double sextuple* as a collection of lines $a_1, \ldots, a_6, b_1, \ldots, b_6$ in a configuration S intersecting as follows:

$$a_i \cdot b_j = 1 - \delta_{ij} \qquad (4.7)$$

(where δ_{ij} is the Kronecker symbol).

Lemma 4.8. *Let $A' := \{a_1, \ldots, a_6, b_1, \ldots, b_5\} \subset S$ be a collection of lines which satisfy (4.7). Then there is a unique line $b_6 \in S$ completing A' to a double sextuple A. Furthermore, all elements of A are pairwise distinct, the lines a_i are pairwise disjoint, the lines b_j are pairwise disjoint, and any other line $c \in S$ intersects exactly three elements of A.*

Proof. The twelfth line is

$$b_6 = 3h - (a_1 + \ldots + a_6 + b_1 + \ldots + b_5),$$

and the other statements are immediate, *cf.* the proof of Lemma 4.6. □

4.3. **Pencils.** Let $X \subset \mathbb{P}^3$ be a nonsingular quartic such that $\operatorname{rk} \mathcal{F}(X) \geqslant 2$. Fix a line $l \subset X$. The pencil of planes through l gives rise to an elliptic pencil $X \to \mathbb{P}^1$. Each fiber containing a line is reducible: it splits either into three lines or a line and a conic; in the former case, the three lines and l form a plane in $\mathcal{F}(X)$. Clearly, the lines in X contained in the fibers of the pencil defined by l are precisely those intersecting l. Motivated by this observation, we define a *pencil* $\mathcal{P}$ in a configuration (S, h) as a set of lines satisfying the following properties:

- all lines in $\mathcal{P}$ intersect a given line l, called the *axis* of $\mathcal{P}$;
- if a_1, $a_2 \in \mathcal{P}$ and $a_1 \cdot a_2 = 1$, then $h - l - a_1 - a_2 \in \mathcal{P}$ (*cf.* Lemma 4.2).

Lemma 4.5 implies that

$$a \sim b \text{ if } a = b \text{ or } a \cdot b = 1$$

is an equivalence relation on $\mathcal{P}$. The equivalence classes are called the *fibers* of $\mathcal{P}$. The number m of lines in a fiber may take values 3 or 1; a fiber consisting of m lines is called an *m-fiber*, and the number of such fibers is denoted by $\#_m(\mathcal{P})$. By Corollary 4.3, $\mathcal{P}$ has a unique axis whenever $\#_3(\mathcal{P}) \geqslant 1$ and $\#_3(\mathcal{P}) + \#_1(\mathcal{P}) \geqslant 2$.

Each line $l \in S$ gives rise to a well-defined pencil

$$\mathcal{P}(l) := \{a \in \operatorname{Fn} S \mid a \cdot l = 1\};$$

such a pencil is called *maximal*. Any line $a \in S$ disjoint from l is called a *section* of $\mathcal{P}(l)$ or any subpencil thereof. The set of sections of $\mathcal{P}$ depends on the ambient (pre-)configuration S; it is denoted by $S(\mathcal{P})$. By definition,

$$S(\mathcal{P}) = \{a \in \operatorname{Fn}(S) \mid a \cdot l = 0\}.$$

Clearly, for any line $l \in S$, one has

$$\operatorname{val} l = |\mathcal{P}(l)| = 3\,\#_3(\mathcal{P}(l)) + \#_1(\mathcal{P}(l)).$$

The number $\operatorname{mult} l := \#_3(\mathcal{P}(l))$ is called the *multiplicity* of l. Alternatively, $\operatorname{mult} l$ is the number of distinct planes containing l.

Two pencils $\mathcal{P}_1$, $\mathcal{P}_2$ are called *obverse* if their axes are disjoint; otherwise, the pencils are called *adjacent*. The following lemma is an immediate consequence of Lemmas 4.5 and 4.6(2).

Lemma 4.9. *Let $\mathcal{P}_1 \neq \mathcal{P}_2$ be two pencils. Then*

1. $|\mathcal{P}_1 \cap \mathcal{P}_2| \leqslant 10$ *if $\mathcal{P}_1$, $\mathcal{P}_2$ are obverse, and*
2. $|\mathcal{P}_1 \cap \mathcal{P}_2| \leqslant 2$ *if $\mathcal{P}_1$, $\mathcal{P}_2$ are adjacent.* ◁

4.4. **Combinatorial invariants.** A pencil $\mathcal{P}$ is often said to be of *type* (p,q), where $p := \#_3(\mathcal{P})$ and $q := \#_1(\mathcal{P})$. If an $\mathbf{L}$-realization ψ is fixed, the pencil is called *primitive* or *imprimitive* if so is the sublattice $\operatorname{span}_h \psi(\mathcal{P}) \subset \mathbf{L}$. In this case, the type is further refined to $(p,q)^\bullet$ and $(p,q)^\circ$, respectively. A geometric configuration containing a maximal pencil $\mathcal{P}$ of type $(p,q)^*$ is called a $(p,q)^*$-*configuration*, and the pair $(S,\mathcal{P})$ is called a $(p,q)^*$-*pair*. The multiset

$$\mathfrak{ps}(S) := \bigl\{\text{type of } \mathcal{P}(l) \bigm| l \in \operatorname{Fn}(S)\bigr\}$$

is called the *pencil structure* of a configuration S. We usually represent $\mathfrak{ps}(S)$ in the partition notation (see, *e.g.*, §6.2 below): a "factor" $(p,q)^a$ means that S has a pencils of type (p,q).

The *linking type* $\operatorname{lk}(\mathcal{P}_1,\mathcal{P}_2)$ of a pair of obverse pencils is the pair (μ_1,μ_3), where $\mu_1 := |\mathcal{P}_1 \cap \mathcal{P}_2|$ and μ_3 is the number of lines in $\mathcal{P}_1 \cap \mathcal{P}_2$ that belong to a 3-fiber both in $\mathcal{P}_1$ and $\mathcal{P}_2$. If $\mathcal{P}_i = \mathcal{P}(l_i)$, $i = 1,2$, we also use the notation $\operatorname{lk}(l_1,l_2)$. The multiset

$$\mathfrak{ls}(S) := \bigl\{\operatorname{lk}(l_1,l_2) \bigm| l_1,l_2 \in \operatorname{Fn}(S),\ l_1 \cdot l_2 = 0\bigr\}$$

is called the *linking structure* of S.

Clearly, both $\mathfrak{ps}(S)$ and $\mathfrak{ls}(S)$ are invariant under isomorphisms.

5. The arithmetics of pencils

In this section, we study the more subtle properties of geometric configurations related to their primitive embeddings to $\mathbf{L}$.

5.1. **Notation and setup.** Throughout this section, we consider a pencil $\mathcal{P}$ of a certain type (p,q). Thus, we have the sets $\operatorname{fb}_3\mathcal{P} = \{1,\ldots,p\}$ and $\operatorname{fb}_1\mathcal{P} = \{1,\ldots,q\}$ of the 3- and 1-fibers of $\mathcal{P}$, respectively, and the full set $\operatorname{fb}\mathcal{P} := \operatorname{fb}_3\mathcal{P} \sqcup \operatorname{fb}_1\mathcal{P}$ of fibers is their disjoint union. We regard $\mathcal{P}$ as a pencil in the "minimal" configuration $P := P_{p,q}$, which is generated over $\mathbb{Z}$ by $\mathcal{P}$ itself, the axis l, and the polarization h. We also keep in mind a geometric realization $\psi\colon P \to \mathbf{L}$, identifying $\mathcal{P}$ and P with their images in $\mathbf{L}$ and denoting by $\tilde{P}$ the primitive hull $(P \otimes \mathbb{Q}) \cap \mathbf{L}$.

When speaking about sections of $\mathcal{P}$, we assume $\mathcal{P}$ embedded to a configuration S, which is usually not specified. (One can consider the minimal configuration generated by P and the sections in question.) However, *we always assume that the realization of P extends to a geometric realization $S \to \mathbf{L}$.*

The group of symmetries of $\mathcal{P}$ is obviously

$$\mathbb{G}_{p,q} := (\mathbb{S}_3^p \rtimes \mathbb{S}_p) \times \mathbb{S}_q.$$

In addition to h and l, consider the following classes in $P_{p,q}$:

- $m_{i,j}$, $i \in \operatorname{fb}_3\mathcal{P}$, $j \in \mathbb{Z}/3$, the lines in the 3-fibers;
- n_k, $k \in \operatorname{fb}_1\mathcal{P}$, the lines in the 1-fibers.

Then $P_{p,q}$ is the hyperbolic lattice freely generated by h, l, $m_{i,j}$, $i \in \operatorname{fb}_3\mathcal{P}$, $j = \pm 1$, and n_k, $k \in \operatorname{fb}_1\mathcal{P}$. For the lines $m_{i,\pm 1}$, we will also use the shortcut $m_{i,\pm}$.

Observation 5.1. One has $\det P_{p,q} = -3^{p+2}(-2)^q$. The 3-primary part $\operatorname{discr}_3 P_{p,q}$ contains the classes represented by the following mutually orthogonal vectors:

- $\lambda := \frac{1}{3}(l-h)$: one has $\lambda^2 = 0$ and $\lambda \cdot h = \lambda \cdot l = -1$;
- $\mu_i = \mu_{i,0} := \frac{1}{3}(m_{i,+} - m_{i,-})$, $i \in \operatorname{fb}_3\mathcal{P}$: one has $\mu_i^2 = -\frac{2}{3}$ and $\mu_i \cdot h = 0$.

If $r := p + q - 1 \neq 0 \bmod 3$, then $\operatorname{discr}_3 P_{p,q}$ is generated by μ_i, $i \in \operatorname{fb}_3\mathcal{P}$, and the order 9 class of the vector

- $\upsilon := \frac{1}{3}\bigl(l - r\lambda + \sum_{i=1}^{p}(m_{i,+} + m_{i,-}) - \sum_{k=1}^{q} n_k\bigr)$;

note that $3\upsilon = -r\lambda \neq 0 \bmod P$. Hence, in this case the subgroup of elements of order 3 is generated by λ and μ_i. If $p + q = 1 \bmod 3$, then $\operatorname{discr}_3 P_{p,q}$ is generated by λ, μ_i, and the order 3 class of

- $\omega := \frac{1}{3}\bigl(l + \sum_{i=1}^{p}(m_{i,+} + m_{i,-}) - \sum_{k=1}^{q} n_k\bigr)$.

The 2-primary part $\operatorname{discr}_2 P_{p,q}$ is generated by the classes of $3\nu_k$, where

- $\nu_k := n_k^* = -\frac{1}{2}(\lambda + n_k)$, $k \in \operatorname{fb}_1 \mathcal{P}$: one has $\nu_k^2 = -\frac{1}{2}$ and $\nu_k \cdot h = 0$.

The class $\mu_i \in \operatorname{discr} P_{p,q}$ is also represented by the vector $\bar\mu_i^+ := \frac{1}{3}(m_{i,+} + 2m_{i,-})$, so that one has $\bar\mu_i^2 = -\frac{2}{3}$ and $\bar\mu_i \cdot h = 1$. The class $-\mu_i \in \operatorname{discr} P_{p,q}$ is also represented by $\bar\mu_i^- := \frac{1}{3}(2m_{i,+} + m_{i,-})$. For any line $a \in \mathcal{P}$, the class λ is represented by the vector $\lambda + a \in h^\perp$, so that one has $(\lambda + a)^2 = -2$.

The following two statements are immediate.

Lemma 5.2. *For any triple of distinct indices $i, j, k \in \operatorname{fb}_3 \mathcal{P}$ and any $u \in \mathbb{Z}/3$, the classes $\pm\lambda$ and $u\lambda \pm \mu_i \pm \mu_j \pm \mu_k$ are represented by vectors of square (-2) in $h^\perp \subset P_{p,q}$. Hence, these classes cannot belong to the pivot $\tilde P/P$.* ◁

Lemma 5.3. *The sum of any four distinct elements of the form $3\nu_k$, $k \in \operatorname{fb}_1 \mathcal{P}$, is represented by a vector of square (-2) in $h^\perp \subset P_{p,q}$. Hence, the class of such a sum cannot belong to the pivot $\tilde P/P$.* ◁

5.2. **Euler's bound.** We start with eliminating very large pencils.

Proposition 5.4. *The type (p, q) of a pencil contained in a geometric configuration satisfies the inequalities*

$$3p + 2q \leqslant 24 \quad \textit{and} \quad 3p + q \leqslant 20.$$

Corollary 5.5 (*cf.* Rams, Schütt [19])**.** *The valency of any line l in a geometric configuration S does not exceed* 20*.* ◁

In the real case, there is an additional restriction to the types of pencils.

Proposition 5.6. *A pencil $\mathcal{P}$ contained in a totally reflexive geometric configuration cannot be of type $(6, 0)^\bullet$ or $(5, q)$, $q \geqslant 2$.*

Proof of Propositions 5.4 and 5.6. Assume that $(p, q) = (7, 0)$. By Observation 5.1, the isotropic elements in $\operatorname{discr}_3 P_{7,0}$ are:

1. the classes mentioned in Lemma 5.2;
2. classes of the form $u\lambda + \sum_{i\in I} \pm\mu_i$, where $u \in \mathbb{Z}/3$ and $I \subset \operatorname{fb}_3 \mathcal{P}$, $|I| = 6$; all these classes form a single orbit of $\mathbb{G}_{7,0}$;
3. classes of the form (up to sign) $\omega + u\lambda - \sum_{i\in I} \pm\mu_i$, where $I \subset \operatorname{fb}_3 \mathcal{P}$ is any subset and $u = (5 - |I|) \bmod 3$.

Each class as in item 3 is represented by a vector of square (-2) orthogonal to h, *viz.* $\omega + (5 - |I|)\lambda - \sum_{i\in I} \bar\mu_i^\pm$. Hence, neither (1) nor (3) can belong to the pivot $\tilde P/P$. On the other hand, by Theorem 3.2, one has $\ell_3(\tilde P/P) \geqslant 2$ and $\tilde P/P$ must contain two distinct nontrivial orthogonal vectors β_1, β_2 as in (2). On the other hand, if both vectors are as in (2), then at least one of their linear combinations is as in (1), *cf.* [10].

Similar arguments apply to the other border cases: by Theorem 3.2, one has

- $\ell_3(\tilde P/P) \geqslant 1$ if $(p, q) = (5, 4)$ (use Lemma 5.2),

- $\ell_2(\tilde{P}/P) \geqslant 1$ if $(p,q) = (3,8)$,
- $\ell_2(\tilde{P}/P) \geqslant 2$ if $(p,q) = (1,11)$ (use Lemma 5.3), and
- $\ell_2(\tilde{P}/P) \geqslant 3$ if $(p,q) = (0,13)$ (use Lemma 5.3).

In the case $(p,q) = (3,8)$, the only isotropic element allowed by Lemma 5.3 is the characteristic element $\nu := \sum_{k=1}^{8} \nu_k$. The discriminant form $\nu^\perp/\nu$ is even, and the new lattice does not embed to $\mathbf{L}$ by Theorem 3.2.

For Proposition 5.6, one uses Observation 5.1 and Theorem 3.2; the latter should be applied to either $P \oplus [2]$ or an appropriate finite index extension of $P \oplus [2]$ or $P \oplus \mathbf{U}(2)$, see Lemma 3.9. □

The conclusion of Proposition 5.4 can be recast as follows: for any line l in a geometric configuration S, one has $\operatorname{val} l \leqslant 20$ and $\operatorname{mult} l \leqslant 6$; furthermore,

$$\begin{aligned} &\text{if } \operatorname{mult} l \leqslant \ \ 0,\ \ 1,\ \ 2,\ \ 3,\ \ 4,\ \ 5,\ \ 6 = \max, \\ &\text{then } \operatorname{val} l \leqslant 12, 13, 15, 16, 18, 18, 20 = \max,\ \text{respectively.} \end{aligned} \tag{5.7}$$

It follows from (5.7) that $\max\{\operatorname{val} l \mid l \in \operatorname{Fn}(S)\} \leqslant 18$ if and only if S does not contain a pencil of type $(6,q)$, $q \geqslant 1$.

Remark 5.8. Interpreting pencil geometrically as in §4.3, one can easily see that the first inequality $3p + 2q \leqslant 24$ in Proposition 5.4 is nothing but the well-known bound on the number and types of singular fibers in an elliptic pencil.

5.3. **Coordinates.** Consider a section s of a pencil $\mathcal{P}$. By Corollary 4.3, for each index $i \in \operatorname{fb}_3 \mathcal{P}$, the section s intersects exactly one of the three lines $m_{i,j}$, $j \in \mathbb{Z}/3$; the corresponding index $\epsilon_i := j \in \mathbb{Z}/3$ is called the *i-th 3-coordinate* of s. Introduce also the *k-th 1-coordinate* as the residue $\varrho_k := (s \cdot n_k) \bmod 2 \in \mathbb{Z}/2$, $k \in \operatorname{fb}_1 \mathcal{P}$.

We will treat the coordinate space $\mathcal{C}_{p,q} := (\mathbb{Z}/3)^p \times (\mathbb{Z}/2)^q$ as an abelian group, even though only few linear combinations of coordinate vectors have invariant meaning. To avoid confusion with the operations in lattices, we will use $\oplus$ and $\ominus$ for the addition and subtraction in $\mathcal{C}_{p,q}$, respectively.

Convention 5.9. Given sections $s, s_1, s_2, \ldots$ of $\mathcal{P}$ and $u = 1, 3$, we will use the following notation:

- $\epsilon_i := \epsilon_i(s)$ and $\varrho_k := \varrho_k(s)$ are, respectively, the 3- and 1-coordinates of s;
- $[s]$ or $\bar{s} := [\epsilon_1, \ldots, \epsilon_p; \varrho_1, \ldots, \varrho_q]$ is the sequence of all coordinates of s;
- $|s|_u$ is the number of non-vanishing u-coordinates of s;
- $|s_1 \ominus s_2|_u$ is the number of positions where the u-coordinates of s_1, s_2 differ;
- $\{s_1 * s_2 * \ldots\}_3 := \{i \in \operatorname{fb}_3 \mathcal{P} \mid \epsilon_i(s_1) = \epsilon_i(s_2) = \ldots\}$;
- $\{s_1 * s_2 * \ldots\}_1 := \{k \in \operatorname{fb}_1 \mathcal{P} \mid \varrho_k(s_1) = \varrho_k(s_2) = \ldots = 1\}$;
- $\{\ldots\} := \{\ldots\}_3 \sqcup \{\ldots\}_1$ (regarded as a set of fibers of $\mathcal{P}$);
- $|\ldots|_*$ is the cardinality of the set $\{\ldots\}_*$ for $* = 1, 3$, or empty;
- $\mathbb{I} := \mathbb{I}_{p,q} = [0, \ldots, 0; 1, \ldots, 1] \in \mathcal{C}_{p,q}$.

The same notation applies if all or some of s, s_1, s_2 are elements of the coordinate space $\mathcal{C}_{p,q}$. The 3-coordinates $\epsilon_i(s)$, numbers $|s|_3$, and element $\mathbb{I} \in \mathcal{C}$ depend on the indexing of the lines in the 3-fibers; however, the sets $\{\ldots\}_3$, numbers $|s_1 \ominus s_2|_3$, and expressions of the form

$$\mathbb{I} \oplus \bar{s}, \quad \bar{s}_1 \oplus \bar{s}_2 \oplus \bar{s}_3 = \mathbb{I}, \quad \text{or} \quad \bar{s}_3 = \mathbb{I} \ominus \bar{s}_1 \ominus \bar{s}_2$$

have invariant meaning. Note also the difference between the definitions of $\{\ldots\}_3$ and $\{\ldots\}_1$: in the former case, we count *all* equal coordinates, whereas in the latter, only the *non-vanishing* ones.

The following statements are immediate consequences of Lemmas 4.5 and 4.6.

Lemma 5.10. *Let s_1, s_2 be two sections of $\mathcal{P}$ and $s_1 \cdot s_2 = 1$. Then $|s_1 * s_2| \leqslant 1$. If $|s_1 * s_2| = 1$, then there is a section s satisfying $\bar{s} \oplus \bar{s}_1 \oplus \bar{s}_2 = \mathbb{I}$; the sections s, s_1, s_2 and the only line $a \in \mathcal{P}$ intersecting all three of them constitute a plane.* ◁

Lemma 5.11. *Let s_1, s_2, s_3 be distinct sections of $\mathcal{P}$. Then*:

(1) *one has $|s_1 * s_2| \leqslant 4$;*
(2) *if $|s_1 * s_2| = 4$, there is a unique section s such that $\bar{s} \oplus \bar{s}_1 \oplus \bar{s}_2 = \mathbb{I}$;*
(3) *if $|s_1 * s_2 * s_3| = 3$, the pencil $\mathcal{P}$ is not maximal.* ◁

Remark 5.12. In Lemmas 5.10 and 5.11, as well as in the other similar places below, the existence statement means that s is a certain (explicit, but not specified) integral linear combination of the other sections involved and generators of $\tilde{P}$.

Corollary 5.13. *If $p \geqslant 5$, then, for any configuration $S \supset P$, the coordinate map $c\colon S(\mathcal{P}) \to \mathcal{C}_{p,q}$, $s \mapsto [s]$, is injective.* ◁

The injectivity of c for types $(4, *)$ and $(3, 7)$ is discussed in §5.7 below.
The next corollary deals with an obverse pencil in a configuration $S \supset \mathcal{P}$.

Corollary 5.14. *Given a section $s_0 \in S(\mathcal{P})$, consider $s, s_1, s_2 \in \mathcal{P}(s_0) \cap S_k(\mathcal{P})$ and assume that $s_1 \cdot s_2 = 1$. Then*:

(1) *one has $|s * s_0| \leqslant 1$;*
(2) $\{s_0 * s_1\} = \{s_0 * s_2\} = \{s_1 * s_2\} = \{s_0 * s_1 * s_2\}$;
(3) *if $\mathcal{P}$ is maximal, then $|s_1 * s_2| = 1$;*
(4) *if $\mathcal{P}$ is maximal, then s is in a 1-fiber of $\mathcal{P}(s_0)$ if and only if $|s * s_0| = 0$.*

Proof. Statement (1) is a paraphrase of Lemma 5.10. For (2) and (3), just observe that s_0, s_1, s_2 span a plane, and the forth line a of this plane must intersect l, see Corollary 4.3; hence, either $a \in \mathcal{P}$ or $\mathcal{P}$ is not maximal. Finally, Statement (4) is a paraphrase of (3). □

Denote $D := 2p + \frac{1}{2}q - 2$ and, given a collection of sections $s_1, \ldots, s_k$, let

$$r_{ij} := (s_1 \cdot s_2) + \tfrac{1}{9}D + \tfrac{1}{2}|s_1 * s_2|_1 - \tfrac{1}{6}(|s_1|_1 + |s_2|_1) - \tfrac{1}{3}|s_1 \ominus s_2|_3, \quad 1 \leqslant i, j \leqslant k,$$

and define the *determinant*

$$\det(s_1, \ldots, s_k) := \det[-r_{ij}]_{1 \leqslant i,j \leqslant k}.$$

The following lemma is a simple sufficient condition for the existence of a collection of sections in terms of their coordinates and pairwise intersections: the orthogonal complement $P^\perp$ in any configuration $S \supset \mathcal{P}$ must be negative definite.

Lemma 5.15. *For any collection $s_1, \ldots, s_k$ of sections one has $\det(s_1, \ldots, s_k) \geqslant 0$. If $\det(s_1, \ldots, s_k) = 0$, then the sections are linearly dependent.* ◁

5.4. **Combinatorial rigidity.** The group $\mathbb{G}_{p,q}$ acts on the coordinate space $\mathcal{C}_{p,q}$. Furthermore, given two configurations $S, S' \supset \mathcal{P}$, any isometry $(S, \mathcal{P}) \to (S', \mathcal{P})$ induces an injection $\bar{S} \hookrightarrow \bar{S}'$, which is the restriction of an element of $\mathbb{G}_{p,q}$. (Here, $\bar{S}$ and $\bar{S}'$ are the images of $S(\mathcal{P})$ and $S'(\mathcal{P})$, respectively, under the coordinate map.) A configuration $S \supset \mathcal{P}$ or, more precisely, pair $(S, \mathcal{P})$ is called (*combinatorially*) *rigid* if, for any configuration $S' \supset \mathcal{P}$, any bijection $g(\bar{S}) = \bar{S}'$ restricted from an element $g \in \mathbb{G}_{p,q}$ is induced by an isometry $(S, \mathcal{P}) \to (S', \mathcal{P})$.

We say that S or $(S,\mathcal{P})$ is *generated by a subset* $\bar{A}\subset\bar{S}$ if $S=\bigl(\tilde{P}+\sum_{\bar{s}\in\bar{A}}\mathbb{Z}s\bigr)/\ker$; if $\bar{A}=\bar{S}$, then S is said to be *generated by sections*. For such a configuration, an obvious sufficient condition for the combinatorial rigidity is that the intersection $s_1\cdot s_2$ of a pair of sections s_1,s_2 such that $\bar{s}_1,\bar{s}_2\in\bar{A}$ is determined by their images $\bar{s}_1,\bar{s}_2$, *i.e.*, for any other configuration $S'\supset\mathcal{P}$ and pair of sections $s'_1,s'_2\in S'(\mathcal{P})$ such that $\bar{S}'=\bar{S}$ and $\bar{s}'_1=\bar{s}_1$, $\bar{s}'_2=\bar{s}_2$, one has $s'_1\cdot s'_2=s_1\cdot s_2$. By Lemma 5.10, an ambiguity may arise only if $|s_1*s_2|\leqslant 1$. The following statement is a partial converse of Lemma 5.10; we do not need to assume that the configuration $S\supset\mathcal{P}$ is geometric.

Lemma 5.16. *Let $p=6$, $(p,q)=(5,3)$, $p=4$ and $q\geqslant 4$, or $(p,q)=(3,7)$. Consider a pair of sections $s_1,s_2\in S(\mathcal{P})$ such that $|s_1*s_2|=1$. Then, $\mathcal{P}$ has a pair of sections s'_1,s'_2 such that $s'_1\cdot s'_2=1$ and $[s'_i]=\bar{s}_i$, $i=1,2$, if and only if there is a section s such that $\bar{s}\oplus\bar{s}_1\oplus\bar{s}_2=\mathbb{I}$.*

Proof. The necessity is given by Lemma 5.10. For the converse, it suffices to show that three sections s,s_1,s_2 as in the statement cannot be pairwise disjoint. Most such triples are eliminated by Lemma 5.15, and the few remaining ones violate condition (1) in Definition 3.7. □

5.5. **Primitivity and rigidity for type** $(6,*)$**.** Primitive and imprimitive pencils of type $(6,*)$ exhibit very different behaviour. Here, we start with a few common observations; imprimitive pencils are treated separately in the next section.

Proposition 5.17. *Assume that $p=6$. Then the following holds*:

1. *if $\mathcal{P}$ is not maximal or $q\geqslant 1$, then $\mathcal{P}$ is imprimitive*;
2. *if $\mathcal{P}$ is imprimitive, then $\tilde{P}/P=\langle\beta\rangle$, $\beta:=\sum_{i=1}^{6}\mu_i$, up to automorphism.*

Proof. The imprimitivity follows from Theorem 3.2, and the only possible nontrivial pivot is given by Observation 5.1 and Lemma 5.2. □

Lemma 5.18. *Let $(p,q)=(6,0)$. Consider a geometric configuration $S\supset\mathcal{P}$, let $\bar{S}$ be the image of $S(\mathcal{P})$ under the coordinate map, and, for a pair $s_1,s_2\in S(\mathcal{P})$, denote $\bar{s}:=\mathbb{I}\ominus\bar{s}_1\ominus\bar{s}_2\in\mathcal{C}_{6,0}$. Then the following holds*:

1. *if $|s_1*s_2|=0$ and $s_1\cdot s_2=0$, then $\mathcal{P}$ is imprimitive and $\frac{1}{3}(s_1-s_2)\in\tilde{P}$*;
2. *if $|s_1*s_2|=0$ or 3 and $\bar{s}\in\bar{S}$, then $\mathcal{P}$ is imprimitive*;
3. *if $|s_1*s_2|=1$, then $\bar{s}\in\bar{S}$ if and only if $s_1\cdot s_2=1$*;
4. *if $|s_1*s_2|=4$, then $\bar{s}\in\bar{S}$.*

Proof. Statement (1): the two vectors are linearly dependent by Lemma 5.15; then $s_1-s_2=\beta$ up to automorphism.

Statement (2), $|s_1*s_2|=0$: if $\mathcal{P}$ is primitive, then $s\cdot s_1=s\cdot s_2=s_1\cdot s_2=1$ by Statement (1); hence, the three sections span a plane, and the forth line of this plane is in $\mathcal{P}(l)\smallsetminus\mathcal{P}$, which contradicts Proposition 5.17(1).

Statement (2), $|s_1*s_2|=3$: the imprimitivity of $\operatorname{span}_h(\mathcal{P},s_1,s_2,s)$ is given by Theorem 3.2, and the enumeration of isotropic elements not realized by vectors e as in Definition 3.7(1) shows that the pivot is generated by β (up to isomorphism).

Statements (3) and (4) follow from Lemmas 5.16 and 4.6(3), respectively. □

Corollary 5.19. *Any $(6,0)^\bullet$-configuration generated by sections is rigid.* ◁

5.6. **Triplets of sections.** In this section, we study in more detail an imprimitive pencil of type $(6,0)^{\circ}$. Thus, we fix a pencil $\mathcal{P}$ and number the lines m_* in the fibers so that the pivot $\tilde{P}/P$ is generated by the element β introduced in Proposition 5.17. Then, for any section s,

$$\epsilon_1(s) + \ldots + \epsilon_6(s) = 0 \bmod 3. \tag{5.20}$$

The group $O_h(\tilde{P}, l)$ is obviously the subgroup

$$\tilde{\mathbb{G}} := ((\mathbb{Z}/3)^5 \rtimes \mathbb{Z}/2) \rtimes \mathbb{S}_6 \subset \mathbb{G}_{6,0}; \tag{5.21}$$

indeed, the choice of β gives rise to a distinguished cyclic order in each fiber, which is well defined up to simultaneous reversal. This group has a distinguished subgroup of order 3: it is generated by the permutations $\sigma^{\pm 1}\colon m_{i,j} \mapsto m_{i,j\pm 1}$, $i \in \operatorname{fb}_3 \mathcal{P}$, $j \in \mathbb{Z}/3$. A choice of one of these two generators makes $\mathcal{C}_{6,0}$ an $\mathbb{F}_3$-affine space.

Consider a configuration $S \supset \tilde{P}$ and let $\bar{S} \subset \mathcal{C}_{6,0}$ be the image of $S(\mathcal{P})$ under the coordinate map.

Lemma 5.22. *The set $\bar{S}$ is σ-invariant,* i.e., *$\bar{s}_{\pm} := \sigma^{\pm 1}\bar{s} \in \bar{S}$ whenever $\bar{s} \in \bar{S}$. The three sections $\bar{s}, \bar{s}_{\pm}$ are pairwise disjoint.*

Proof. Up to automorphism, one can assume that $\bar{s} = [1, \ldots, 1]$. Then the two other sections are $s + \beta$ and $s - 2h + 2l + \sum_{i=1}^{6}(m_{i,1} + m_{i,-1}) - \beta$. $\square$

A subset $\{s, s_{\pm}\} \subset S(\mathcal{P})$ or $\{\bar{s}, \bar{s}_{\pm}\} \subset \bar{S}$ as in Lemma 5.22 is called a *triplet*. Two sections $s_1, s_2 \in S(\mathcal{P})$ are said to be *equivalent*, $s_1 \sim s_2$, if they belong to one triplet. Note that $|s_1 * s_2| = 0$ whenever $s_1 \sim s_2$.

Lemma 5.23. *For a pair of sections $s_1, s_2 \in S(\mathcal{P})$, one has $s_1 \cdot s_2 = 1$ if and only if $|s_1 * s_2| \leqslant 1$ and $s_1 \not\sim s_2$.*

Proof. If $|s_1 * s_2| = 0$ and $s_1 \cdot s_2 = 0$, Lemma 5.18(1) and the fact that $\ell_3(\tilde{P}/P) = 1$ imply that $s_1 \sim s_2$. If $|s_1 * s_2| = 1$, then, using (5.20) and Lemmas 5.18(4) and 5.22, one can easily show that $\mathbb{I} \ominus \bar{s}_1 \ominus \bar{s}_2 \in \bar{S}$; then, $s_1 \cdot s_2 = 1$ by Lemma 5.18(3). $\square$

Corollary 5.24. *Any $(6,*)^{\circ}$-configuration generated by sections is rigid.* $\triangleleft$

Note that, for $(6,*)^{\circ}$-configurations, the rigidity holds in a very strong sense: the intersection of two sections is completely determined by their coordinates.

The set of triplets can be coordinatized by the affine space

$$\mathcal{A} := \{\bar{s} \in \mathcal{C}_{6,0} \,|\, \bar{s} \text{ satisfies } (5.20)\}/\sigma.$$

In fact, $\mathcal{A}$ is naturally a principal homogeneous space over the subquotient $\lambda^{\perp}/\lambda$ of the discriminant $\operatorname{discr} \tilde{P}$. Denote by $\mathfrak{q}$ the descent of the discriminant form of $\tilde{P}$ reduced modulo $\mathbb{Z}$; then, clearly, $\mathfrak{q}(\bar{s}_1 - \bar{s}_2) = \frac{1}{3}(|s_1 \ominus s_2| \bmod 3)$. Comparing the orders, one can see that the group $\tilde{\mathbb{G}}/\sigma$ is isomorphic to the full group $O(\mathfrak{q}) \rtimes \mathcal{A}$ of $\mathfrak{q}$-isometries of $\mathcal{A}$. In other words, any $\tilde{\mathbb{G}}$-invariant property of a set of sections $\bar{S} \subset \mathcal{C}_{6,0}$ satisfying (5.20) and Lemma 5.22 can be stated as a "metric" (with respect to $\mathfrak{q}$) property of the projection $\bar{\mathcal{S}}$ of this set to $\mathcal{A}$.

Below, we state two properties that hold for any configuration S, not necessarily geometric. Recall that the lines in $\lambda^{\perp}/\lambda$ can be subdivided into

- 15 *positive* lines ℓ^+ and 15 *negative* lines ℓ^-, with $q|_{\ell^{\pm}} \cong \langle \pm\frac{1}{3} \rangle$, and
- 10 *isotropic* lines ℓ^0, with $q|_{\ell^0} \equiv 0$.

The planes in $\lambda^{\perp}/\lambda$ can be subdivided into

- 20 *positive* planes π^+ and 20 *negative* planes π^-, with $\pi^\pm \cong \ell^\pm \oplus \ell^0$,
- 45 *hyperbolic* planes, isomorphic to $\ell^+ \oplus \ell^-$, and
- 45 *definite* planes, isomorphic to $\ell^+ \oplus \ell^+ \cong \ell^- \oplus \ell^-$.

(There are no isotropic planes.) The same terminology applies to the lines/planes in $\mathcal{A}$, according to the underlying vector space. The group $O(\mathfrak{q})$ acts transitively on the set of lines/planes of the same type.

Lemma 5.25. *For any configuration $S \supset \tilde{P}$, the set $\bar{S} \subset \mathcal{A}$ is "convex": whenever a negative line $\ell^- \subset \mathcal{A}$ has two common points with $\bar{S}$, it is contained in $\bar{S}$.*

Lemma 5.26. *Let $S \supset \tilde{P}$ be a configuration and $\pi^- \subset \mathcal{A}$ a negative plane. Then the intersection $\bar{S} \cap \pi^-$ is contained in a line; equivalently, $\pi^- \not\subset \bar{S}$.*

Proof of Lemmas 5.25 and 5.26. Lemma 5.25 is a restatement of Lemma 5.18(4). By Lemma 5.25, the two restrictions in Lemma 5.26 are equivalent: $\bar{S} \supset \pi^-$ if and only if S contains three non-collinear points of π^-. If this is the case, the points can be chosen to form an equilateral triangle with side $-\frac{1}{3}$; by Lemma 5.22, we can find three sections s_1, s_2, s_3 so that $|s_i * s_j| = 1$ for all $i \neq j$ but $|s_1 * s_2 * s_3| = 0$. Then $s_i \cdot s_j = 1$, see Lemma 5.23, and the three sections span a plane. This plane must contain three more lines, *viz.* the elements of $\mathcal{P}$ intersecting the three pairs s_i, s_j, $1 \leqslant i < j \leqslant 3$. This is a contradiction to Lemma 4.5. □

Remarkably, Lemmas 5.25 and 5.26 almost characterize the sets of sections in configurations (not necessarily geometric) containing a pencil of type $(6,0)^\circ$: this fact is established experimentally during the proof of Theorem 6.4. There is but one extra restriction, stated below without proof.

Lemma 5.27. *Let $S \supset \tilde{P}$ be a configuration and ℓ', ℓ'' two parallel isotropic lines in a positive plane in $\mathcal{A}$. If $\ell' \subset \bar{S}$ and $\bar{S}$ contains two points of ℓ'', then $\ell'' \subset \bar{S}$.* ◁

5.7. **Primitivity and rigidity for types** $(4,*)$ **and** $(3,*)$**.** As above, we fix a configuration $S \supset \mathcal{P}$ and denote by $\bar{S} \subset \mathcal{C}_{p,q}$ the image of the set of sections $S(\mathcal{P})$ under the coordinate map. It follows from Observation 5.1 and Lemma 5.2 that any pencil of type $(4,q)$, $q \leqslant 5$, or $(3,q)$, $q \leqslant 6$, is primitive. Below, we consider in detail the two extremal cases.

Proposition 5.28. *If $(p,q) = (4,6)$, then $\mathcal{P}$ is imprimitive and has a unique, up to isomorphism, geometric finite index extension. Furthermore,*

1. *$\mathcal{P}$ has a unique section $l^* \in S$ intersecting all ten fibers;*
2. *as a consequence, $\mathcal{P}$ is maximal in any configuration;*
3. *if a section s intersects l^*, then the lines s and l^* span a plane;*
4. *the set $\bar{S}$ is invariant under the involution $\bar{s} \mapsto \bar{s}^\vee := \mathbb{I} \ominus l^* \ominus s$.*

If $(p,q) = (3,7)$, then $\mathcal{P}$ is imprimitive if and only if there is a section l^* as in (1) above; if this is the case, Statements (3) and (4) also hold.

Proof. Let $(p,q) = (4,6)$. The pivot $\tilde{P}/P$ must have 3-torsion by Theorem 3.2, whereas its 2-torsion is trivial by Lemma 5.3. In addition to the classes mentioned in Lemma 5.2, the isotropic elements in $\operatorname{discr}_3 P_{4,6}$ are those constituting the $\mathbb{G}_{4,6}$ orbits of the classes of $\pm\omega$, see Observation 5.1. Hence, up to automorphism, $\tilde{P}/P$ is generated by ω, and it is immediate that ω is a section l^* as in (1). A section with these properties is unique due to Lemma 5.11(1).

If $(p,q) = (3,7)$, the only nontrivial elements that may be contained in the pivot are the orbits of the classes of $\pm(\omega - \lambda)$, and $\omega - \lambda$ is a section l^* as in (1).

With the above choice of l^*, we have $[l^*] = [0, \dots, 0; 1, \dots, 1]$ and

$$|s * l^*| = 4 - 3(s \cdot l^*) \tag{5.29}$$

for any other section s. (In particular, this relation restricts the coordinate vectors realized by sections.) Clearly, $s \cdot l^* = 1$ if and only if $|s * l^*| = 1$, in which case s and l^* intersect a third common line $a \in \mathcal{P}$ and thus span a plane; in fact, this plane is $\{l^*, a, s, s^\vee\}$. Statement (4) follows from Lemma 5.10 or Lemma 5.11(2) if $s \cdot l^* = 1$ or 0, respectively. □

Proposition 5.30. *Let $(p, q) = (3, 7)$. If $\mathcal{P}$ is not maximal, then there is a section s of $\mathcal{P}$ such that $|s|_2 \leqslant 6$. Conversely, if there is a section s such that $|s|_2 = 6$, then $\mathcal{P}$ is not maximal.*

Proof. The only pencil $\mathcal{P}'$ that may properly contain $\mathcal{P}$ is one of type $(4, 6)$, and the section s as in the statement is the restriction of l^* given by Proposition 5.28. If $\mathcal{P}$ has a section s such that $|s|_2 = 6$, then s and l intersect nine disjoint lines; by Lemma 4.6(2), they must intersect a tenth line. □

Proposition 5.31. *Let $(p, q) = (4, 5)$. Then $\mathcal{P}$ is primitive, and $\mathcal{P}$ is maximal in a geometric configuration S if and only $|s|_1 \leqslant 4$ for each section $s \in S(\mathcal{P})$.*

Proof. The primitivity is essentially given by Observation 5.1 and Lemmas 5.2 and 5.3. By Lemma 4.6(3), if there is a section s with $|s|_1 = 5$, the pencil has a tenth fiber. Conversely, the only pencil that can properly contain $\mathcal{P}$ is one of type $(4, 6)$, and its section l^* given by Proposition 5.28(1) restricts to $\mathcal{P}$. □

Proposition 5.32. *Let $p = 4$, $q \geqslant 4$ or $(p, q) = (3, 7)$, and assume that $\mathcal{P}$ is maximal. Then, for any ambient geometric configuration $S \supset \mathcal{P}$, the coordinate map $c\colon S(\mathcal{P}) \to \mathcal{C}_{p,q}$, $s \mapsto [s]$, identifies at most one pair of sections. Furthermore, if such a pair s_1, s_2 identified by c does exist, then there also is a (unique) section $l^* \in S(\mathcal{P})$ such that $\bar{s}_1 + \bar{s}_2 + \bar{l}^* = \mathbb{I}$, and, for this section l^*, one has $|l^*|_2 = q$.*

Proof. Let $s_1 \neq s_2$ be a pair of sections such that $\bar{s}_1 = \bar{s}_2$. By Lemma 4.6(3), we have $|\bar{s}_i|_2 + p \leqslant 4$ and, if $|\bar{s}_i|_2 + p = 4$, there also is a section l^* as in the statement. The number of sections l^* with $|l^*|_2 = q \geqslant 4$ is

- one if $(p, q) = (4, 6)$ or at most one if $(p, q) = (3, 7)$, see Proposition 5.28,
- zero if $(p, q) = (4, 5)$, see Proposition 5.31, and
- zero, one, or three if $(p, q) = (4, 4)$, see Lemma 4.6(3).

Furthermore, a given section l^* cannot share all 3-coordinates with any section other than s_1, s_2, see Lemma 4.6(3) again.

If $(p, q) = (4, 4)$ and $\mathcal{P}$ has three sections l_1^*, l_2^*, l_3^* with $|l_i^*|_2 = 4$, one can easily show that only one pull-back $c^{-1}(\bar{l}_i^* + \mathbb{I})$ may be nonempty, as otherwise S does not admit a geometric **L**-realization.

In the remaining case $(p, q) = (3, 7)$ and $|\bar{s}_i|_2 = 0$, one can use Theorem 3.2 to show that the image of any geometric realization of S must contain a section s of $\mathcal{P}$ such that $|s|_2 = 6$; hence, $\mathcal{P}$ is not maximal, see Proposition 5.30. □

Till the rest of this section, we assume that $(p, q) = (4, 6)$.

Denote $S^*(\mathcal{P}) := \{s \in S(\mathcal{P}) \mid s \cdot l^* = 1\}$. According to (5.29), the image of this set in $\mathcal{C}_{4,6}$ can be characterized as

$$\bar{S}^* = \bigl\{\bar{s} \in \bar{S} \bigm| |\bar{s} * l^*| = 1\bigr\}. \tag{5.33}$$

Let also

$$\bar{S}^{\circ} := \bigl\{ s \in \bar{S} \bigm| |s * s'| = 0 \text{ and } |s|_1 + |s'|_1 = 1 \text{ for some } s' \in \bar{S}^{*} \bigr\}.$$

The following statement complements Lemma 5.16; we do not need to assume that the configuration $S \supset \tilde{P}$ is geometric.

Lemma 5.34. *Let $(p, q) = (4, 6)$. Consider a pair of sections $s_1, s_2 \in S(\mathcal{P})$ such that $|s_1 * s_2| = 0$ and let $\bar{s}'_1 := \mathbb{I} \ominus \bar{s}_1^{\vee} \ominus \bar{s}_2$ and $\bar{s}'_2 := \mathbb{I} \ominus \bar{s}_1 \ominus \bar{s}_2^{\vee} = (\bar{s}'_1)^{\vee}$. Then:*

(1) *one has $1 \leqslant |s_1|_1 + |s_2|_1 \leqslant 5$;*
(2) *if $|s_1|_1 + |s_2|_1 = 5$, then also $\bar{s}'_1, \bar{s}'_2 \in \bar{S}$.*

If the pair s_1, s_2 is "homogeneous", then:

(3) *if $\bar{s}_1, \bar{s}_2 \in \bar{S}^{*}$, one has $s_1 \cdot s_2 = 0$, and*
(4) *if $\bar{s}_1, \bar{s}_2 \notin \bar{S}^{*}$, one has $s_1 \cdot s_2 = 1$.*

If the pair is "mixed", $\bar{s}_1 \in \bar{S}^{*}$ and $\bar{s}_2 \notin \bar{S}^{*}$, then:

(5) *if $|s_1|_1 + |s_2|_1 \geqslant 3$, one has $s_1 \cdot s_2 = 1$, and*
(6) *if $|s_1|_1 + |s_2|_1 = 2$, one has $s_1 \cdot s_2 = 0$ if and only if $\bar{s}'_1, \bar{s}'_2 \in \bar{S}$.*

If S is required to be geometric, then one can also state that $|s_1|_1 + |s_2|_1 \leqslant 4$ whenever $\bar{s}_1 \in \bar{S}^{*}$. We do not use this restriction explicitly.

Proof of Lemma 5.34. Statement (3) is obvious, as s_1, s_2 are in distinct fibers of the pencil $\mathcal{P}(l^{*})$. In all other cases, by Lemma 4.6(4), the section s_1 must intersect exactly one (if $\bar{s}_1 \in \bar{S}^{*}$) or two (if $\bar{s}_1 \notin \bar{S}^{*}$) of the lines s_2, $s_2^{\vee}$; with (5.29) taken into account, the intersection $s_1 \cdot s_2^{\vee}$ is given by Lemmas 5.10 and 5.16. □

Corollary 5.35 (of Lemmas 5.16 and 5.34)**.** *Any $(4, 6)$-configuration S generated by $\bar{S} \smallsetminus \bar{S}^{\circ}$ is rigid.* ◁

Remark 5.36. For many configurations, the hypotheses of Corollary 5.35 can also be verified combinatorially, using Lemmas 5.11(2) and 5.16: assuming that $S \supset \tilde{P}$ is generated by sections, it is generated by $\bar{S} \smallsetminus \bar{S}^{\circ}$ if, for any $\bar{s} \in \bar{S}^{\circ}$, there is a pair $\bar{s}_1, \bar{s}_2 \subset \bar{S} \smallsetminus \bar{S}^{\circ}$ such that $\bar{s} \oplus \bar{s}_1 \oplus \bar{s}_2 = \mathbb{I}$ and $|\bar{s}_1 * \bar{s}_2| = 1$ or 4.

5.8. **Rigidity for type $(5, 3)$.** As an immediate consequence of Observation 5.1 and Lemma 5.2, any pencil of type $(5, *)$ is primitive.

In the next two statements, S does not need to be geometric.

Lemma 5.37. *Let $p = 5$, $q \geqslant 1$, and assume that $\mathcal{P}$ has a section. Then $\mathcal{P}$ is contained in a pencil $\mathcal{P}'$ of type $(6, *)^{\circ}$ if and only if $\mathcal{P}$ has a pair of sections s_1, s_2 such that $s_1 \cdot s_2 = 0$, $|s_1 * s_2|_3 = 0$, and $|s_1 \ominus s_2|_1 > 0$.*

Proof. If $\mathcal{P} \subset \mathcal{P}'$, then s_1, s_2 are two appropriate equivalent sections of $\mathcal{P}'$, see Lemma 5.22. For the sufficiency, assume that $(p, q) = (5, 1)$ and

$$\bar{s}_1 = [0, 0, 0, 0, 0; 1], \quad \bar{s}_2 = [1, 1, 1, 1, 1; 0].$$

Then an extra member of $\mathcal{P}'$ is $h - l + \sum_{i=1}^{5}(m_{i,+} - m_{i,0}) - 2n_1 - 3s_1 + 3s_2$. □

Corollary 5.38. *Let $p = 5$, $q \geqslant 1$, and assume that $\mathcal{P}$ is maximal. Then, for any pair $s_1, s_2 \in S(\mathcal{P})$ such that $|s_1 * s_2| = 0$ and $|s_1|_1 + |s_2|_1 > 0$, one has $s_1 \cdot s_2 = 1$.* ◁

Let $(p, q) = (5, 3)$ and assume that $\mathcal{P}$ is maximal (see Lemma 5.37 for a criterion). Then, according to Lemma 5.16 and Corollary 5.38, the intersection $s_1 \cdot s_2$ may not be determined by the coordinates $\bar{s}_1, \bar{s}_2 \in \bar{S}$ only if

- one has $|\bar{s}_1 * \bar{s}_2| = |\bar{s}_1|_1 = |\bar{s}_2|_1 = 0$ and
- for any $\bar{s} \in \bar{S}$, if $|\bar{s} * \bar{s}_1| = |\bar{s} * \bar{s}_2| = 0$, then $|\bar{s}|_1 = 0$.

(For the latter condition, if $|\bar{s}|_1 > 0$, then $s \cdot s_1 = s \cdot s_2 = 1$ by Corollary 5.38 and, hence, $s_1 \cdot s_2 = 0$, see Corollary 5.14(3).) Denote by $\bar{S}^\circ \subset \bar{S}$ the union of all such pairs $(\bar{s}_1, \bar{s}_2)$.

Corollary 5.39. *Any* $(5,3)$*-configuration* S *generated by* $\bar{S} \smallsetminus \bar{S}^\circ$ *is rigid.* ◁

For another sufficient rigidity condition, consider a section $s_0 \in S(\mathcal{P})$ and let $\mathfrak{S}(s_0) := \mathcal{P}(s_0) \cap S_k(\mathcal{P})$. If $|s_0|_1 > 0$, this set is determined by the coordinates: by Lemma 5.16 and Corollary 5.38, one has $s \in \mathfrak{S}(s_0)$ if and only if $|\bar{s} * \bar{s}_0| = 0$ or $|\bar{s} * \bar{s}_0| = 1$ and $\mathbb{I} \ominus \bar{s} \ominus \bar{s}_0 \in \bar{S}$. Furthermore, the intersections $s_1 \cdot s_2$, $s_1, s_2 \in \mathfrak{S}$, are also known: they are given by Corollary 5.14.

Corollary 5.40. *Any* $(5,3)$*-pair* $(S, \mathcal{P})$ *generated by the union* $\{\bar{s}_0\} \cup \bar{\mathfrak{S}}(s_0)$ *for some section* $s_0 \in S(\mathcal{P})$ *such that* $|s_0|_1 > 0$ *is rigid.* ◁

5.9. **Other types.** For completeness, we discuss the primitivity of the other types of pencils. We treat the 3- and 2-torsion of the pivot separately.

Proposition 5.41. *Let* $\mathcal{P}$ *be a pencil of type* (p,q) *with* $p \leqslant 2$*. If the pivot* $\tilde{P}/P$ *has* 3*-torsion, then*

- $p + q = 10$, i.e., $(p,q) = (2,8)$, $(1,9)$, *or* $(0,10)$, *and*
- $\mathcal{P}$ *has a section* l^* *as in Proposition 5.28(1).*

Conversely, if $\mathcal{P}$ *has a section* l^* *as in Proposition 5.28(1), then* $p+q=10$*, one has* $\tilde{P}/P = \mathbb{Z}/3$*, and Statements (3) and (4) of Proposition 5.28 also hold.*

A section l^* *as above (or, equivalently, a geometric index* 3 *extension* $\tilde{P} \supset P$*) is unique up to automorphism.*

Proof. The proof repeats literally that of Proposition 5.28; the section l^* is the class $\frac{1}{3}[\omega + (p-4)\lambda]$ (*cf.* also Lemma 4.6(2)). A direct computation shows that, whenever the pivot $\tilde{P} \ni l^*$, one has $\tilde{P}/P = \mathbb{Z}/3$, *i.e.*, no further finite index extension satisfies the conditions of Definition 3.7. □

By Observation 5.1, any 2-torsion element $\alpha \in \tilde{P}/P$ is of the form $\sum 3\nu_k$, where the index k runs over a certain subset $\operatorname{supp}\alpha \subset \operatorname{fb}_1 \mathcal{P}$, called the *support* of α. It is clear that $\operatorname{supp}(\alpha+\beta)$ is the symmetric difference $(\operatorname{supp}\alpha) \bigtriangleup (\operatorname{supp}\beta)$.

Proposition 5.42. *Let* $\mathcal{P}$ *be a pencil of type* (p,q)*, and let* $\alpha \in \tilde{P}/P$ *be a nonzero* 2*-torsion element. Then*

(1) *one has* $|\operatorname{supp}\alpha| = 8$ *and, in particular,* $q \geqslant 8$;
(2) $|\operatorname{supp}\alpha \cap \{s\}_1| = 0$, 2, *or* 4 *for any section* s *of* $\mathcal{P}$.

Besides, the 2*-torsion of the pivot is as follows:*

- $(\mathbb{Z}/2) \oplus (\mathbb{Z}/2)$ *if* $(p,q) = (0,12)$,
- $\mathbb{Z}/2$ *if* $(p,q) = (0,11)$, $(1,10)$, *or* $(2,9)$,
- 0 *or* $\mathbb{Z}/2$ *in all other cases with* $q \geqslant 8$.

A geometric index 2 *(index* 4 *in the case* $q = 12$*) extension* $\tilde{P} \supset P$ *is unique up to automorphism.*

Proof. Clearly, $|\operatorname{supp}\alpha| = 0 \bmod 4$; hence, $|\operatorname{supp}\alpha| = 8$ or 12 by Lemma 5.3. The last statement is proved by a direct computation using Theorem 3.2. In particular, it follows that, in the case $q = 12$, there are three distinct nonzero elements and, hence, none of them can have support of length 12. This proves Statement (1). For statement (2), it suffices to consider the minimal pencil of type $(0, 8)$, so that $\operatorname{supp}\alpha = \operatorname{fb}_1 \mathcal{P}$. Then, clearly, $|s|_1$ is even, as otherwise $s \notin P$, and the values $|s|_1 = 6$ and 8 are ruled out by Definition 3.7(1) and (2), respectively.

The uniqueness is immediate. In the case of index 2, an extension is determined by a choice of the octet $\operatorname{supp}\alpha \subset \operatorname{fb}_1 \mathcal{P}$. If $q = 12$, three octets $\operatorname{supp}\alpha_i \subset \operatorname{fb}_1 \mathcal{P}$, $i = 1, 2, 3$, should be chosen so that $|\operatorname{supp}\alpha_i \cap \operatorname{supp}\alpha_j| = 4$ whenever $i \neq j$. This choice is equivalent to partitioning $\operatorname{fb}_1 \mathcal{P}$ into three quadruples. □

Corollary 5.43 (*cf.* Lemma 4.6(2))**.** *If a pencil $\mathcal{P}$ has a section s intersecting ten fibers of $\mathcal{P}$, then $\mathcal{P}$ has no other fibers.*

Proof. Assuming that $\mathcal{P}$ is of type $(0, 11)$, Proposition 5.42(2) applied to s and the nontrivial element $\alpha \in \tilde{P}/P$ leads to a contradiction. The existence of α is also guaranteed by Proposition 5.42. □

As another consequence of the results of this section, the type (p, q) and the primitivity bit almost determine a geometric realization $P \to \mathbf{L}$ up to isomorphism. The pivot $\tilde{P}/P$ may (must if $q > 10$) have 2-torsion if and only if $q \geqslant 8$ (see Proposition 5.42), and it may (must if $(p, q) = (4, 6)$ or $p = 6$ and $q > 0$) have 3-torsion if and only if $p = 6$ (see Proposition 5.17) or $p + q = 10$ (see Propositions 5.28 and 5.41). The case $p + q = 10$ and $q \geqslant 8$ is exceptional: here, the pivot may be trivial, $\mathbb{Z}/2$, or $\mathbb{Z}/3$, *i.e.*, there are three geometric realizations $P \to \mathbf{L}$. In this latter case, it makes sense to subdivide the type $(p, q)^\circ$ into $(p, q)^2$ and $(p, q)^3$.

Conjecture 5.44. The pivot $\tilde{P}/P$ has 3-torsion if and only if the axis of the pencil is a line of the second kind in the sense of Segre [24].

6. Counting sections of pencils

The goal of this section is a computer aided estimate on the size of a geometric configuration containing a pair of large obverse pencils. Even though most extra restrictions in the "counting" lemmas seem purely technical, for the moment we do need them to keep the computation under control.

6.1. **The algorithm.** Fix a pencil $\mathcal{P} := \mathcal{P}(l)$ of type (p, q) and a section s_0 of $\mathcal{P}$. Let $\bar{s}_0 := [s_0] \in \mathcal{C}_{p,q}$ and denote by $\mathbb{G}(\bar{s}_0)$ the stabilizer of $\bar{s}_0$. (Up to automorphism, there are $q + 1$ possibilities for $\bar{s}_0$; we usually choose for $\bar{s}_0$ the vector with several last 1-coordinates equal to 1 and all other coordinates equal to 0.) More sections $s_1, s_2, \ldots$ are added one by one, building the obverse pencil $\mathcal{P}(s_0)$. Thus, we assume that

$$s_0 \cdot s_i = 1 \quad \text{and} \quad s_i \cdot s_j = 0 \quad \text{for } i > j \geqslant 1, \tag{6.1}$$

i.e., all new sections are in separate fibers of $\mathcal{P}(s_0)$. Our goal is adding sufficiently many sections, so that, in the resulting configuration, $\mathcal{P}$ is still a maximal pencil and the multiplicity and valency of s_0 satisfy certain prescribed bounds

$$p_{\min} \leqslant \operatorname{mult} s_0 \leqslant p_{\max}, \qquad v_{\min} \leqslant \operatorname{val} s_0 \leqslant v_{\max}.$$

It is essential that most of the time we deal with coordinates rather than sections themselves: we choose certain elements $\bar{s}_i \in \mathcal{C}_{p,q}$ and consider the pre-configuration

$$S_k := P(\bar{s}_0, \ldots, \bar{s}_k) = (\tilde{P} + \mathbb{Z}s_0 + \ldots + \mathbb{Z}s_k)/\ker,$$

where $[s_i] = \bar{s}_i$ for all $i \geqslant 0$ and the intersection matrix of P is extended using (6.1) and the definitions of sections and coordinates. By Corollary 5.14, for each $i \geqslant 1$ we must have $\bar{s}_i \in \mathcal{C}_0(\bar{s}_0) \cup \mathcal{C}_1(\bar{s}_0)$, where

$$\mathcal{C}_r(\bar{s}_0) := \big\{\bar{s} \in \mathcal{C}_{p,q} \bigm| |\bar{s} * \bar{s}_0| = r\big\};$$

furthermore, s_i is contained in a 1-fiber of $\mathcal{P}(s_0)$ if and only if $\bar{s}_i \in \mathcal{C}_0(\bar{s}_0)$.

Once a lattice S_k has been constructed, we denote by

$$G_k := O_h(S_k, l, s_0)$$

the group of its isometries preserving h, l and s_0. The computation of this group is discussed in §6.1.3 below. (At the expense of a certain overcounting, we compute separately the stabilizers in $\mathbb{S}_3^p$ and $\mathbb{S}_p \times \mathbb{S}_q$.)

The algorithm runs in several steps.

6.1.1. *Step 1: collecting the candidates.* Assume S_{k-1} known and denote by $\bar{S}_{k-1}$ the multiset $\{[s] \,|\, s \in S_{k-1}(\mathcal{P})\}$. The group G_{k-1} acts on $\mathcal{C}_0(\bar{s}_0) \cup \mathcal{C}_1(\bar{s}_0) \smallsetminus \bar{S}_{k-1}$ and, when passing to S_k, it suffices to take for $\bar{s}_k$ one representative from each orbit of this action. We can also assume that all explicit 3-fibers are added first and avoid adding too many 3-fibers:

(1) $\bar{s}_k \in \mathcal{C}_1(\bar{s}_0)$ if $\operatorname{mult} s_0 < p_{\min}$ and $\bar{s}_k \in \mathcal{C}_0(\bar{s}_0)$ if $\operatorname{mult} s_0 \geqslant p_{\max}$.

There is an obvious injective map from the set of 3-fibers of $\mathcal{P}(s_0)$ to $\operatorname{fb}\mathcal{P}$ (each 3-fiber contains a unique line $a \in \mathcal{P}$); this map should remain injective:

(2) if $s \in S_{k-1}(\mathcal{P})$ is contained in a 3-fiber of $\mathcal{P}(s_0)$, then $|\bar{s}_0 * \bar{s}_k * s| = 0$.

Other restrictions taken into account when choosing $\bar{s}_k$ are as follows:

(3) Lemma 5.15 (in fact, we check that $[-r_{ij}]$ is negative semi-definite);
(4) $\operatorname{rk}[-r_{ij}] + 2p + q \leqslant 18$ (as S_k should admit an embedding to $\mathbf{L}$);
(5) $\mathbf{S}_4(\bar{s}_k)$: $|\bar{s}_k * \bar{s}| \leqslant 4$ for any $\bar{s} \in \bar{S}_{k-1}$, see Lemma 5.11(1);
(6) $\mathbf{S}_3(\bar{s}_k)$: $|\bar{s}_k * \bar{s}' * \bar{s}''| \leqslant 3$ for all $\bar{s}' \neq \bar{s}'' \in \bar{S}_{k-1}$, see Lemma 5.11(2), (3);
(7) $\mathbf{S}_h(\bar{s}_k)$: if $|\bar{s}_k * \bar{s}| = 4$ for some $\bar{s} \in \bar{S}$, then $\mathbf{S}_4(\bar{s}')$ and $\mathbf{S}_3(\bar{s}')$ hold, where $\bar{s}' := \mathbb{I} \ominus \bar{s} \ominus \bar{s}_k$, see Lemma 5.11(2);
(8) if $\bar{s}_k \in \mathcal{C}_1(\bar{s}_0)$, then $\mathbf{S}_4(\bar{s}')$, $\mathbf{S}_3(\bar{s}')$, and $\mathbf{S}_h(\bar{s}')$ hold for $\bar{s}' := \mathbb{I} \ominus \bar{s}_0 \ominus \bar{s}_k$.

In cases (7) and (8), we also exclude from further consideration the G_{k-1}-orbit of the respective section $\bar{s}'$, as its presence in $\bar{S}_k$ would imply the presence of $\bar{s}_k$.

6.1.2. *Step 2: validating a section $\bar{s}_k$.* Now, for each candidate $\bar{s}_k$ collected at the previous step, we compute the pre-configuration $S_k = (S_{k-1} + \mathbb{Z}s_k)/\ker$, consider the orthogonal complement $h^\perp$ in S_k, and use `GAP` [11] function `ShortestVectors` to compute the sets $\mathfrak{V}_2(S_k)$ and $\mathfrak{V}_4(S_k)$, where

$$\mathfrak{V}_r(S_k) := \{v \in h^\perp \subset S_k \,|\, v^2 = -r\}.$$

(Note that the lattice S_k is hyperbolic, hence $h^\perp$ is elliptic, by §6.1.1(3).)

A candidate $\bar{s}_k$ is rejected as invalid (not leading to a geometric configuration) if one of the following holds:

(1) $\mathfrak{V}_2 \neq \varnothing$, see Definition 3.7(1);
(2) there is $v \in \mathfrak{V}_4$ such that $v + h \in 2S_k$, see Definition 3.7(2).

Otherwise, the new set of sections $S_k(\mathcal{P})$ is computed *via*

$$S_k(\mathcal{P}) = \{v + l \,|\, v \in \mathfrak{V}_4,\ v \cdot l = 2\}.$$

At this point, the full intersection matrix is known, and we can compute and record the set

$$\mathfrak{S}_k := \mathfrak{S}(\bar{s}_0, \ldots, \bar{s}_k) = \mathcal{P}(s_0) \cap S_k(\mathcal{P}),$$

including types of the fibers. This set is used for the further validation. Namely, we reject $\bar{s}_k$ if

(3) $\operatorname{mult} s_0 > p_{\max}$ (too many 3-fibers), or
(4) $\operatorname{val} s_0 > v_{\max}$ (too many lines in $\mathcal{P}(s_0)$), or
(5) there is a pair $s' \neq s'' \in \mathfrak{S}_k$ such that $s' \cdot s'' = 1$ and $|s_0 * s' * s''| = 0$, see Corollary 5.14, or
(6) any other type specific restriction is not satisfied (whenever used, this extra restriction is specified explicitly in the respective proof).

To conserve space, for each candidate $\bar{s}_k$ that passed the validation, we record

- the elements $\bar{s}_0, \ldots, \bar{s}_k \in \mathcal{C}_{p,q}$,
- the multiset $\bar{S}_k$ (sections in terms of coordinates), and
- the image $\bar{\mathfrak{S}}_k \subset \bar{S}_k$ of $\mathfrak{S}_k$ under the coordinate map,

disregarding all other information.

6.1.3. *Step 3*: *eliminating repetitions.* Before further processing, we eliminate the repetitions in the obtained list of lattices S_k by retaining a single representative of each orbit of the $\mathbb{G}(\bar{s}_0)$-action. To compute the orbits or, equivalently, the stabilisers G_k, we use on of the following two approaches.

(1) The stabilizers are computed *via* $G_k = \operatorname{stab} \bar{\mathfrak{S}}_k \subset \mathbb{G}(\bar{s}_0)$, and the lattices are compared by means of the orbits of $\bar{\mathfrak{S}}_k$. This approach works if each S_k is exactly as in the construction above, *i.e.*, generated over $\tilde{P}_{p,q}$ by the set $\{s_0\} \cup \mathfrak{S}_k$, on which *the intersection matrix is known.*
(2) The stabilizers are computed *via* $G_k = \operatorname{stab} \bar{S}_k \subset \mathbb{G}(\bar{s}_0)$, and the lattices are compared by means of the orbits of $\bar{S}_k$. This approach applies if each S_k is known to be combinatorially rigid.

By default, we use approach (1).

6.1.4. *Step 4*: *checking the* **L**-*realizability.* For each configuration S_k obtained at Step 3, we check if it admits a geometric **L**-realization. To this end, we start with the lattice S_k itself and apply Theorem 3.2 to see if S_k admits a primitive **L**-realization. If not, we replace S_k with a finite index extension $\tilde{S}_k \supset S_k$ defined by an isotropic vector $v \in \operatorname{discr} S_k$ of prime order. (This and subsequent steps are repeated for each isotropic vector found.) The new lattice $\tilde{S}_k$ is rejected if it fails to satisfy one of the conditions in §6.1.2; otherwise, we apply Theorem 3.2 again. The algorithm stops when a primitive embedding is found (and then S_k is accepted) or all isotropic vectors are exhausted; in the latter case, the original lattice S_k is rejected as not admitting a geometric **L**-realization. Admittedly ineffective, this algorithm works reasonably well for the vast majority of configurations.

6.1.5. *Increasing the rank.* We repeat Steps 1–4 above until either nothing else can be added or the desired bounds $\operatorname{mult} s_0 \geqslant p_{\min}$, $\operatorname{val} s_0 \geqslant v_{\min}$ have been achieved. Most lattices S_k obtained have rank 20 and, hence, each geometric configuration containing S_k is a finite index extension of S_k. In the cases where $\operatorname{rk} S_k \leqslant 19$, we keep S_k on the list, but we allow also the addition of an extra section s_{k+1} disjoint from s_0. (Certainly, in this case we have to switch to approach (2) in §6.1.3, *i.e.*, we need to know that the configurations obtained are combinatorially rigid. If the latter property cannot be asserted, configurations with extra sections are excluded from Step 3.) This time, we have $s_0 \cdot s_{k+1} = 0$, but the intersections $\iota_i := s_i \cdot s_{k+1}$, $i = 1, \ldots, k$, should be given as part of the input; for each pair $(\bar{s}_{k+1}, [\iota_i])$, we check conditions (3)–(8) in §6.1.1, requiring in addition that $\operatorname{rk} S_{k+1} > \operatorname{rk} S_k$, *i.e.*, the same lattice cannot be obtained as a finite index extension of S_k. Then, Steps 2–4 are repeated and, at Step 2, we require that

(1) the valency of s_0 in S_{k+1} must be equal to that in S_k,

as otherwise the same lattice can be obtained by adding a section intersecting s_0.

6.1.6. *Final step*: *computing* **L**-*realizations.* There remains to enumerate, for each lattice S_k, its geometric **L**-realizations. This is done similar to §6.1.4, except that we do not stop at the first valid realization; on the other hand, we require that

(1) the valency of s_0 in $\tilde{S}_k$ must be equal to that in S_k, *cf.* §6.1.5(1).

At this step, for all consecutive extensions $S_k = \tilde{S}_k^0 \subset \tilde{S}_k^1 \subset \ldots$ of prime index, we can also check that $|\mathrm{Fn}(\tilde{S}_k^i)| > |\mathrm{Fn}(S_k^{i-1})|$; this fact implies that all configurations found are generated by sections.

For each finite index extension $\tilde{S}_k \supset S_k$ found in this way, *assuming that* $\mathcal{P}$ *is maximal in* $\tilde{S}_k$, we have

$$|\mathrm{Fn}(\tilde{S}_k)| = |\tilde{S}_k(\mathcal{P})| + 3p + q + 1. \tag{6.2}$$

In extreme cases (when too many lines have been found), we recompute the maximal pencil *via*

$$\mathcal{P}(l) = \{v + l \mid v \in \mathfrak{V}_6(\tilde{S}_k),\ v \cdot l = 3\}$$

and compute the pencil structure of $\tilde{S}_k$. (The computation of $\mathfrak{V}_6$ is rather expensive and we try to avoid it as much as possible.)

6.2. **A list of configurations.** For further references, we collect in Table 1 a list of large configurations found in the experiments. (We list all known configurations with more that 48 lines; for the moment, we do not assert that the list is complete.) The notation refers to certain particular configurations found in the computation. We will also speak about configurations of *type* $\mathbf{X}_*$, $\mathbf{Y}_*$, *etc.*, meaning that the pencil structures of the two configurations are equal. The configurations marked with a * in the table (most notably, the **Y**-series) admit totally reflexive **L**-realizations; the others do not. One has $\operatorname{rk} \mathbf{Z}_* = 19$; the other configurations listed in the table are of rank 20. There is no particular difference between **X** and **Q**.

Theorem 6.3. *A geometric configurations of each type listed in Table 1 is unique up to isomorphism.*

Proof. Each configuration S satisfies the hypotheses of the respective classification statement cited in the table (with pencils of type $(6,0)^\bullet$ ruled out by Theorem 6.5), and the uniqueness follows from the classification.

Indeed, the essential part of the hypotheses is the existence of a certain pair of obverse pencils. Let $v := \max\{\operatorname{val} l \mid l \in \mathrm{Fn}(S)\}$, and denote ny n the number of lines of valency v. If $v > 18$, then, in view of (5.7), the configuration is covered by Theorem 6.4. If $n \geqslant 5$ or $n \geqslant 4$ and $|\mathrm{Fn}(S)| < 4v - 8$, then, referring in the latter case to Corollary 4.4, we obtain a pair of skew lines of valency v, which suffices for all statements. In the remaining four cases ($\mathbf{X}'_{52}$, $\mathbf{X}''_{52}$, $\mathbf{Y}'_{52}$, and $\mathbf{Z}_{49}$), a similar argument gives us a pair of lines of valency $v = 18$ and $\geqslant 15$. □

TABLE 1. Known large geometric configurations

S	$\lvert\mathrm{Fn}\rvert$	Pencil structure (see §4.4), reference, remarks
$\mathbf{X}_{64}$	64	$(6,0)^{16}(4,6)^{48}$, see Theorem 6.4 and Theorem 6.7
$\mathbf{X}'_{60}$	60	$(6,2)^{10}(4,4)^{30}(3,7)^{20}$, see Theorem 6.4
$\mathbf{X}''_{60}$	60	$(4,5)^{60}$, see Lemma 6.8
$\mathbf{X}_{56}$	56	$(4,6)^{8}(4,4)^{32}(2,8)^{16}$, see Theorem 6.7
$^*\mathbf{Y}_{56}$	56	$(4,4)^{32}(3,7)^{24}$, see Lemma 6.14
$\mathbf{Q}_{56}$	56	$(4,4)^{24}(3,7)^{32}$, see Lemma 6.14
$\mathbf{X}_{54}$	54	$(6,2)^{4}(4,6)^{6}(4,4)^{6}(4,2)^{24}(2,8)^{12}(0,10)^{2}$, see Theorem 6.4
$\mathbf{Q}_{54}$	54	$(4,4)^{24}(4,3)^{24}(0,12)^{6}$, see Lemma 6.15
$\mathbf{X}'_{52}$	52	$(6,0)^{1}(4,4)^{12}(4,3)^{12}(4,2)^{3}(3,5)^{18}(0,12)^{6}$, see Theorem 6.4
$\mathbf{X}''_{52}$	52	$(6,0)^{1}(4,4)^{9}(4,3)^{18}(3,5)^{18}(0,12)^{6}$, see Theorem 6.4
$\mathbf{X}'''_{52}$	52	$(4,6)^{10}(3,5)^{40}(0,10)^{2}$, see Theorem 6.7
$\mathbf{X}^{\mathrm{v}}_{52}$	52	$(5,3)^{8}(3,5)^{32}(2,8)^{12}$, see Theorem 6.12
$^*\mathbf{Y}'_{52}$	52	$(4,6)^{2}(4,4)^{16}(3,5)^{20}(2,8)^{14}$, see Theorem 6.7
$^*\mathbf{Y}''_{52}$	52	$(4,5)^{8}(4,3)^{12}(3,6)^{16}(2,7)^{16}$, see Lemma 6.8
$^*\mathbf{Z}_{52}$	52	$(6,0)^{4}(4,4)^{12}(4,2)^{24}(2,8)^{12}$, see Theorem 6.4; $\operatorname{rk}\mathbf{Z}_{52}=19$
$\mathbf{Q}'_{52}$	52	$(4,4)^{16}(4,3)^{16}(4,2)^{16}(0,12)^{4}$, see Lemma 6.15
$\mathbf{Q}''_{52}$	52	$(4,4)^{8}(4,3)^{32}(4,2)^{8}(0,12)^{4}$, see Lemma 6.15
$\mathbf{X}_{51}$	51	$(6,2)^{1}(5,3)^{6}(4,3)^{3}(3,6)^{6}(3,4)^{8}(2,7)^{27}$, see Theorem 6.4
$\mathbf{X}'_{50}$	50	$(6,1)^{1}(4,4)^{9}(4,3)^{9}(4,2)^{9}(3,4)^{18}(0,12)^{3}(0,10)^{1}$, see Theorem 6.4
$\mathbf{X}''_{50}$	50	$(6,1)^{1}(4,4)^{6}(4,3)^{15}(4,2)^{6}(3,4)^{18}(0,12)^{3}(0,10)^{1}$, see Theorem 6.4
$\mathbf{X}'''_{50}$	50	$(5,3)^{4}(4,4)^{8}(3,5)^{16}(2,8)^{4}(2,6)^{18}$, see Theorem 6.12
$\mathbf{Z}_{50}$	50	$(4,4)^{10}(3,5)^{40}$, see Lemma 6.15; $\operatorname{rk}\mathbf{Z}_{50}=19$
$\mathbf{Z}_{49}$	49	$(6,0)^{1}(4,3)^{18}(4,2)^{9}(3,4)^{18}(0,12)^{3}$, see Theorem 6.4; $\operatorname{rk}\mathbf{Z}_{49}=19$
$^*\mathbf{Y}'_{48}$	48	$(5,1)^{2}(3,7)^{6}(3,5)^{24}(2,6)^{12}(1,9)^{4}$, see Lemma 6.14
$^*\mathbf{Y}''_{48}$	48	$(4,4)^{4}(4,2)^{16}(3,6)^{8}(2,7)^{12}(2,6)^{8}$, see Lemma 6.15

Among others, Table 1 lists all geometric configurations S containing a pair of obverse pencils $\mathcal{P}_1$, $\mathcal{P}_2$ such that

$$\lvert\mathrm{Fn}(S)\rvert > 48 \quad\text{and}\quad \lvert\mathcal{P}_1\rvert + \lvert\mathcal{P}_2\rvert \geqslant 33.$$

6.3. **Pencils of type** $(6,*)$**.** For the moment, $(6,*)^\circ$-configurations is the only class that is sufficiently well understood. The properties of such configurations are summarized in the next theorem.

Theorem 6.4. *There are* 300 *isomorphism classes of* $(6,q)^\circ$*-pairs*:

- *for* $q=0$: 62 *classes, of which* 43 *are totally reflexive*;
- *for* $q=1$: 107 *classes, none totally reflexive*;
- *for* $q=2$: 131 *classes, none totally reflexive.*

Let $(S,\mathcal{P})$ *and* $(S',\mathcal{P}')$ *be two* $(6,*)^\circ$*-pairs. Then*:

(1) S *is generated by sections and combinatorially rigid*;
(2) *with one exception, one has* $(S',\mathcal{P}')\cong(S,\mathcal{P})$ *if and only if* $\mathfrak{ls}(S')=\mathfrak{ls}(S)$;
(3) *either one has* $\lvert\mathrm{Fn}(S)\rvert<52$ *or* $S\cong\mathbf{X}_{64}$, $\mathbf{X}'_{60}$, $\mathbf{X}_{54}$, $\mathbf{X}'_{52}$, $\mathbf{X}''_{52}$, *or* $\mathbf{Z}_{52}$.

Furthermore, for any $n \in \{19, \dots, 52, 54, 60, 64\}$*, there is a* $(6,*)^\circ$*-configuration* S *such that* $|\mathrm{Fn}(S)| = n$.

As an addendum to Theorem 6.4(2), note that, with the exception of eleven pairs, any two distinct $(6,*)^\circ$-configurations are distinguished by the pencil structure.

Proof of Theorem 6.4. We start with a pencil $\mathcal{P}$ of type $(6,0)^\circ$ and apply the algorithm of §6.1, introducing a number of modifications:

- we do not fix a section $\bar{s}_0$ and use the group $\tilde{\mathbb{G}}$ instead of $\mathbb{G}(\bar{s}_0)$, see (5.21); the intersection matrices are computed by means of Lemma 5.23;
- at Step 1, all restrictions are lifted: instead, we construct the "convex hull" (in the sense of Lemma 5.25) of the set $\bar{S}_{k-1} \cup \bar{s}_k$ and check whether the resulting set $\bar{\mathcal{S}}_k$ satisfies Lemma 5.26; certainly, $\bar{s}_k$ must satisfy (5.20);
- at Step 2, all restrictions except (1) and (2) are lifted;
- at Step 3, approach (2) can be used due to Corollary 5.24;
- since all sets of sections are to be tried, we replace condition (1) in §6.1.6 with $|\mathrm{Fn}(\tilde{S}_k)| = |\mathrm{Fn}(S_k)|$. It turns out that such extensions do not exist; hence, any geometric configuration is generated by sections.

As a result, we obtain 84 configurations (of which 25 are extremal with respect to inclusion) generated by sections of $\mathcal{P}$; in these configurations, $\mathcal{P}$ is not always maximal. Then, we try to add up to two extra 1-fibers. The procedure is similar to §6.1.5: we specify the intersection of the fiber added with sections generating S_k and repeat Steps 1–4 of the algorithm; a new configuration S'_k is accepted only if $|\mathrm{Fn}(S'_k)| = |\mathrm{Fn}(S_k)|$. Repetitions are eliminated using approach (2) of §6.1.3 and appropriate subgroup $\tilde{\mathbb{G}} \subset \mathbb{G}_{6,q}$.

All other statements of the theorem follow directly from the classification. □

Theorem 6.5. *There are* 69 *isomorphism classes of* $(6,0)^\bullet$*-pairs* $(S, \mathcal{P})$ *admitting a section* $s_0 \in S(\mathcal{P})$ *such that* $15 \leqslant \operatorname{val} s_0 \leqslant 18$. *Let* $(S, \mathcal{P})$ *and* $(S', \mathcal{P}')$ *be two such pairs. Then*:

(1) *S is generated by sections and combinatorially rigid*;
(2) $(S', \mathcal{P}') \cong (S, \mathcal{P})$ *if and only if* $\mathfrak{ls}(S') = \mathfrak{ls}(S)$;
(3) *one has* $|\mathrm{Fn}(S)| < 44$.

Proof. The sections are enumerated using the algorithm of §6.1, letting

$$p_{\min} = 2, \quad p_{\max} = 6, \quad v_{\min} = 15, \quad v_{\max} = 18. \tag{6.6}$$

Here, the lower bound $p_{\min} = 2$ follows from (5.7), and the seemingly redundant upper bound $p_{\max} = 6$ helps us eliminate a number of configurations before any further processing. We introduce also a few modifications to the algorithm. First, by Corollary 5.19, we can use approach (2) in §6.1.3: this is necessary since some of the configurations S_k with $\operatorname{val} s_0 \geqslant 16$ have rank 19, see §6.1.5. Besides, we can

- use Lemma 5.18(2) for condition (6) in §6.1.2, and
- in §6.1.5, consider only the candidates $\bar{s}_{k+1}$ satisfying $1 \leqslant |\bar{s}_{k+1} * \bar{s}_0| \leqslant 4$, see Lemma 5.18(1) and Lemma 5.11(1).

we obtain 81 configurations, each with a distinguished section s_0. Switching to the full automorphism group $\mathbb{G}_{6,0}$ and resorting reduces the list down to 69 classes. The maximal number of lines in the configurations obtained is 44. □

6.4. **Pencils of type** $(4, *)$**.** A complete classification of $(4,6)$-configurations also seems feasible; however, for the moment we confine ourselves to a partial statement. similar to Theorem 6.5.

Theorem 6.7. *There are* 195 *isomorphism classes of* $(4,6)$*-pairs* $(S, \mathcal{P})$ *admitting a section* $s_0 \in S(\mathcal{P})$ *such that* $15 \leqslant \operatorname{val} s_0 \leqslant 18$*. If* $(S, \mathcal{P})$ *is such a pair, then*:

1. *S is generated by sections and combinatorially rigid*;
2. *either one has* $|\operatorname{Fn}(S)| \leqslant 48$ *or* $S \cong \mathbf{X}_{64}$, $\mathbf{X}_{56}$, $\mathbf{X}_{54}$, $\mathbf{X}'''_{52}$, *or* $\mathbf{Y}'_{52}$.

Proof. First, assume that $\operatorname{mult} l^* \leqslant 2$, hence $s_0 \neq l^*$. We need to consider seven cases: $|s_0|_1 \in \{0, \ldots, 4\}$ and $s_0 \cdot l^* = 0$ or 1 for the first two values $|s_0|_1 = 0, 1$. In each case, we employ the algorithm of §6.1, using parameters (6.6), restricting the candidates in §6.1.1 to satisfy (5.29), and imposing the restriction $|\bar{S}^*| \leqslant 4$, see (5.33), as condition (6) in §6.1.2. All pairs obtained are rigid by Corollary 5.35, and resorting the list with the full automorphism group $\mathbb{G}_{4,6}$ reduces it to 20 classes.

Let $s_0 = l^*$. To avoid complications with large pivots, we start with a manual classification of configurations $S \supset \mathcal{P}$ generated by up to four sections s_i such that $s_i \cdot l^* = 1$ and $|s_i|_1 = 0$. It is easily shown that, in addition to P itself, there are six isomorphism classes of such configurations S, each admitting a unique, up to automorphism, geometric finite index extension $\tilde{S} \supset S$. Briefly, they are as follows:

- 1 class with $\operatorname{mult} l^* = 1$, $\operatorname{rk} S = 17$, and $\ell_3(\tilde{S}/S) = 0$,
- 2 classes with $\operatorname{mult} l^* = 2$, $\operatorname{rk} S = 18$, and $\ell_3(\tilde{S}/S) = 1$,
- 1 class with $\operatorname{mult} l^* = 3$, $\operatorname{rk} S = 19$, and $\ell_3(\tilde{S}/S) = 2$,
- 1 class with $\operatorname{mult} l^* = 4$, $\operatorname{rk} S = 19$, and $\ell_3(\tilde{S}/S) = 2$,
- 1 class with $\operatorname{mult} l^* = 4$, $\operatorname{rk} S = 20$, and $\ell_3(\tilde{S}/S) = 3$.

Starting, instead of $\tilde{P}$, with one of these geometric configurations $\tilde{S}$, we build a separate list, replacing $\mathbb{G}(\bar{s}_0)$ with $O_h(\tilde{S}, l)$ and inhibiting sections $\bar{s}$ with $|\bar{s}|_1 = 0$ at Step 1. Running the algorithm, we obtain a large number of configurations (due to the lack of sorting in §6.1.5 and §6.1.6). All but one are rigid by Corollary 5.35, and the remaining one has an "ambiguous" pair of sections s_1, s_2, but the assumptions $s_1 \cdot s_2 = 0$ or 1 result in configurations S_0, S_1 with non-isomorphic sets of sections (in fact, S_0 is generated by $\bar{S}_0 \smallsetminus \bar{S}_0^\circ$, whereas S_1 is not; this phenomenon is similar to Lemma 5.16). Thus, *a posteriori*, all configurations are rigid; switching to approach (2) in §6.1.3 and resorting the list reduces it to 175 classes. □

Lemma 6.8. *If a* $(4,5)$*-pair* $(S, \mathcal{P})$ *admits a section* s_0 *such that* $16 \leqslant \operatorname{val} s_0 \leqslant 17$*, then either one has* $|\operatorname{Fn}(S)| \leqslant 48$ *or* $S \cong \mathbf{X}''_{60}$ *or* $\mathbf{Y}''_{52}$*. Furthermore, a geometric configuration of type* $\mathbf{X}''_{60}$ *is unique up to isomorphism.*

Proof. We apply the algorithm of §6.1, letting

$$p_{\min} = 3, \quad p_{\max} = 5, \quad v_{\min} = 16, \quad v_{\max} = 17 \tag{6.9}$$

and using for (6) in §6.1.2 the extra requirement that $|\bar{s}|_1 \leqslant 4$ for any $\bar{s} \in \bar{S}_k$, see Proposition 5.31. We also suppress the sorting in §6.1.6, which results in a rather large number of classes in the case where $|\bar{s}_0|_1 = 4$. Disregarding the pairs $(S, \mathcal{P})$ with $|S(\mathcal{P})| \leqslant 30$, we arrive at a number of configurations of type $\mathbf{Y}''_{52}$ and several dozens of those of type $\mathbf{X}''_{60}$; crucial is the fact that *only two configurations of type* $\mathbf{X}''_{60}$ *appear in the case where* $|\bar{s}_0|_1 = 0$.

For the uniqueness, we compute the linking structure of each configuration S of type $\mathbf{X}''_{60}$. The result is the same for all configurations:

$$\mathfrak{ls}(S) = (4,4)^{150}(5,3)^{360}(6,2)^{360}(7,2)^{240}(8,0)^{30}(8,3)^{120}.$$

Since $(4,4) \in \mathfrak{ls}(S)$, it follows that S has a pair of skew lines l, s_0 such that $|s_0|_1 = 0$ with respect to $\mathcal{P}(l)$; in particular, there are at most two isomorphism classes.

A further computation in (any) one of the configurations shows that there are at least two classes of pairs $\mathcal{P}_1, \mathcal{P}_2$ such that $|\mathcal{P}_1 \cap \mathcal{P}_2| = 4$. Namely, in each 3-fiber of $\mathcal{P}_2$, consider the two lines s', s'' that are sections of $\mathcal{P}_1$ and compute $|s'|_1$, $|s''|_1$ with respect to $\mathcal{P}_1$. The resulting multiset of four unordered pairs is obviously an invariant of $\mathcal{P}_1, \mathcal{P}_2$; it turns out to be symmetric, and it can take values

$$(1,4)^2(2,3)^2 \text{ (120 pairs)} \quad \text{or} \quad (1,4)^4 \text{ (30 pairs)}. \tag{6.10}$$

Thus, we conclude that the two classes obtained in the case $|s_0|_1 = 0$ correspond, in fact, to two distinct pairs of obverse pencils in the same configuration.

All configurations of type $\mathbf{Y}''_{52}$ (obtained in the computation) are isomorphic, as only one configuration is obtained when $|s_0|_1 = 2$ and each configuration has a pair of obverse pencils $\mathcal{P}_1$, $\mathcal{P}_2$ of type $(4,5)$ and such that $|\mathcal{P}_1 \cap \mathcal{P}_2| = 6$. ☐

Corollary 6.11 (of the proofs)**.** *For any $n \in \{18,\ldots,48,52,54,56,60,64\}$, there exists a $(4,*)$-configuration S such that $|\mathrm{Fn}(S)| = n$. If $n \in \{18,\ldots,47,52\}$, this configuration S can be chosen totally reflexive.*

Proof. By Propositions 5.28 and 5.31, we can reliably detect the maximality of a pencil $\mathcal{P}$ of type $(4,6)$ or $(4,5)$ in a configuration S by the set of sections $S(\mathcal{P})$, without recomputing the full set $\mathrm{Fn}(S)$. Hence, (6.2) applies to any geometric finite index extension $\tilde{S}_k \supset S_k$ accepted in §6.1.4; recording the values obtained, we obtain the first statement of the corollary. The second one is obtained by using, in addition, Lemma 3.9, *cf.* the proof of Proposition 5.6. ☐

6.5. **Pencils of type** $(5,*)$**.** As in the case $(4,*)$, we have a partial classification for the maximal type $(5,3)$ and certain bounds for the submaximal type $(5,2)$.

Theorem 6.12. *There are 421 isomorphism classes of $(5,3)$-pairs $(S,\mathcal{P})$ admitting a section $s_0 \in S(\mathcal{P})$ such that $15 \leqslant \operatorname{val} s_0 \leqslant 18$. If $(S,\mathcal{P})$ is such a pair, then either one has $|\mathrm{Fn}(S)| \leqslant 48$ or $S \cong \mathbf{X}^{\mathrm{v}}_{52}$, $\mathbf{X}_{51}$, or $\mathbf{X}'''_{50}$.*

Proof. The computation runs exactly as outlined in §6.1, with the parameters as in (6.6) and Lemma 5.37 used for condition (6) in §6.1.2. (Note that, since the only pencil that can properly contain $\mathcal{P}$ is that of type $(6,2)$, Lemma 5.37 gives us a criterion of maximality of $\mathcal{P}$.) With two exceptions, all configurations obtained are rigid by Corollary 5.39 or 5.40, and we can resort the combined list (the union over all four values $|\bar{s}_0|_1 = 0,\ldots,3$) using approach (2) in §6.1.3 and the full group $\mathbb{G}_{5,3}$. Each of the two configurations whose rigidity could not be established differs from all others by its linking structure. ☐

Lemma 6.13. *If a $(5,2)$-pair $(S,\mathcal{P})$ admits a section s_0 such that $16 \leqslant \operatorname{val} s_0 \leqslant 17$, then one has $|\mathrm{Fn}(S)| \leqslant 48$.*

Proof. The computation runs as outlined in §6.1, using parameters as in (6.9) and Lemma 5.37 for condition (6) in §6.1.2. There are a few configurations S_k of rank 19, to which we add extra sections (see §6.1.5) but *do not sort the results*, *i.e.*, skip

Step 3. Apart from several configurations of type $\mathbf{X}^{\mathrm{v}}_{52}$ or $\mathbf{X}'''_{50}$, one has $|\tilde{S}_k(\mathcal{P})| \leqslant 30$ and the statement follows from (6.2). ☐

6.6. **Pencils of size** 16**.** In this section we deal with geometric configurations containing a pair of obverse maximal pencils $\mathcal{P} := \mathcal{P}(l)$ and $\mathcal{P}' := \mathcal{P}(s_0)$ such that $|\mathcal{P}| = |\mathcal{P}'| = 16$. Since we are interested in the configurations themselves rather than triples $(S, \mathcal{P}, \mathcal{P}')$, we make several additional assumptions.

First of all, we assume that $\operatorname{mult} l \leqslant \operatorname{mult} s_0$; hence, when applying the algorithm outlined in §6.1, we can use the parameters

$$p_{\min} = p := \operatorname{mult} l, \quad p_{\max} = 5, \quad v_{\min} = v_{\max} = 16.$$

The next few restrictions are considered as part of the type specific condition (6) in §6.1.2; the necessary computation uses the set $\mathfrak{V}_4(S_k)$.

(1) We require that $\max\{\operatorname{val} l \mid l \in \operatorname{Fn}(S)\} \leqslant 17$.

This restriction is part of all statements: on the one hand, it helps us eliminate a number of configurations covered by other theorems and, on the other hand, it is sufficient for the proof of Proposition 8.1 in its current form.

Besides, we list all pairs $l_1, l_2 \in \operatorname{Fn}(S)$ of skew lines such that $\operatorname{val} l_1 = \operatorname{val} l_2 = 16$ and compute the refined types of the pencils $\mathcal{P}_i := \mathcal{P}(l_i)$, $i = 1, 2$, and the linking types $\operatorname{lk}(l_1, l_2)$. For each pair l_1, l_2, assuming that $\operatorname{mult} l_1 \leqslant \operatorname{mult} l_2$, we require that

(2) $\operatorname{mult} l_1 \geqslant p$, and
(3) if $\operatorname{mult} l_1 = p$, then $|\mathcal{P}_1 \cap \mathcal{P}_2| \geqslant |s_0|_2 + p$.

(If these two conditions are not satisfied, we can obtain the same configuration S replacing l, s_0 with the "smaller" pair l_1, l_2.)

In §6.1.3, approach (1) is used for sorting. In §6.1.5, we may need to add up to two extra sections; since the combinatorial rigidity is not known, the configurations containing extra sections are excluded from the sorting algorithm. Finally, at the final step we only keep the configurations S such that $|\operatorname{Fn}(S)| > 48$ or $|\operatorname{Fn}(S)| = 48$ and S is totally reflexive.

Lemma 6.14. *Let $(S, \mathcal{P})$ be a $(3, 7)$-pair and $s_0 \in S(\mathcal{P})$ a section such that*

$$\max\{\operatorname{val} l \mid l \in \operatorname{Fn}(S)\} \leqslant 17 \quad \textit{and} \quad \operatorname{val} s_0 = 16.$$

Then either one has $|\operatorname{Fn}(S)| \leqslant 48$ or $S \cong \mathbf{Y}_{56}$ or $\mathbf{Q}_{56}$. If S is totally reflexive, then either $|\operatorname{Fn}(S)| < 48$ or $S \cong \mathbf{Y}_{56}$ or $\mathbf{Y}'_{48}$.

Proof. The computation runs as outlined above. In addition to (1)–(3), we inhibit all configurations in which $\mathcal{P}$ has a section s such that $|s|_2 = 6$, see Proposition 5.30. We obtain several configurations of type $\mathbf{Y}_{56}$, $\mathbf{Q}_{56}$, or $\mathbf{Y}'_{48}$; furthermore,

- if $|s_0|_2 = 0$, there is a single configuration S; this configuration S is of type $\mathbf{Q}_{56}$, and the pencils $\mathcal{P}$ and $\mathcal{P}'$ are of type $(3, 7)^\bullet$;
- if $|s_0|_2 = 1$, there is a unique configuration S of type $\mathbf{Y}_{56}$ in which $\mathcal{P}$ is of type $(3, 7)^\circ$ and $\mathcal{P}'$ is of type $(4, 4)$;
- if $|s_0|_2 = 2$, there is a unique configuration S of type $\mathbf{Y}'_{48}$ in which $\mathcal{P}$ is of type $(3, 7)^\circ$ and $\mathcal{P}'$ is of type $(3, 7)$.

On the other hand, a direct computation shows that each configuration S obtained has a pair $\mathcal{P}$, $\mathcal{P}'$ of obverse pencils whose types and intersection $|\mathcal{P} \cap \mathcal{P}'|$ are as above. (Recall that $|\mathcal{P}(l) \cap \mathcal{P}(s_0)| = |s_0|_2 + 3$.) Replacing l and s_0 with the axes of these pencils, we conclude that, up to isomorphism *and under the assumptions of the lemma*, each type $\mathbf{Y}_{56}$, $\mathbf{Q}_{56}$, $\mathbf{Y}'_{48}$ is represented by a unique configuration. ☐

Lemma 6.15. *Let $(S, \mathcal{P})$ be a $(4,4)$-pair and $s_0 \in S(\mathcal{P})$ a section such that*

$$\max\{\operatorname{val} l \mid l \in \operatorname{Fn}(S)\} \leqslant 17 \quad \textit{and} \quad \operatorname{val} s_0 = 16.$$

Then either one has $|\operatorname{Fn}(S)| \leqslant 48$ or $S \cong$ $\mathbf{Y}_{56}$, $\mathbf{Q}_{56}$, $\mathbf{Q}_{54}$, $\mathbf{Q}'_{52}$, $\mathbf{Q}''_{52}$, or $\mathbf{Z}_{50}$. If S is totally reflexive, then either $|\operatorname{Fn}(S)| < 48$ or $S \cong \mathbf{Y}_{56}$ or $\mathbf{Y}''_{48}$.

Proof. The configurations of type (hence, isomorphic to) $\mathbf{Y}_{56}$ or $\mathbf{Q}_{56}$ are given by Lemma 6.14. The other types are obtained by a computation outlined above, which returns several dozens of configurations with $|s_0|_2 \leqslant 2$. Switching to approach (2) in §6.1.3 and the full automorphism group $\mathbb{G}_{4,4}$ *and checking explicitly that each isomorphism $\bar{S}' \to \bar{S}''$ lifts to an isometry $S' \to S''$*, one can show that, for any two configurations S', S'' in the lists obtained, $S' \cong S''$ if only if $\mathfrak{ps}(S') = \mathfrak{ps}(S'')$. The pencil structures realized are those listed in the statement. □

Lemma 6.16. *Let $(S, \mathcal{P})$ be a $(5,1)$-pair and $s_0 \in S(\mathcal{P})$ a section such that*

$$\max\{\operatorname{val} l \mid l \in \operatorname{Fn}(S)\} \leqslant 17 \quad \textit{and} \quad \operatorname{val} s_0 = 16.$$

Then one has $|\operatorname{Fn}(S)| \leqslant 48$.

Proof. The computation runs as outlined at the beginning of this section, with Lemma 5.37 used to rule out some non-maximal pencils. This computation results in an empty list of configurations. □

6.7. **Triangle free configurations.** A configuration S is called *triangle* (respectively, *quadrangle*) *free* if the graph $\operatorname{Fn} S$ has no cycles of length 3 (respectively, 3 or 4). By Lemma 4.5, a configuration is triangle free if and only if it contains no planes. Clearly, all pencils in such a configurations are of type $(0, *)$.

Lemma 6.17. *Let $\mathcal{P}, \mathcal{P}'$ be a pair of obverse pencils in a geometric triangle free configuration S, and assume that $|\mathcal{P} \cap \mathcal{P}'| \geqslant 2$. Then one has either $|\mathcal{P} \cup \mathcal{P}'| \leqslant 18$ or $|\operatorname{Fn}(S)| \leqslant 33$.*

Proof. Assuming that $|\mathcal{P}| \geqslant |\mathcal{P}'|$, denote by s_0 the axis of $\mathcal{P}'$; it is a section of $\mathcal{P}$ and $r := |s_0|_1 \geqslant 2$. Clearly, $\mathcal{P}$ is of type $(0, q)$, and we can assume that $q \geqslant 11$ and $r \leqslant 2q - 19$, as otherwise the inequality $|\mathcal{P} \cup \mathcal{P}'| \leqslant 18$ holds immediately. The structure of the extension $\tilde{P} \supset P$ is given by Proposition 5.42 (the pivot has no 3-torsion by Proposition 5.41) and, depending on the values of q, r, there are up to two (up to automorphism) possibilities for the section s_0.

We apply the algorithm outlined in §6.1, using the parameters

$$p_{\min} = p_{\max} = 0, \quad v_{\min} = 19 - q, \quad v_{\max} = q - r$$

and introducing a few modifications. Namely, at Step 1 we allow repetition when collecting sections $\bar{s}_i$, as the coordinate map (*cf.* Corollary 5.13) is not injective for $\mathcal{P}$; on the other hand, only the sections satisfying Proposition 5.42(2) are to be considered. At Step 2, as condition (6) in §6.1.2, we check that the configuration is still triangle free. Adding, if necessary, up to two extra sections disjoint from s_0 (see §6.1.5; such records are not sorted), we arrive at a number of configurations, each containing at most 33 lines. □

Lemma 6.18. *Let S be a geometric quadrangle free configuration. Consider three lines $l_0 \in \operatorname{Fn}(S)$ and $l_1, l_2 \in \mathcal{P}(l_0)$ such that $\operatorname{val} l_0 \geqslant \operatorname{val} l_1 \geqslant \operatorname{val} l_2$. Then either*

- $\operatorname{val} l_0 + \operatorname{val} l_1 \leqslant 14$ *and* $\operatorname{val} l_2 = 1$*, or*
- $\operatorname{val} l_0 \leqslant 7$ *and* $\operatorname{val} l_2 \leqslant \operatorname{val} l_1 \leqslant 5$*, or*

Table 2. **L**-configurations with more than 52 lines (see Lemma 6.19)

S	$\lvert\mathrm{Fn}\rvert$	t.r.	ref	sym	$\lvert O_h(S)\rvert$	$\operatorname{discr} S$	$T := S^\perp$
$\mathbf{X}_{64}$	64		✓	✓	4608	$\mathcal{V}_4 \oplus \langle\frac{4}{3}\rangle$	$[8,4,8]$
$\mathbf{X}'_{60}$	60		✓	✓	480	$\mathcal{U}_2 \oplus \langle\frac{4}{3}\rangle \oplus \langle\frac{2}{5}\rangle$	$[4,2,16]$
$\mathbf{X}''_{60}$	60				240	$\langle\frac{6}{5}\rangle \oplus \langle\frac{10}{11}\rangle$	$[4,1,14]$
$\mathbf{X}_{56}$	56				128	$\langle\frac{15}{8}\rangle \oplus \langle\frac{15}{8}\rangle$	$[8,0,8]$
$\mathbf{Y}_{56}$	56	✓	✓	✓	64	$\langle\frac{3}{2}\rangle \oplus \langle\frac{63}{32}\rangle$	$[2,0,32]$
$\mathbf{Q}_{56}$	56		✓	✓	384	$\mathcal{U}_2 \oplus \langle\frac{4}{3}\rangle \oplus \langle\frac{2}{5}\rangle$	$[4,2,16]$
$\mathbf{X}_{54}$	54		✓	✓	384	$\langle\frac{1}{4}\rangle \oplus \langle\frac{3}{8}\rangle \oplus \langle\frac{4}{3}\rangle$	$[4,0,24]$
$\mathbf{Q}_{54}$	54		✓	✓	48	$\mathcal{V}_2 \oplus \langle\frac{2}{19}\rangle$	$[4,2,20]$

- $\operatorname{val} l_1 \leqslant \operatorname{val} l_0 \leqslant 6$ *and* $\operatorname{val} l_2 \leqslant 5$.

Proof. It is convenient to consider the pencil $\mathcal{P} := \mathcal{P}(l_1)$, of which l_0 is a fiber and l_2 is a section. Since S is quadrangle free, each section of $\mathcal{P}$ intersects at most one fiber, and two sections intersecting l_2 cannot intersect the same fiber. In addition to l_2, the pencil $\mathcal{P}$ has $(\operatorname{val} l_0 - 2)$ sections intersecting l_0 (all disjoint from l_2) and $(\operatorname{val} l_2 - 1)$ sections intersecting l_2 (all disjoint from l_0); all these sections are pairwise disjoint. An extra parameter is the number of the sections intersecting l_2 that also intersect a fiber of $\mathcal{P}$. A direct computation (applying Theorem 3.2 to the finite index extensions allowed by Definition 3.7) rules out the values

$$(6,6,6),\ (7,6,1),\ (8,5,1),\ (10,4,2),\ (11,3,2),\ (11,4,1),\ (12,2,1)$$

for the triple $(\operatorname{val} l_0, \operatorname{val} l_1, \operatorname{val} l_2)$. □

6.8. **Existence and uniqueness.** We conclude this section with two statements related to the uniqueness of large configurations and the existence of configurations with a prescribed number of lines.

Lemma 6.19. *Each pencil structure listed in Table 2 is realized by a unique, up to weak isomorphism,* **L**-*configuration* S. *This* **L**-*configuration* S *is totally reflexive if and only if* $S = \mathbf{Y}_{56}$; *it is reflexive unless* $S = \mathbf{X}''_{60}$ *or* $\mathbf{X}_{56}$, *whereas* $\mathbf{X}''_{60}$ *and* $\mathbf{X}_{56}$ *are not symmetric.*

Proof. By Theorem 6.3, each pencil structure as in the statement is realized by a unique geometric configuration S; hence, there only remains to verify that each of the three configurations admits a unique primitive **L**-realization.

All configurations are known explicitly, and one can compute their automorphism groups, discriminants, and perspective transcendental lattices $T := S^\perp$; they are as shown in Table 2. (The lattice T is generated by two vectors u, v so that $u^2 = a$, $u \cdot v = b$, and $v^2 = c$, where $[a,b,c]$ is the triple given in the table. Each lattice is unique in its genus, which follows from the classical theory of binary forms [12].)

With two exceptions, the homomorphism $\rho\colon\ O_h(S) \to \operatorname{Aut} \operatorname{discr} S$ is surjective. The exceptions are:

- $S = \mathbf{Q}_{54}$, where $\operatorname{Im}\rho = \operatorname{Aut} \operatorname{discr}_2 S$, and
- $S = \mathbf{X}_{56}$, which will be treated separately.

Furthermore, each involution in $\operatorname{Im}\rho$ lifts to an involution in $O_h(S)$. (This is not a common property of configurations, a counterexample being $\mathbf{Z}_{50}$, see §9.4.4.)

In each case (other than $S = \mathbf{X}_{56}$), it is immediate that the image of $O^+(T)$ intersects each coset modulo $\operatorname{Im}\rho$ and, hence, a primitive $\mathbf{L}$-realization is unique up to weak isomorphism (see the description of primitive extensions in §3.1). Besides, whenever T has an orientation reversing isometry (*i.e.*, in all cases except $S = \mathbf{X}''_{60}$, see Table 2), this isometry, which is necessarily involutive, can be chosen to induce an element in the image $\operatorname{Im}\rho$ and, thus, lift to an involution in $O_h(S)$. Hence, the $\mathbf{L}$-configuration is symmetric and reflexive.

In the exceptional case $S = \mathbf{X}_{56}$, the image of $O_h(S)$ (respectively, $O(T)$) is the index 2 subgroup of $\operatorname{Aut}\operatorname{discr} S$ generated by the reflections t_α, where $\alpha \in \operatorname{discr} S$ and $\alpha^2 = \frac{3}{8}$ or $\frac{3}{4} \bmod 2\mathbb{Z}$ (respectively, $\alpha^2 = \frac{15}{8}$ or $\frac{7}{4} \bmod 2\mathbb{Z}$). The intersection of the two subgroups has index 4 and coincides with the image of $O^+(T)$. It follows that there is a single weak isomorphism class, which is not symmetric.

The only totally reflexive configuration is $\mathbf{Y}_{56}$, as $\mathbf{Y}_{56}{}^\perp$ is the only transcendental lattice containing a vector of square 2, see Table 2. □

Remark 6.20. The computation of the automorphism groups make use of the pencil structure: we list all pencils of a given type (usually, the first one listed in Table 1) and then enumerate the isometries taking one fixed pencil to another one similar to the sorting algorithm in §6.1.3.

Remark 6.21. Not every configuration S listed in Theorem 6.3 admits a unique $\mathbf{L}$-realizations. Simplest examples are $\mathbf{Y}_*$, see §9.4.2 and Table 6 below. More examples are found in Table 6 in §9.

Lemma 6.22. *For any number $n \in \{0, \ldots, 52, 54, 56, 60, 64\}$, there is a geometric configuration S such that $|\operatorname{Fn}(S)| = n$. If $n \in \{0, \ldots, 48, 52, 56\}$, this configuration can be chosen totally reflexive.*

Proof. Any count $n \leqslant 17$ is easily realized by the span of a single pencil. Hence, the first statement follows from Theorem 6.4, and the second one mostly follows from Corollary 6.11. The missing values $n = 48$, 56 for totally reflexive configurations are given by Lemma 6.14. □

7. Triangle free configurations

Recall that a configuration S is said to be triangle free if it contains no planes. The principal goal of this section is a proof of a bound to the number of lines in such a configuration, see Theorem 7.9 in §7.3 below. Throughout the section, we fix a configuration S and a geometric $\mathbf{L}$-realization $\psi\colon S \to \mathbf{L}$.

7.1. **Adjacency graphs.** Given a graph Γ, we denote by $\mathbb{Z}\Gamma$ the lattice freely generated by the vertices of Γ, so that $v^2 = -2$ for each vertex v and $u \cdot v = 1$ (respectively, 0) if the vertices $u \neq v$ are (respectively, are not) adjacent in Γ. If $\Gamma \subset \operatorname{Fn}(S)$, we also consider the images $S\Gamma := \mathbb{Z}\Gamma/\ker \subset S$ and $\mathbf{L}\Gamma := \psi(S\Gamma) \subset \mathbf{L}$ of this lattice in S and $\mathbf{L}$, denoting by $\psi_\Gamma\colon \mathbb{Z}\Gamma \to \mathbf{L}$ the composed map.

A graph Γ is called *elliptic* (respectively, *parabolic*) if $\mathbb{Z}\Gamma$ is negative definite (respectively, negative semi-definite). The *Milnor number* $\mu(\Gamma)$ of an elliptic or parabolic graph Γ is the rank of the lattice $\mathbb{Z}\Gamma/\ker$. A connected elliptic (parabolic) graph is called a *Dynkin diagram* (respectively, *affine Dynkin diagram*). A Dynkin diagram D extends to a unique affine Dynkin diagram, which we denote by $\tilde{D} \supset D$; we refer to [7] for a detailed treaty of Dynkin diagrams and their affine counterparts. Recall that any graph Γ such that $\mathbb{Z}\Gamma$ is not negative definite contains an affine

Dynkin diagram as an induced subgraph. For any affine Dynkin diagram $\tilde{D}$, the kernel $\ker \mathbb{Z}\tilde{D}$ is spanned by a single distinguished generator $k_{\tilde{D}} = \sum \kappa(e)e$, $e \in \tilde{D}$, with each coefficient $\kappa(e)$ *strictly positive*. The coefficient sum $\kappa(\tilde{D}) := \sum \kappa(e)$ of this linear combination is as follows:

$$\kappa(\tilde{\mathbf{A}}_p) = p+1, \quad \kappa(\tilde{\mathbf{D}}_q) = 2q-2, \quad \kappa(\tilde{\mathbf{E}}_6) = 12, \quad \kappa(\tilde{\mathbf{E}}_7) = 18, \quad \kappa(\tilde{\mathbf{E}}_8) = 30. \tag{7.1}$$

We extend this κ-notation to elliptic Dynkin diagrams letting $\kappa(D) := \kappa(\tilde{D})$.

Lemma 7.2. *Let* $\Gamma \subset \operatorname{Fn}(S)$ *be a parabolic subgraph such that* $\operatorname{rk}\ker \mathbb{Z}\Gamma = 1$. *Then, the isometry* $\psi_\Gamma \colon \mathbb{Z}\Gamma \to \mathbf{L}$ *is a monomorphism.*

Proof. By the assumption, Γ is a disjoint union of several Dynkin diagram and a single affine Dynkin diagram $\tilde{D}$. Since ψ_Γ is an isometry, one has $\operatorname{Ker}\psi_\Gamma \subset \ker \mathbb{Z}\Gamma$, and, as explained above, the latter subgroup is spanned by a single vector $k_{\tilde{D}}$ so that $\psi_\Gamma(k_{\tilde{D}}) \cdot h = \kappa(\tilde{D}) > 0$. Hence, $\psi_\Gamma(k_{\tilde{D}}) \neq 0$ and $\operatorname{Ker}\psi_\Gamma = 0$. ☐

7.2. **Pseudo-pencils.** Given a nonzero isotropic vector $v \in S$, the *pseudo-pencil* defined by v is the set

$$\mathcal{K}(v) := \bigl\{a \in \operatorname{Fn}(S) \bigm| a \cdot v = 0\bigr\}.$$

Since S is hyperbolic, $v \cdot h \neq 0$ and we can assume $v \cdot h > 0$. We can also assume v primitive. Then, the integer $\deg \mathcal{K} := v \cdot h$ is called the *degree* of $\mathcal{K}$. The connected components of $\mathcal{K}$ are called its *fibers*. A *section* (more generally, n-*section*, $n > 0$) of $\mathcal{K}$ is a line $s \in \operatorname{Fn}(s)$ such that $s \cdot v = 1$ (respectively, $s \cdot v = n$). The set of sections of $\mathcal{K}$, depending on the ambient configuration S, is denoted by $S(\mathcal{K})$.

Each pencil is a pseudo-pencil of degree 3: one has $\mathcal{P}(l) = \mathcal{K}(h - l)$. Conversely, if $v \cdot h = 3$, then $l := h - v \in \operatorname{Fn}(s)$ and $\mathcal{K}(v) = \mathcal{P}(l)$.

As another example, fix an affine Dynkin diagram $\tilde{D} \subset \operatorname{Fn}(S)$ and let $v \in S$ be the image of $k_{\tilde{D}}$; by Lemma 7.2, $\psi(v) \neq 0$ and $\mathcal{K}(\tilde{D}) := \mathcal{K}(v)$ is a pseudo-pencil. Clearly, $\tilde{D} \subset \mathcal{K}(\tilde{D})$. Since $k_{\tilde{D}}$ is a *positive* linear combinations of the vertices of $\tilde{D}$ and the intersection of two lines is nonnegative (see Lemma 4.1), it follows that

$$\mathcal{K}(\tilde{D}) = \{a \in \operatorname{Fn}(S) \mid a \cdot v = 0 \text{ for each vertex } v \in \tilde{D}\}. \tag{7.3}$$

Proposition 7.4. *For each pseudo-pencil* $\mathcal{K}$ *the following statements hold*:

1. *either* $\deg \mathcal{K} = 1$ *and* $|\operatorname{Fn}(S)| = 1$, *or* $\deg \mathcal{K} \geqslant 3$;
2. *as a graph,* $\mathcal{K}$ *is elliptic or parabolic and* $\mu(\mathcal{K}) \leqslant 18$;
3. *if* $D \subset \mathcal{K}$ *is a Dynkin diagram and* $(\deg \mathcal{K}) \mid \kappa(D)$, *then* $\tilde{D}$ *is a fiber of* $\mathcal{K}$.

Furthermore, if $s \in S(\mathcal{K})$, *then, for any parabolic fiber* $\tilde{D}$ *of* $\mathcal{K}$,

4. $\sum \kappa(e)(s \cdot e) = \deg \mathcal{K}$, *the summation running over* $e \in \tilde{D}$;
5. *in particular, if* $S(\mathcal{K}) \neq \varnothing$, *then* $\kappa(\tilde{D}) = \deg \mathcal{P}$ *and* $k_{\tilde{D}} = v$.

Proof. Let $\mathcal{K} = \mathcal{K}(v)$ with $v \cdot h = \deg \mathcal{K}$. The possibility $v \cdot h = 2$ is excluded by item 2 in Definition 3.7. If $v \cdot h = 1$, then $a := h - 3v$ is a line. Consider another line $b \in \operatorname{Fn}(S)$. If $b \cdot v \neq 0$ or 1, then $\sigma_+(\mathbb{Z}h + \mathbb{Z}v + \mathbb{Z}b) = 2$. If $b \cdot v = 0$, then $e := b - v$ is as in item 1 in Definition 3.7. In the remaining case $b \cdot v = 1$ one has $\operatorname{rk}\ker(\mathbb{Z}h + \mathbb{Z}v + \mathbb{Z}b) = 2$ and, hence, $b = a$, *i.e.*, a is the only line.

The assumption that $v \neq 0$ implies that $v^\perp$ has a non-trivial kernel and, hence, is parabolic; since also $\operatorname{rk}\psi(\mathcal{K}) \leqslant 19 = \sigma_-\mathbf{L}$, this proves item 2.

For item 3, observe that $\kappa(e_0) = 1$ for the only vertex $e_0 \in \tilde{D} \smallsetminus D$, see, *e.g.*, [7]. Hence, e_0 is an integral linear combination of v and the vertices of D, *i.e.*, $e_0 \in S$.

Clearly, e_0 is a line and, thus, $\tilde{D} \subset \mathcal{K}$. Finally, any affine Dynkin diagram is a whole connected component of any parabolic graph in which it is contained.

The last two statements follow from the definitions and the fact that, for each parabolic fiber $\tilde{D}$ of $\mathcal{K}$, the vector $k_{\tilde{D}}$ is a multiple of v (as $k_{\tilde{D}} \cdot v = 0$); on the other hand, $\sum_{e \in \tilde{D}} \kappa(e)(s \cdot e) = s \cdot k_{\tilde{D}}$. □

Corollary 7.5. *For a pseudo-pencil $\mathcal{K}$, one has $|\mathcal{K}| \leqslant 18(1 + 1/\mu)$, where μ is the minimal Milnor number of the parabolic fibers of $\mathcal{K}$. In particular, $|\mathcal{K}| \leqslant 24$.*

Proof. The first bound follows from the obvious identity

$$|\mathcal{K}| = \mu(\mathcal{K}) + |\{\text{parabolic fibers of } \mathcal{K}\}|.$$

If $\mathcal{K}$ has a fiber of type $\tilde{\mathbf{A}}_2$, it is an ordinary pencil and $|\mathcal{K}| \leqslant 20$ by Corollary 5.5. Otherwise, $\mu \geqslant 3$ and we have $|\mathcal{K}| \leqslant 24$. □

Geometrically, if $S = \mathcal{F}(X)$ for a nonsingular quartic $X \subset \mathbb{P}^3$, a pseudo-pencil $\mathcal{K}$ can often be interpreted as an elliptic pencil $\pi\colon X \to \mathbb{P}^1$ whose fibers are curves of degree $\deg \mathcal{K}$ in $\mathbb{P}^3$. For example, this is so in the important special case where $\mathcal{K}$ has a parabolic fiber $\tilde{D}$. Indeed, in this case, the class $v = \sum \kappa(e)e$, $e \in \tilde{D}$, regarded as a divisor, is obviously numerically effective and, hence, does define a linear system of arithmetic genus 1 without fixed points or components. From this geometric point of view, $\mathcal{K}$ is the union of lines contained in the fibers of π. More precisely, if *all* components of a reducible fiber F of π are lines, these lines form a parabolic fiber of $\mathcal{K}$; otherwise, the lines contained in F constitute one or several elliptic fibers of $\mathcal{K}$. Furthermore, in this interpretation, the bound $|\mathcal{K}| \leqslant 24$ given by Corollary 7.5 follows from the inequality

$$|\{\text{components in the singular fibers of } F\}| \leqslant \chi(X) = 24.$$

Using this geometric interpretation, one can partially extend Statements (4) and (5) of Proposition 7.4 to the elliptic fibers of $\mathcal{K}$. Namely, for each section $s \in S(\mathcal{K})$ and each elliptic fiber D of $\mathcal{K}$, one has

(4) $\sum \kappa(e)(s \cdot e) \leqslant \deg \mathcal{K}$, the summation running over $e \in D$;
(5) in particular, if $S(\mathcal{K}) \neq \varnothing$, then $\kappa(D) < \deg \mathcal{P}$.

As we do not use these statements, we will not try to prove them arithmetically. (Unlike Proposition 7.4, these statements may depend on the requirement that S should be geometric and involve a case-by-case analysis, *cf.* the discussion below.)

The *type* of a pseudo-pencil $\mathcal{K}$ is the isomorphism type of the lattice $\mathbb{Z}\mathcal{K}$; by Proposition 7.4, it is an orthogonal direct sum of elliptic and parabolic root lattices. (For example, in this new language, an ordinary pencil of type (p, q) has type $p\tilde{\mathbf{A}}_2 \oplus q\mathbf{A}_1$.) Using Proposition 7.4 and arguing as in §5, *i.e.*, applying Nikulin's Theorem 3.2 to all finite index extensions of the lattice $P := (\mathbb{Z}\mathcal{K} + \mathbb{Z}h)/\ker$ that are not ruled out by Definition 3.7, it should not be difficult to obtain a complete classification of pseudo-pencils appearing in geometric configurations; in particular, one can probably improve the bound $|\mathcal{K}| \leqslant 24$ given by Corollary 7.5. However, we confine ourselves to just the two special cases used in the proof of Theorem 7.9.

Lemma 7.6. *Assume that S is triangle free, and let $\mathcal{K} \subset S$ be a pseudo-pencil with a fiber of type $\tilde{\mathbf{A}}_3$. Then either $|\mathcal{K}| \leqslant 20$ or $\mathcal{K}$ is of type $5\tilde{\mathbf{A}}_3 \oplus \mathbf{A}_1$; in the latter case, one has $|\mathrm{Fn}(S)| \leqslant 45$.*

Proof. By Proposition 7.4, one has $\deg\mathcal{K} = \kappa(\tilde{\mathbf{A}}_3) = 4$ and all fibers of $\mathcal{K}$ are of types $\tilde{\mathbf{A}}_3$, $\mathbf{A}_2$, or $\mathbf{A}_1$. Arguing as explained above, we conclude that the only pseudo-pencil $\mathcal{K}$ such that $|\mathcal{K}| > 20$ and the lattice $P := (\mathbb{Z}\mathcal{K} + \mathbb{Z}h)/\ker$ admits a geometric $\mathbf{L}$-realization is that of type $5\tilde{\mathbf{A}}_3 \oplus \mathbf{A}_1$. Assuming this type, consider the quadrangle $\tilde{D} := \{l_1, \ldots, l_4\}$ constituting one of the type $\tilde{\mathbf{A}}_3$ fibers. Letting $\mathcal{P}_i := \mathcal{P}(l_i)$, by (7.3) we have

$$|\mathrm{Fn}(S)| = |\mathcal{P}_1 \cup \mathcal{P}_3| + |\mathcal{P}_2 \cup \mathcal{P}_4| + |\mathcal{K}| - 4. \tag{7.7}$$

(Since S is triangle free, a line $a \in \mathrm{Fn}(S)$ cannot intersect two adjacent vertices of the quadrangle.) Due to Lemma 4.6(2) and Corollary 5.43, for each of the two pairs $(i,j) = (1,3)$ or $(2,4)$, either $|\mathcal{P}_i \cup \mathcal{P}_j| = |\mathcal{P}_i \cap \mathcal{P}_j| = 10$ or $|\mathcal{P}_i \cap \mathcal{P}_j| \leqslant 8$; thus, letting $n_i := |\mathcal{P}_i \smallsetminus \mathcal{P}_j|$, we get $|\mathcal{P}_i \cup \mathcal{P}_j| \leqslant \max\{20, 16 + n_i + n_j\}$ and, if $n_i \leqslant 3$ for all $i = 1, \ldots, 4$, from (7.7) we obtain $|\mathrm{Fn}(S)| \leqslant 45$, as stated.

What remains is the case where one of the integers n_i, say, n_1, is at least 4, *i.e.*, there are at least four lines intersecting l_1 and disjoing from the three other lines. In this case, we run an algorithm similar to that described in §6.1, adding to S up to three sections intersecting l_1 in order to increase the rank from $\operatorname{rk} P = 18$ to the maximum 20. By Proposition 7.4(4), each section intersects exactly one line of each other parabolic fiber; given the rich automorphism group, this observation leaves relatively few possibilities for pairs and triples of sections. Then, as in §6.1.6, we enumerate the geometric realizations of each configuration of maximal rank and compute the number of lines, arriving at the inequality $|\mathrm{Fn}(S)| \leqslant 33$. □

Lemma 7.8. *If $\mathcal{K} \subset S$ is a pseudo-pencil with a fiber of type $\tilde{\mathbf{D}}_4$, then $|\mathcal{K}| \leqslant 19$.*

Proof. By Proposition 7.4, one has $\deg\mathcal{K} = \kappa(\tilde{\mathbf{D}}_4) = 6$ and all fibers of $\mathcal{K}$ are of types $\tilde{\mathbf{D}}_4$, $\tilde{\mathbf{A}}_5$, or $\mathbf{A}_p$, $1 \leqslant p \leqslant 4$. Trying all combinations one by one and arguing as explained prior to Lemma 7.6, we arrive at the inequality stated. (In fact, the only type with $|\mathcal{K}| = 19$ is $2\tilde{\mathbf{D}}_4 \oplus \tilde{\mathbf{A}}_5 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$.) □

7.3. **The bound.**
The following theorem is the principal result of this section.

Theorem 7.9. *If a geometric configuration S is triangle free, then $|\mathrm{Fn}(S)| \leqslant 52$.*

Proof. We consider separately several cases, each time picking an appropriate affine Dynkin diagram $\tilde{D} \subset \mathrm{Fn}(S)$ and using (7.3) to estimate the number of lines, which is $|\mathcal{K}(\tilde{D})| + |\{\text{lines intersecting a vertex of } \tilde{D}\}|$.

First, assume that the maximal valency of a line in S is at most 3. If $\mathrm{Fn}(S)$ is elliptic, then $|\mathrm{Fn}(S)| \leqslant 19$. Otherwise, $\mathrm{Fn}(S)$ contains an affine Dynkin diagram; pick one $\tilde{D} \subset \mathrm{Fn}(S)$ of the minimal Milnor number μ. Using the classification of affine Dynkin diagrams, we conclude that the number of lines that are not in $\tilde{D}$ and adjacent to a vertex of $\tilde{D}$ is at most $2n_1 + n_2 \leqslant \mu + 3$, where n_i is the number of vertices of $\tilde{D}$ of valency i. Since $2 \leqslant \mu \leqslant 18$, by (7.3) and Corollary 7.5,

$$|\mathrm{Fn}(S)| \leqslant \mu + 3 + |\mathcal{K}(D)| \leqslant \mu + \frac{18}{\mu} + 21 \leqslant 40.$$

Now, assume that S has a line of valency at least 4 and is quadrangle free. Let l_0 be a line of maximal valency, and pick four lines $l_1, \ldots, l_4$ adjacent to l_0 so that $\operatorname{val} l_1 \geqslant \ldots \geqslant \operatorname{val} l_4$. Then, $\tilde{D} := \{l_0, \ldots, l_4\}$ is a subgraph of type $\tilde{\mathbf{D}}_4$ and, by (7.3) and Lemma 7.8,

$$|\mathrm{Fn}(S)| \leqslant \operatorname{val} l_0 + \operatorname{val} l_1 + \operatorname{val} l_2 + \operatorname{val} l_3 + \operatorname{val} l_4 + 11.$$

The sum of the valencies in the latter expression is estimated using Lemma 6.18 (and the assumption $\operatorname{val} l_3, \operatorname{val} l_4 \leqslant \operatorname{val} l_2$), and we obtain $|\operatorname{Fn}(S)| \leqslant 38$.

Finally, assume that $\operatorname{Fn}(S)$ has a quadrangle, *i.e.*, a 4-cycle l_1, l_2, l_3, l_4, which can be regarded as a subgraph $\tilde{D}$ of type $\tilde{\mathbf{A}}_3$. Assume that $|\operatorname{Fn}(S)| \geqslant 46$ and apply (7.7): each of the first two terms is bounded by 18 by Lemma 6.17, and $|\mathcal{K}| \leqslant 20$ by Lemma 7.6; hence, $|\operatorname{Fn}(S)| \leqslant 52$. ☐

Remark 7.10. The idea that triangle free configurations of lines in quartics should be treated separately is also due to B. Segre, and his geometric proof [24] of the bound $|\operatorname{Fn}(S)| \leqslant 64$ for such configurations can easily be modified to get $|\operatorname{Fn}(S)| \leqslant 60$. Our bound $|\operatorname{Fn}(S)| \leqslant 52$ given by Theorem 7.9 can be improved to $|\operatorname{Fn}(S)| \leqslant 50$: in Lemma 7.6, the few types with $|\mathcal{K}| = 19$ or 20 can be ruled out similar to $5\tilde{\mathbf{A}}_3 \oplus \mathbf{A}_1$. Probably, this better bound is still not sharp: currently, the best known example of triangle free configurations has 37 lines.

8. Proofs

In this section, we prove the principal results of the paper, *viz.* Theorem 1.1, Corollary 1.3, and Addendum 1.4.

8.1. **Large configurations.** All proofs are based on the following statement, which bounds the number of lines in a geometric configuration containing a plane. With further applications in mind, we state it in a slightly stronger form.

Proposition 8.1. *If a geometric configuration S contains a plane, then either*

- *S is isomorphic to $\mathbf{X}_{64}$, $\mathbf{X}'_{60}$, $\mathbf{X}''_{60}$, $\mathbf{X}_{56}$, $\mathbf{Y}_{56}$, $\mathbf{Q}_{56}$, $\mathbf{X}_{54}$, $\mathbf{Q}_{54}$, $\mathbf{X}'_{52}$, $\mathbf{X}''_{52}$, $\mathbf{X}'''_{52}$, $\mathbf{X}^{\mathrm{v}}_{52}$, $\mathbf{Y}'_{52}$, or $\mathbf{Z}_{52}$, or*
- *one has $|\operatorname{Fn}(S)| \leqslant 52$ and $\max\{\operatorname{val} l \mid l \in \operatorname{Fn}(S)\} \leqslant 17$, or*
- *one has $|\operatorname{Fn}(S)| < 52$.*

Proof. Assume that $|\operatorname{Fn}(S)| \geqslant 52$. If S has a pencil of type $(6, *)^\circ$, Theorem 6.4 implies that $S \cong \mathbf{X}_{64}$, $\mathbf{X}'_{60}$, $\mathbf{X}_{54}$, $\mathbf{X}'_{52}$, $\mathbf{X}''_{52}$, or $\mathbf{Z}_{52}$. Hence, from now on we can also assume that S does not have such a pencil. In particular,

$$v := \max\{\operatorname{val} l \mid l \in \operatorname{Fn}(S)\} \leqslant 18;$$

if $v \leqslant 15$, then $|\operatorname{Fn}(S)| = 52$ by Corollary 4.4.

Pick a maximal pencil $\mathcal{P}$ such that $|\mathcal{P}| = v$. By (5.7), this pencil $\mathcal{P}$ has a 3-fiber $\{m_1, m_2, m_3\}$, which we order so that $\operatorname{val} m_1 \leqslant \operatorname{val} m_2 \leqslant \operatorname{val} m_3$. We have

$$\operatorname{val} m_1 + \operatorname{val} m_2 + \operatorname{val} m_3 = |\operatorname{Fn}(S)| + 8 - v \geqslant 42;$$

hence $\operatorname{val} m_3 \geqslant 14$. Then $\operatorname{mult} m_3 \geqslant 2$ by (5.7) again, and one can find another plane $\{s_0, s_1, s_2, m_3\}$ containing m_3. The lines s_0, s_1, s_2 are sections of $\mathcal{P}$, and they satisfy the inequality

$$\operatorname{val} s_0 + \operatorname{val} s_1 + \operatorname{val} s_2 = |\operatorname{Fn}(S)| + 8 - \operatorname{val} m_3.$$

Assuming that $\operatorname{val} s_0 \geqslant \operatorname{val} s_1 \geqslant \operatorname{val} s_2$, we obtain

$$3 \operatorname{val} s_0 \geqslant |\operatorname{Fn}(S)| + 8 - \operatorname{val} m_3. \tag{8.2}$$

Let $v = 18$. We need to show that $\operatorname{val} s_0 \geqslant 15$; then, Theorems 6.5, 6.7, and 6.12, would imply that $S \cong \mathbf{X}_{56}$, $\mathbf{X}'''_{52}$, $\mathbf{X}^{\mathrm{v}}_{52}$, or $\mathbf{Y}'_{52}$. If $\operatorname{val} m_3 \leqslant 17$, the desired inequality $\operatorname{val} s_0 \geqslant 15$ follows from (8.2). If $\operatorname{val} m_3 = 18$ *and* $\operatorname{val} s_0 \leqslant 14$, we repeat the same argument, taking m_3 and s_0 for l and m_3, respectively, and obtaining a section s_0' of the new pencil $\mathcal{P}(m_3)$ of valency $\operatorname{val} s_0' \geqslant 16$.

If $v = 16$ and $|\operatorname{Fn}(S)| > 52$, the same argument as above produces a pencil $\mathcal{P}'$ (not necessarily the original one) and section s_0' of $\mathcal{P}'$ such that $|\mathcal{P}'| = \operatorname{val} s_0' = 16$; hence, Lemmas 6.14, 6.15, and 6.16 imply that $S \cong \mathbf{Y}_{56}$, $\mathbf{Q}_{56}$, or $\mathbf{Q}_{54}$.

Finally, let $v = 17$. If $|\operatorname{Fn}(S)| \geqslant 54$, we use the same argument to get a pencil $\mathcal{P}'$ and section s_0' of $\mathcal{P}'$ such that $|\mathcal{P}'| = 17$ and $\operatorname{val} s_0' \geqslant 16$; hence, by Lemmas 6.8 and 6.13, we have $S \cong \mathbf{X}''_{60}$. If $|\operatorname{Fn}(S)| = 53$, the argument may fail as one may have $\operatorname{val} s_0 \leqslant 15$ and $\operatorname{val} m_3 = 16$. But in the latter case, starting with $\mathcal{P}' := \mathcal{P}(m_3)$, we obtain a section s_0' of $\mathcal{P}'$ such that $\operatorname{val} s_0' \geqslant 16$; this is a contradiction to Lemmas 6.14, 6.15, and 6.16 (if $\operatorname{val} s_0' = 16$) or 6.8 and 6.13 (if $\operatorname{val} s_0' = 17$; in this latter case, when applying the lemmas, we regard m_3 as a section of $\mathcal{P}(s_0')$). □

8.2. **Real configurations.** In the next statement, we consider a configuration S equipped with a "real structure", *i.e.*, involutive automorphism $S \to S$, $a \mapsto \bar{a}$. For such a configuration, the *real part* is the subconfiguration $S_{\mathbb{R}} := \{a \in S \mid \bar{a} = a\}$. We let $\operatorname{Fn}_{\mathbb{R}}(S) := \operatorname{Fn}(S_{\mathbb{R}})$ and call the lines contained in $\operatorname{Fn}_{\mathbb{R}}(S)$ *real*.

Proposition 8.3. *Let S be a geometric configuration equipped with an involutive automorphism $a \mapsto \bar{a}$, and assume that $|\operatorname{Fn}_{\mathbb{R}}(S)| > 48$. Then any plane $\alpha \subset \operatorname{Fn}(S)$ is* totally real*, i.e., $\alpha \subset \operatorname{Fn}_{\mathbb{R}}(S)$.*

Proof. Consider a plane $\alpha = \{a_1, a_2, a_3, a_4\}$. Let r be the number of real lines in α, and let r_i be the number of real lines in $\mathcal{P}(a_i)$, $i = 1, \ldots, 4$. The following formula is a straightforward modification of the conclusion of Corollary 4.4:

$$|\operatorname{Fn}_{\mathbb{R}}(X)| = r_1 + r_2 + r_3 + r_4 - 2r.$$

If a_i is real, then $r_i \leqslant |\mathcal{P}(a_i)| \leqslant 20$ by (5.7). Otherwise, $r_i \leqslant |\mathcal{P}(a_i) \cap \mathcal{P}(\bar{a}_i)|$, which does not exceed 2 or 10 if $a_i \cdot \bar{a}_i = 1$ or 0, respectively, see Lemma 4.9.

Consider the conjugate plane $\bar{\alpha}$. If $\alpha \cap \bar{\alpha} = \varnothing$, then $r = 0$ and $|\operatorname{Fn}_{\mathbb{R}}(X)| \leqslant 40$. If $|\alpha \cap \bar{\alpha}| = 1$ (*i.e.*, $r = 1$), then $|\operatorname{Fn}_{\mathbb{R}}(X)| \leqslant 48$. If $|\alpha \cap \bar{\alpha}| > 1$, then $\alpha = \bar{\alpha}$ by Lemma 4.5 and $r_i \leqslant 2$ for each non-real line a_i; hence, $|\operatorname{Fn}_{\mathbb{R}}(X)| \leqslant 16r + 8$ and, since $r \neq 3$, we conclude that $r = 4$, *i.e.*, $\alpha \subset \operatorname{Fn}_{\mathbb{R}}(S)$. □

The following corollary is a real counterpart of Theorem 7.9.

Corollary 8.4. *Let $X \subset \mathbb{P}^3$ be a nonsingular real quartic. If $|\operatorname{Fn}_{\mathbb{R}}(X)| > 52$, then $\mathcal{F}_{\mathbb{R}}(X)$ contains a plane; moreover, any plane in $\mathcal{F}(X)$ is contained in $\mathcal{F}_{\mathbb{R}}(X)$.*

Proof. Clearly, $\mathcal{F}_{\mathbb{R}}(X)$ is the real part of the Fano configuration $\mathcal{F}(X)$ with respect to the involution $a \mapsto -\operatorname{conj}_* a$ induced by the real structure. The configuration $\mathcal{F}(X)$ is geometric (see Theorem 3.10) and it contains a plane (see Theorem 7.9); there remains to apply Proposition 8.3. □

8.3. **Proof of Theorem 1.1.** According to Theorem 3.10, the Fano configuration $\mathcal{F}(X)$ is geometric and, since we assume $|\operatorname{Fn}(X)| > 52$, Theorem 7.9 implies that this configuration contains a plain. Then, by Proposition 8.1, $\mathcal{F}(X)$ is isomorphic to one of the configurations listed in Table 2, and the statement of the theorem follows from Lemma 6.19 and Addendum 3.12. (The quartic corresponding to $\mathbf{X}_{64}$ is identified as Schur's quartic since both contain 64 lines.) □

8.4. **Proof of Corollary 1.3.** The real Fano configuration $\mathcal{F}_{\mathbb{R}}(X)$ is geometric (see Theorem 3.10) and, assuming that $|\mathrm{Fn}_{\mathbb{R}}(X)| > 52$, this configuration contains a plain due to Corollary 8.4. Then, the statement of the corollary follows from Proposition 8.1 and Corollary 3.14. ☐

8.5. **Proof of Addendum 1.4.** The statement is an immediate consequence of Lemma 6.22 and Theorem 3.10 (for lines in complex quartics) or Corollary 3.14 (for real lines in real quartics). ☐

9. The known examples

9.1. **Schur's quartic.** The following example is more than 130 years old: it goes back to F. Schur [23] (see also [2, 6]). According to our Theorem 1.1, this is the *only* nonsingular quartic containing 64 lines, and its configuration of lines is $\mathbf{X}_{64}$.

Consider the quartic X_{64} given by the equation

$$\varphi(z_0, z_1) = \varphi(z_2, z_3), \qquad \varphi(u, v) := u(u^3 - v^3). \tag{9.1}$$

Let $k_0 := 0$, $k_1 := 1$, and $k_{2,3} := (-1 \pm i\sqrt{3})/2$ be the four roots of $\varphi(u/v, 1)$. Then, X_{64} contains the sixteen lines

$$z_1 = k_r z_0, \quad z_3 = k_s z_2, \quad r, s = 0, \ldots, 3. \tag{9.2}$$

Besides, X_{64} contains the line

$$l_0 := \{z_0 = z_2,\ z_1 = z_3\}.$$

Finally, observe that φ is the "most symmetric" polynomial of degree four: its zero locus $\{k_0, k_1, k_2, k_3\} \subset \mathbb{P}^1$ has j-invariant 0, *i.e.*, φ is invariant under a subgroup $G \cong \mathbb{A}_4 \subset PGL(2, \mathbb{C})$. This subgroup lifts a a subgroup $\tilde{G} \subset GL(2, \mathbb{C})$ preserving φ literally, not just up to a factor; it is generated by

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} \in GL(2, \mathbb{C}), \quad \epsilon^3 = 1, \epsilon \neq 1,$$

and the kernel of the projection $\tilde{G} \twoheadrightarrow G$ is the central subgroup $H \cong \mathbb{Z}/4$ generated by $i\,\mathrm{id}$. Letting $\tilde{G}$ act separately on (z_0, z_1) and (z_2, z_3), we obtain a subgroup $\mathrm{Aut}_0 X_{64} := \tilde{G} \odot \tilde{G} \subset \mathrm{Aut}\, X_{64}$, where the central product is the quotient of $\tilde{G} \times \tilde{G}$ by the diagonal $H \subset H \times H$. The stabilizer of l_0 is the diagonal $\tilde{G}/H \subset \mathrm{Aut}_0 X_{64}$; hence, its orbit consists of 48 distinct lines, and X contains $16 + 48 = 64$ lines.

A computation of the intersection matrix reveals that the sixteen lines (9.2) are distinguished: each is contained in six planes α such that $X_{64} \cap \alpha$ splits into four lines, whereas any other line is contained in four such planes. Hence, any (anti-) automorphism of X_{64} preserves the pair of lines $m_{ij} := \{z_i = z_j = 0\}$, $(i, j) = (0, 1)$ or $(2, 3)$. It follows that $\mathrm{Aut}\, X_{64}$ is an extension of the group $\mathrm{Aut}_0 X_{64}$ preserving each of m_{01}, m_{23} by the involution $z_0 \leftrightarrow z_2$, $z_1 \leftrightarrow z_3$ interchanging $m_{01} \leftrightarrow m_{23}$. This group has order 1152. As a consequence, we have the following statement.

Proposition 9.3. *Up to automorphism, the quartic X_{64} has four real structures,* viz. *those sending $[z_0 : z_1 : z_2 : z_3]$ to*

$$[\bar{z}_0 : \bar{z}_1 : \bar{z}_2 : \bar{z}_3], \quad [\bar{z}_0 : \bar{z}_1 : i\bar{z}_2 : i\bar{z}_3], \quad [\bar{z}_2 : \bar{z}_3 : \bar{z}_0 : \bar{z}_1], \quad [\bar{z}_2 : \bar{z}_3 : -\bar{z}_0 : -\bar{z}_1].$$

The numbers of real lines are 8, 4, 28, and 4, respectively.

TABLE 3. The solutions to $\xi_1(u,v) = \xi_2(u,v) = 0$

$P_1(-1+\epsilon, -1+\epsilon)$	$A_1(1/\epsilon, -2)$	$B_1(\infty, \infty)$
$P_2(1+\epsilon, -1-\epsilon)$	$A_2(1/2, \epsilon)$	$B_2(0, 0)$
$P_3(1-\epsilon, 1-\epsilon)$	$C_1(-1/\epsilon, 2)$	
$P_4(-1-\epsilon, 1+\epsilon)$	$C_2(-1/2, -\epsilon)$	

Proof. Denote by $\bar{\ }$ the standard complex conjugation, and extend its action to matrices. Then, any real structure on X_{64} is $\sigma_g\colon z \mapsto g\bar{z}$, where $g \in \operatorname{Aut} X_{64}$ is such that $g\bar{g} = \mathrm{id}$. Two real structures σ_g, $\sigma_{g'}$ are isomorphic if and only if one has $g' = h^{-1} g \bar{h}$ for some $h \in \operatorname{Aut} X_{64}$.

The set of lines real with respect to a real structure σ_g is found as follows. A line $l \subset X_{64}$ as in (9.2) is uniquely determined by its "endpoints" $l \cap m_{01}$, $l \cap m_{23}$, and the set of all eight endpoints is preserved by any (anti-)automorphism of X_{64}. Hence, such a line is real if and only if σ_g preserves its pair of endpoints; there are four such lines for any g. The other lines constitute the orbit $\operatorname{Aut}_0 X_{64}/G$ of l_0, where $G = \tilde{G}/H$ is the diagonal. Since $\bar{l}_0 = l_0$, a line hl_0 is σ_g-real if and only if $h^{-1} g \bar{h} \in G$. Now, both statements are easily proved using `GAP` [11]. □

9.2. **A real quartic with 56 real lines.** To our knowledge, this example is new.

Below, we make use of bihomogeneous polynomials, *i.e.*, algebraic curves in the product $\mathbb{P}^1 \times \mathbb{P}^1$. For the sake of simplicity, we use the affine coordinates $u := z_0/z_1$, $v := z_2/z_3$ in the two copies of $\mathbb{P}^1$.

Fix $\epsilon := \pm\sqrt{2}$ and consider the polynomials

$$\xi_1(u,v) := -3v + v^3 + 2\epsilon u, \qquad \xi_2(u,v) := 2\epsilon u^3 - v + 3u^2 v$$

of bidegree $(1,3)$ and $(3,1)$, respectively. The quartic $Y := Y_{56}$ in question is given by the polynomial

$$z_1 z_3^3 \xi_1\Big(\frac{z_0}{z_1}, \frac{z_2}{z_3}\Big) - z_1^3 z_3 \xi_2\Big(\frac{z_0}{z_1}, \frac{z_2}{z_3}\Big), \tag{9.4}$$

or, explicitly,

$$3\epsilon z_0^2 z_1 z_2 + 3\epsilon z_1 z_2 z_3^2 - \epsilon z_1^3 z_2 - \epsilon z_1 z_2^3 + 4 z_0^3 z_3 - 4 z_0 z_3^3.$$

Below, we show that Y contains 56 real lines; by Theorem 1.1, this configuration of lines is $\mathbf{Y}_{56}$, and Y is the only real quartic with this property.

The quartic Y contains the two lines

$$m_1 := \{z_0 = z_1 = 0\}, \quad m_2 := \{z_2 = z_3 = 0\}. \tag{9.5}$$

The curves $\{\xi_k = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$, $k = 1, 2$, intersect at 10 real points, see Table 3. Each such point $L(u,v)$ gives rise to the line

$$l := \{z_0 = u z_1,\ z_2 = v z_3\} \tag{9.6}$$

through $[u : 1 : 0 : 0] \in m_2$ and $[0 : 0 : v : 1] \in m_1$; it is contained in Y.

The intersection of Y with each of the six planes shown in Table 4 splits into four lines; twelve of the resulting 24 lines (some of which coincide) are among (9.5) and (9.6), see Table 4, and the twelve others are new and distinct.

Table 4. The six special planes

Plane	New lines	Old lines
$z_1 = 0$:	$(z_0 - z_3)(z_0 + z_3)$	m_1, b_1
$z_2 = 0$:	$(z_0 - z_3)(z_0 + z_3)$	m_2, b_2
$z_1 = \epsilon z_0$:	$(z_0 + z_2 - z_3)(z_0 - z_2 + z_3)$	m_1, a_1
$z_1 = -\epsilon z_0$:	$(z_0 + z_2 + z_3)(z_0 - z_2 - z_3)$	m_1, c_1
$z_2 = \epsilon z_3$:	$(z_0 + z_1 + z_3)(z_0 + z_1 - z_3)$	m_2, a_2
$z_2 = -\epsilon z_3$:	$(z_0 - z_1 - z_3)(z_0 - z_1 + z_3)$	m_2, c_2

Table 5. The sixteen special quadrics

(a_1, b_1, c_1, b_2):	$u + 2\epsilon v$
(a_2, b_2, c_2, b_1):	$u - 2\epsilon v$
(p_1, p_3, b_1, b_2):	$u - v$
(p_2, p_4, b_1, b_2):	$u + v$
(p_3, p_4, b_1, a_1):	$1 + \epsilon u + v$
(p_1, p_2, b_1, c_1):	$1 - \epsilon u - v$
(p_2, p_3, b_2, a_2):	$\epsilon u - v + uv$
(p_1, p_4, b_2, c_2):	$\epsilon u - v - uv$
(p_1, p_4, a_1, c_1):	$\epsilon - 2\epsilon u - v + uv$
(p_2, p_3, a_1, c_1):	$\epsilon + 2\epsilon u + v + uv$
(p_1, p_2, a_2, c_2):	$1 - 2\epsilon u + v - \epsilon uv$
(p_3, p_4, a_2, c_2):	$1 + 2\epsilon u - v - \epsilon uv$
(p_1, p_3, a_1, a_2):	$-3\epsilon + 4 + (2\epsilon - 2)u - (2\epsilon - 2)v + \epsilon uv$
(p_1, p_3, c_1, c_2):	$-3\epsilon + 4 - (2\epsilon - 2)u + (2\epsilon - 2)v + \epsilon uv$
(p_2, p_4, a_1, c_2):	$3\epsilon + 4 + (2\epsilon + 2)u + (2\epsilon + 2)v + \epsilon uv$
(p_2, p_4, a_2, c_1):	$3\epsilon + 4 - (2\epsilon + 2)u - (2\epsilon + 2)v + \epsilon uv$

Finally, the ten skew lines (9.6) constitute sixteen quadruples (l_1, l_2, l_3, l_4), each lying in a quadric, see Table 5. The equation of this quadric Q is

$$z_1 z_3\, \chi\Big(\frac{z_0}{z_1}, \frac{z_2}{z_3}\Big) = 0,$$

where $\chi(u, v)$ is the polynomial given in Table 5 (see also Remark 9.7 below). The intersection $Y \cap Q$ is a bidegree $(4, 4)$ curve in Q. Since it contains four skew generatrices of Q, it must split into $l_1, \ldots, l_4$ and four generatrices of the other family. Two of them are m_1, m_2, and the two others are new. It is straightforward that the $16 \times 2 = 32$ lines thus obtained are all real (see also Remark 9.8 below), pairwise distinct (as the sixteen quadrics are distinct), and distinct from (9.5), (9.6), and the lines in Table 4 (as they are disjoint from $m_1 \cup m_2$).

Summarizing, we obtain $2 + 10 + 12 + 32 = 56$ real lines in Y.

Remark 9.7. Let $u_1, \ldots, u_4 \in m_1$ and $v_1, \ldots, v_4 \in m_2$ be two quadruples, where, as above, we let $u := z_0/z_1$ and $v := z_2/z_3$. Then the lines $l_i := (u_i v_i)$, $i = 1, \ldots, 4$, *cf.* (9.6), lie in a quadric if and only if the quadruples (u_i) and (v_i) are isomorphic, *i.e.*, their cross-ratios are equal. When this is the case, the quadruples are related by a fractional linear transformation, $v_i = f(u_i)$ for $i = 1, \ldots, 4$, and the equation of the quadric is obtained from $z_2/z_3 = f(z_1/z_0)$ by clearing the denominators.

9.3. **Further properties of Y_{56}.** Let $Y := Y_{56}$ be the quartic constructed in the previous section. The following statements are straightforward.

(1) The lines m_1 and m_2 are disjoint.
(2) The lines (9.6) are pairwise disjoint; each of them intersects m_1 and m_2.

Let α be a plane as in Table 4. Then $Y \cap \alpha$ splits into m_i, l, and a pair r_1, r_2, where $i = 1$ or 2 and l is one of the lines (9.6), see Table 4.

(3) The lines r_1 and r_2 intersect m_i, l, and each other; they are disjoint from m_{3-i} and any line $l' \neq l$ as in (9.6).

This observation confirms the fact that all twelve lines thus obtained are pairwise distinct and distinct from (9.5) and (9.6). Note that, according to Table 4, the plane α is completely determined by the line $l \subset \alpha$ as in (9.6); hence, we can use the notation $\alpha(l)$ and $r_{1,2}(l)$.

Finally, pick a quadruple (l_1, l_2, l_3, l_4) as in Table 5, let Q be the corresponding quadric, and let n_1, n_2 be the two extra lines (other than m_1, m_2) in $Y \cap Q$. The remaining observations follow from the properties of the generatrices of Q; in particular, the intersection $Y \cap Q$ may contain at most four generatrices of each family and, if a line intersects three generatrices of the same family, it lies in Q.

(4) The lines n_1 and n_2 are disjoint from $m_1 \cup m_2$; they intersect each of l_1, l_2, l_3, l_4 and are disjoint from all other lines as in (9.6).
(5) If a line l as in (9.6) is distinct from all l_i, $i = 1, \ldots, 4$, the lines n_1, n_2 and $r_{1,2}(l)$ can be indexed so that $\#(n_i \cap r_j) = \delta_{ij}$ is the Kronecker symbol.

In more details, the intersection matrix can be computed using explicit equations of all lines. We leave this exercise to the reader.

Remark 9.8. Statement (5) proves also that n_1 and n_2 are real: if they were complex conjugate, they would have to intersect the same real line r_1 or r_2.

We conclude with a description of the automorphism group $\operatorname{Aut} Y_{56}$.

Proposition 9.9. *The group* $\operatorname{Aut} Y_{56} \subset PGL(4, \mathbb{C})$ *is generated by*

- *the reflections* $z_i \mapsto \rho_i z_i$ *with* $\rho_i = \pm 1$ *and* $\rho_0 \rho_3 = \rho_1 \rho_2$,
- *the transposition* $z_1 \leftrightarrow z_2$,
- *the order* 4 *map* $z_0 \mapsto z_3$, $z_3 \mapsto -z_0$, *and*
- *the involution* $z_0 \mapsto (z_0 + z_3)/\epsilon$, $z_3 \mapsto (z_0 - z_3)/\epsilon$.

This group has order 32; *it acts faithfully on the set of lines contained in* Y_{56}.

Proof. Computing the intersection matrix, one can see that there are exactly four pairs (l_1, l_2) of skew lines such that l_1 and l_2 intersect ten other common lines. In turn, these pairs split into four quadrangles: one is (m_1, m_2), (b_1, b_2), and the other is formed by the four remaining lines in the planes $\{z_1 = 0\}$ and $\{z_2 = 0\}$, see Table 4. The last involution in the statement interchanges the two quadrangles. The other transformations preserve the quadrangle (m_1, m_2), (b_1, b_2) and, hence, the coordinate tetrahedron; they can easily be listed. The last two statements are straightforward. □

Remark 9.10. All automorphisms of Y_{56} are real with respect to the standard complex conjugation $c\colon [z_i] \mapsto [\bar{z}_i]$. Hence, the last statement of Proposition 9.9 implies that c is the only real structure on Y_{56} with respect to which all 56 lines are real. (In fact, up to automorphism Y_{56} has six real structures: they are enumerated by the conjugacy classes of the involutions in $\operatorname{Aut} Y_{56}$.)

Remark 9.11. By rescaling $u \mapsto \epsilon u$, one can make the quartic Y_{56} defined over $\mathbb{Q}$; however, some of the lines are still defined over the quadratic number field $\mathbb{Q}(\epsilon)$ only. To see this, one can observe that the cross-rations of some of the quadruples of points in m_1 cut by the lines as in (9.6) are irrational, see Table 3.

9.4. **A few other quartics.** In this concluding section, we describe briefly a few other quartics with large configurations of lines, for which we do not know explicit equations. The existence (and uniqueness, when it holds) is given by the existence of the corresponding **L**-configurations, see Table 1, and the results of §3.4. Other properties, *e.g.*, groups of projective automorphisms, classes of real structures, *etc.*, can easily be computed using the corresponding properties of configurations and Nikulin's theory of lattice extensions; however, we omit these details.

9.4.1. *The quartics menioned in Theorem 1.1.* By Lemma 6.19 and Theorem 3.10, for each of the four configurations $S = \mathbf{X}'_{60}$, $\mathbf{Q}_{56}$, $\mathbf{X}_{54}$, $\mathbf{Q}_{54}$, there exists a unique, up to projective equivalence, quartic X such that $\mathcal{F}(X) \cong S$; this quartic can be chosen real, see Proposition 3.11. We denote these quartics by X'_{60}, Q_{56}, X_{54}, Q_{54}, respectively. Besides, for $S = \mathbf{X}''_{60}$ or $\mathbf{X}_{56}$, there is a unique pair of nonequivalent complex conjugate quartics X, $\bar{X}$ such that $\mathcal{F}(X) \cong \mathcal{F}(\bar{X}) \cong S$; these pairs are denoted by X''_{60}, $\bar{X}''_{60}$ and X_{56}, $\bar{X}_{56}$, respectively. Together with X_{64} (see §9.1) and Y_{56} (see §9.2 and §9.3), these surfaces make a complete list of quartics containing more than 52 lines.

9.4.2. *Large configurations of real lines.* Arguing as in the proof of Lemma 6.19, it is not difficult to classify the **L**-realizations of the four other **Y**-type configurations listed in Table 1; we summarize the results in Table 6. This table is organized similar to Table 2, with the last column showing the numbers r, c of, respectively, real and pairs of complex conjugate quartics with the given configuration of lines. Note, though, that, with the only exception of Proposition 9.12 below, we never assert the uniqueness of the real form: considering the large automorphism groups, it is likely not unique, *cf.* Proposition 9.3 and Remark 9.10.

If $S = \mathbf{Y}''_{48}$, the natural homomorphism $\rho\colon O_h(S) \to \operatorname{Aut}\operatorname{discr} S$ maps $O_h(S)$ onto the index 2 subgroup $\operatorname{Aut}\operatorname{discr}_5 S$; in the other three cases, ρ is surjective. It follows that, in all four cases, the weak isomorphism classes of **L**-realizations are classified by the transcendental lattices $T := S^\perp$. In three cases, there are several isomorphism classes; however, only one of them is totally reflexive. In view of Addendum 3.15, this fact merits a separate statement.

Proposition 9.12. *For each* **Y***-type configuration* S *listed in Table 1, there is a unique, up to real projective equivalence, real quartic* Y *such that* $\mathcal{F}_{\mathbb{R}}(Y) \cong S$*. The real part of this real quartic is a connected surface of genus* 10. ◁

Table 6. Configurations with many $\mathbf{L}$-realizations

S	$\lvert\mathrm{Fn}\rvert$	t.r.	ref	sym	$\lvert O_h(S)\rvert$	discr S	$T := S^\perp$	(r, c)
$\mathbf{X}''_{60}$	60				240	$\langle\frac{6}{5}\rangle \oplus \langle\frac{10}{11}\rangle$	$[4, 1, 14]$	$(0, 1)$
$\mathbf{X}_{56}$	56				128	$\langle\frac{15}{8}\rangle \oplus \langle\frac{15}{8}\rangle$	$[8, 0, 8]$	$(0, 1)$
$\mathbf{Y}'_{52}$	52	✓	✓	✓	8	$\langle\frac{1}{2}\rangle \oplus \langle\frac{3}{2}\rangle \oplus \langle\frac{4}{19}\rangle$	$[2, 0, 38]$	$(1, 1)$
							$[8, 2, 10]$	
$\mathbf{Y}''_{52}$	52	✓	✓	✓	8	$\langle\frac{6}{79}\rangle$	$[2, 1, 40]$	$(1, 2)$
							$[4, 1, 20]$	
							$[8, 1, 10]$	
$\mathbf{Q}''_{52}$	52				64	$\langle\frac{1}{4}\rangle \oplus \langle\frac{5}{4}\rangle \oplus \langle\frac{2}{5}\rangle$	$[8, 4, 12]$	$(0, 1)$
$\mathbf{X}_{51}$	51		✓	✓	12	$\langle\frac{4}{3}\rangle \oplus \langle\frac{2}{29}\rangle$	$[6, 3, 16]$	$(1, 1)$
							$[4, 1, 22]$	
$\mathbf{X}''_{50}$	50		$\times 2$	$\times 2$	12	$\langle\frac{7}{4}\rangle \oplus \langle\frac{5}{8}\rangle \oplus \langle\frac{4}{3}\rangle$	$[4, 0, 24]$	$(2, 0)$
$\mathbf{X}'''_{50}$	50				16	$\langle\frac{7}{4}\rangle \oplus \langle\frac{5}{8}\rangle \oplus \langle\frac{4}{3}\rangle$	$[4, 0, 24]$	$(0, 1)$
$\mathbf{Y}'_{48}$	48	✓	✓	✓	8	$\langle\frac{1}{2}\rangle \oplus \langle\frac{5}{16}\rangle \oplus \langle\frac{2}{3}\rangle$	$[2, 0, 48]$	$(1, 0)$
$\mathbf{Y}''_{48}$	48	✓	✓	✓	8	$\langle\frac{2}{5}\rangle \oplus \langle\frac{4}{19}\rangle$	$[2, 1, 48]$	$(2, 1)$
							$[8, 1, 12]$	
			✓	✓			$[10, 5, 12]$	

The configuration $S = \mathbf{Y}''_{48}$ admits another reflexive $\mathbf{L}$-realization, which is not totally reflexive; thus, the corresponding quartic X can be chosen real, but some of the lines contained in X are necessarily complex conjugate. (Note that, unlike the case of Schur's quartic X_{64}, see Proposition 9.3, or the maximizing real quartic Y_{56}, see Remark 9.10, this quartic X and the quartic Y given by Proposition 9.12 are not just distinct real forms of the same surface: X and Y are not projectively equivalent even over $\mathbb{C}$.)

Remark 9.13. Table 6 suggests also that the quartics Y_* realizing each of the configurations $\mathbf{Y}_* := \mathbf{Y}'_{52}$, $\mathbf{Y}''_{52}$, $\mathbf{Y}'_{48}$, $\mathbf{Y}''_{48}$ should be Galois conjugate over a certain algebraic number field $\mathbb{K}$ of degree 3, 5, 1, 4, respectively, so that this field $\mathbb{K}$ is the minimal field of definition of Y_*. In particular, Y'_{52} and Y''_{52} are probably not defined over $\mathbb{Q}$, *cf.* open problems at the end of §1.1.

9.4.3. *Configurations with many realizations.* For completeness, we describe also the few configurations from Table 1 that admit more than one geometric realization. For all configurations of maximal rank, the computation runs exactly as in the proof of Lemma 6.19 and can easily be automated. Omitting the straightforward details, we summarize the results in Table 6. (The meaning of the columns is explained in §9.4.2.) In the line containing $\mathbf{X}''_{50}$, the symbol "×2" means that there are two distinct geometric $\mathbf{L}$-realizations, which are both reflexive.

Any other configuration in Table 1 admits a unique geometric $\mathbf{L}$-realization, and this realization is reflexive. In particular, for a configuration S as in Table 1, a geometric $\mathbf{L}$-realization is reflexive if and only if it is symmetric. Currently, we do not know whether this is a common property of configurations: in some similar $K3$-related problems, it may not hold (*cf.* the existence of a connected real equisingular family of simple plane sextics containing no real curves discovered in [1].)

Remark 9.14. The isomorphism type of a singular $K3$-surface (*i.e.*, one of Picard rank 20) is determined by its transcendental lattice. Analyzing Tables 2 and 6, one can observe that the quartics X'_{60} and Q_{56} are isomorphic as abstract $K3$-surfaces; a similar statement holds for the seven quartics realizing the configurations $\mathbf{X}_{54}$, $\mathbf{X}''_{50}$, and $\mathbf{X}'''_{50}$. On the other hand, each of the configurations $\mathbf{Y}'_{52}$, $\mathbf{Y}''_{52}$, $\mathbf{X}_{51}$, $\mathbf{Y}''_{48}$ is realized by several distinct $K3$-surfaces.

9.4.4. *Families with parameters.* Finally, worth mentioning are the configurations $S = \mathbf{Z}_{52}$, $\mathbf{Z}_{50}$, $\mathbf{Z}_{49}$ in Table 1. Recall that the dimension of the equilinear moduli space $\Omega'(S)/PGL(4,\mathbb{C})$, *cf.* the proof of Theorem 3.10, equals $20 - \operatorname{rk} S$; hence, we obtain 1-parameter families of distinct quartics sharing the same combinatorial type of configurations of lines.

The connectedness of each family follows from Theorem 3.10 and a computation based on the results of [15], covering indefinite transcendental lattices. We have

- if $S = \mathbf{Z}_{52}$, then $\mathcal{S} = \langle\frac{1}{2}\rangle \oplus \langle\frac{1}{2}\rangle \oplus \langle\frac{5}{8}\rangle \oplus \langle\frac{4}{3}\rangle$ and $O^+(T) \twoheadrightarrow \operatorname{Aut}\operatorname{discr} T$;
- if $S = \mathbf{Z}_{50}$, then $\mathcal{S} = \langle\frac{7}{4}\rangle \oplus \langle\frac{2}{5}\rangle \oplus \langle\frac{2}{5}\rangle$ and $O_h(S) \twoheadrightarrow \operatorname{Aut}\mathcal{S}$;
- if $S = \mathbf{Z}_{49}$, then $\mathcal{S} = \mathcal{V}_2 \oplus \langle\frac{5}{4}\rangle \oplus \langle\frac{6}{7}\rangle$ and $\operatorname{Im}[O_h(S) \to \operatorname{Aut}\mathcal{S}] = \operatorname{Aut}\mathcal{S}_2$.

The uniqueness of $T := S^\perp$ in its genus and the assertion on $O^+(T)$ for $S = \mathbf{Z}_{52}$ follow from [15]. Thus, in each case, there is a unique geometric $\mathbf{L}$-realization. If $S = \mathbf{Z}_{52}$, this realization is totally reflexive, *i.e.*, there is a 1-parameter family (not necessarily connected) of real quartics Z such that $\mathcal{F}_{\mathbb{R}}(Z) \cong \mathbf{Z}_{52}$. For the other two configurations, for each involution $a \in \operatorname{Aut}\operatorname{discr} S$, exactly one of $\pm a$ admits an involutive lift to $O_h(S)$. Hence, these configurations are reflexive (not totally) and the corresponding equilinear families also contain real quartics.

The existence of the family corresponding to $\mathbf{Z}_{52}$, with exactly 52 lines in each quartic, as well as the non-uniqueness of $\mathbf{L}$-realizations discussed in §9.4.2 and §9.4.3, can be regarded as yet another justification for the assumption $|\operatorname{Fn}(X)| > 52$ in Theorem 1.1: quartics with fewer lines are probably more difficult to control.

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