THE BERTINI INVOLUTION

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ABSTRACT. We summarize and extend E. Moody's results on the explicit equations related to the Bertini involution.

These notes are the result of my attempt to understand E. Moody's paper [1]. I correct a few misprints in [1] and take the computation a bit further.

I express my admiration to Ethel I. Moody, who managed to perform this tedious computation in the pre-`Maple` era. A `Maple` implementation of most equations is found at `http://www.fen.bilkent.edu.tr/~degt/papers/Bertini.zip`.

This text is not intended as an 'official' publication; it is distributed in the hope that it may be useful. It can be cited as arXiv:1212.0991.

Whenever possible, I try to keep the original notation of [1].

1. THE BERTINI INVOLUTION

1.1. **The results of** [1]. Consider the pencil of cubics

$$(1.1) \qquad \lambda w(x) + \mu w'(x) = 0,$$

where

$$w(x) = x_3^2(a_1 x_1 + a_2 x_2) + x_3(b_1 x_1^2 + b_2 x_1 x_2 + b_3 x_2^2) + (c_1 x_1^2 x_2 + c_2 x_1 x_2^2)$$

and similar for w', so that the coordinate vertices are amongst the basepoints of the pencil. The point $(0:0:1)$ will play a special rôle.

The curve of the pencil passing through a point y is given by

$$(1.2) \qquad W_3(x) := w(x)w'(y) - w'(x)w(y) = 0.$$

Clearly,

$$W_3(x) = x_3^2(A_1 x_1 + A_2 x_2) + x_3(B_1 x_1^2 + B_2 x_1 x_2 + B_3 x_2^2) + (C_1 x_1^2 x_2 + C_2 x_1 x_2^2),$$

where $A_i(y) := a_i w'(y) - a_i' w(y)$ and similar for B_i, C_i.

The tangent to (1.2) at $(0:0:1)$ meets the curve again at $r = (r_1 : r_2 : r_3)$, where

$$(1.3) \qquad \begin{aligned} r_1 = A_2 r_1', \quad r_2 = -A_1 r_1', \quad r_3 = A_1 A_2 r_3', \\ r_1' := B_1 A_2^2 - B_2 A_1 A_2 + B_3 A_1^2, \qquad r_3' := A_2 C_1 - A_1 C_2. \end{aligned}$$

The locus of these points is

$$(1.4) \qquad \gamma_4(y) := y_1 A_1 + y_2 A_2 = 0.$$

Apart from the basepoints, the locus (1.4) meets (1.2) at a single point r. The line (ry) meets (1.2) at a third point z, and the *Bertini involution* can be defined as the map $y \mapsto z$. Let $\kappa := a_1 b_1' - a_1' b_1$ and

$$C_5(y) := A_2[B_1 + \kappa y_1 y_3^2]_{y_2} + [A_1 - \kappa y_1^2 y_3]_{y_2}[A_2 y_3 + B_3 y_2]_{y_1} + \kappa B_3 y_1 y_3,$$
$$\phi_6(y) := A_1 C_2 + y_3 C_5(y),$$
$$\psi_6(y) := A_2 C_1 + y_3 C_5(y).$$

(Following [1], we use $[e]_u$ do indicate that e has a common factor u and this factor has been removed.) In these notations, the Bertini involution is

(1.5) $\qquad z_1 = \phi_6[A_2^2 \phi_6 + B_3 r_1']_{y_1}, \quad z_2 = \psi_6[A_1^2 \psi_6 + B_1 r_1']_{y_2}, \quad z_3 = \psi_6 \phi_6 C_5.$

Apart from the basepoint $(0 : 0 : 1)$ of the pencil, the fixed point locus of this involution is the curve

(1.6) $\qquad\qquad K(y) := \psi_6[A_1 y_3 + B_1 y_1]_{y_2} - \phi_6[A_2 y_3 + B_3 y_2]_{y_1} = 0.$

Remark 1.7. The expressions for r, C_5, and K found in [1] contain a number of misprints. The corrections suggested are verified by the identities in §1.2 below, as well as by (2.2) and (2.3).

1.2. **Further observations.** The expression for the Bertini involution, see (3.1), is obtained by substituting $z = lr + my$ and solving $W_3(lr + my) = 0$, see (1.2), in $l : m$. (This equation is linear since $W_3(r) = W_3(y) = 0$.) Note that $\{\psi_6 = 0\}$ and $\{\phi_6 = 0\}$ are the curves contracted to the basepoints $(1 : 0 : 0)$ and $(0 : 1 : 0)$, respectively. Hence, they can also be found from the identities

$$y_3 r_1 - y_1 r_3 = A_2 \gamma_4 \phi_6, \qquad y_2 r_3 - y_3 r_2 = A_1 \gamma_4 \psi_6.$$

Besides, one has

$$y_1 r_2 - y_2 r_1 = -r_1' \gamma_4.$$

A point y is fixed by (3.1) if and only if the tangent at y to the member (1.1) of the pencil passing through y meets the curve again at r. In [1], the equation (1.6) of the fixed point locus is obtained by eliminating $\lambda : \mu$ from (1.1) and the polar conic to (1.1) with respect to r (after the substitution $x \mapsto y$). Alternatively, K can be found as the common factor of $y_3 z_1 - y_1 z_3$ and $y_2 z_3 - y_3 z_2$, using the identities

$$y_3 z_1 - y_1 z_3 = -\phi_6 K A_2, \qquad y_2 z_3 - y_3 z_2 = -\psi_6 K A_1.$$

Note that the rightmost factors are just two particular members of the pencil.

2. THE MAP $\mathbb{P}^2 \dashrightarrow \Sigma_2$

From now on, we assume that the distinguished basepoint $(0 : 0 : 1)$ is simple.

2.1. **The anti-bicanonical map.** Let Y be the plane $\mathbb{P}^2$ blown up at all basepoints (including infinitely near) of the pencil other than $(0 : 0 : 1)$. It is a (nodal, in general) Del Pezzo surface of degree 1, and the anti-bicanonical linear system maps Y to a quadric cone in $\mathbb{P}^3$. According to [1], the proper transforms of the sextics $\{\phi_6 = 0\}$ and $\{\psi_6 = 0\}$ are in $|-2K_Y|$. Hence, the space of sections $H^0(Y; -2K_Y)$ is generated by ϕ_6 (or ψ_6) and w^2, ww', w'^2, and the map $y \mapsto \bar{z} \in \mathbb{P}^3$ is given by

$$\bar{z}_0 = \phi_6(y), \quad \bar{z}_1 = w^2(y), \quad \bar{z}_2 = w(y)w'(y), \quad \bar{z}_3 = w'^2(y).$$

Its image is the cone $\bar{z}_1 \bar{z}_3 = \bar{z}_2^2$. The passage to the affine coordinates $\bar{x} := \bar{z}_1/\bar{z}_2$, $\bar{y} := \bar{z}_0/\bar{z}_2$ blows up the vertex and maps the cone to the Hirzebruch surface Σ_2

with the exceptional section E of self-intersection (-2) (the exceptional divisor over the vertex). The composed rational map $\mathbb{P}^2 \dashrightarrow \Sigma_2$ is

$$(2.1) \qquad \bar{x} = w(y)/w'(y), \qquad \bar{y} = \phi_6(y)/w'^2(y).$$

Alternatively, Y with the remaining basepoint $(0:0:1)$ blown up is a rational Jacobian elliptic surface: the elliptic pencil is (1.1) and the distinguished section is the exceptional divisor over $(0:0:1)$. The Bertini involution becomes the fiberwise multiplication by (-1), and the quotient blows down to the Hirzebruch surface Σ_2. The quotient map is generically two-to-one; its ramification locus is the union of the exceptional section $E \subset \Sigma_2$ and a certain proper trigonal curve, *viz.* the image of $\{K = 0\}$. The pull-backs of the fibers of Σ_2 are the anti-canonical curves in Y (*i.q.* the members of the original pencil (1.1) of cubics), and the pull-backs of the *proper* (*i.e.*, disjoint from E) sections of Σ_2 are the anti-bicanonical curves other than those representable in the form $\{\alpha_1 w^2 + \alpha_2 w w' + \alpha_3 w'^2 = 0\}$.

2.2. The ramification locus.

Since $\{\psi_6 = 0\}$ is the pull-back of a section of Σ_2, there must be a relation (after the substitution $x \mapsto y$) of the form

$$(2.2) \qquad \psi_6 = \phi_6 + S_2(w, w'),$$

where S_2 is a certain homogeneous polynomial of degree 2, see below.

The curve $\{\phi_6 = 0\}$ is contracted by the Bertini involution. Hence, the pull-back in Y of its image $\{\bar{y} = 0\}$ splits into two components (sections of the elliptic pencil), of which one is contracted by the blow down map $Y \to \mathbb{P}^2$. It follows that the free term R_3^2 in equation (2.3) below is indeed a perfect square.

Since $\{K = 0\}$ is the pull-back of the ramification locus (other than E), which is a proper trigonal curve, there must be a relation

$$(2.3) \qquad K^2 = -4\phi_6^3 + \phi_6^2 P_2(w, w') + \phi_6 Q_4(w, w') + R_3^2(w, w'),$$

where P_2, Q_4, and R_3 are certain homogeneous polynomials of degree 2, 4, and 3, respectively. Let $S_2(t_1, t_2) = \sum_{i=0}^2 s_i t_1^i t_2^{2-i}$ *etc.* The coefficients p_i, q_i, r_i, s_i are found by a direct computation. They reduce to a remarkably simple form:

$$s_0 = a_2 c_1 - a_1 c_2,$$
$$r_0 = -a_1 b_2 c_2 + a_1 b_3 c_1 + a_2 b_1 c_2,$$
$$q_0 = 4(a_1 c_2 - b_1 b_3)s_0 + 2b_2 r_0,$$
$$p_0 = b_2^2 - 4a_2 c_1 - 4b_1 b_3 + 8a_1 c_2$$

and

$$p_i = (-1)^i \{p_0\}_i, \quad q_i = (-1)^i \{q_0\}_i, \quad r_i = (-1)^i \{r_0\}_i, \quad s_i = (-1)^i \{s_0\}_i,$$

where $\{ \cdot \}_m$ is defined as follows: if e is a degree n monomial in $a_1, \ldots, c_2$, then $\{e\}_m$ is the sum of $\binom{n}{m}$ monomials, each obtained from e by replacing m of its n factors with their primed versions. (For example, one has $\{a_1 c_2\}_1 = a_1 c_2' + a_1' c_2$, $\{b_2^2\}_1 = 2b_2 b_2'$, and $\{a_1 b_1 c_1\}_2 = a_1 b_1' c_1' + a_1' b_1 c_1' + a_1' b_1' c_1$.) This definition extends to homogeneous polynomials by linearity.

Warning 2.4. The operation $\{ \cdot \}_m$ is used only to shorten the notation. As with the derivative, this operation should be performed *before* any substitution of any particular values of the coefficients (see, *e.g.*, the substitution $a_1 = a_2 = 0$ in §3).

Remark 2.5. Observe that $S_2(w, w')$ remains unchanged under the transformation $a_i \leftrightarrow a'_i$, $b_i \leftrightarrow b'_i$, $c_i \leftrightarrow c'_i$, $w_i \leftrightarrow w'_i$. The same holds for $P_2(w, w')$ and $Q_4(w, w')$, whereas $R_3(w, w')$ changes sign.

Problem 2.6. The symmetry in Remark 2.5 is easily explained by interchanging w and w'. However, is there a geometric explanation for the 'regular' behaviour of the other coefficients?

Summarizing, we see that the map $\mathbb{P}^2 \dashrightarrow \Sigma_2$ given by (2.1) takes the sextics $\{\phi_6 = 0\}$ and $\{\psi_6 = 0\}$ to the sections $\{\bar{y} = 0\}$ and $\{\bar{y} = -S_2(\bar{x})\}$, respectively. The map is generically two-to-one (the deck translation being the Bertini involution), and its ramification locus in Σ_2 is the union of E and the trigonal curve

$$-4\bar{y}^3 + \bar{y}^2 P_2(\bar{x}) + \bar{y} Q_4(\bar{x}) + R_3^2(\bar{x}) = 0.$$

As usual, we treat the homogeneous bivariate polynomials S_2, P_2, Q_4, and R_3 as univariate ones via $S_2(\bar{x}) := S_2(\bar{x}, 1) = \sum_{i=0}^2 s_i \bar{x}^i$ etc.

Problem 2.7. Can one express coefficients s_i in terms of p_i, q_i, and r_i? In other words, does a choice of the ramification locus in Σ_2 and one of the sections select automatically the other section?

2.3. **Other sextics contracted by the involution.** The basepoint $(0 : 0 : 1)$ plays a special rôle in the definition of the Bertini involution. The other basepoints are not special. In particular, any other basepoint $(u_1 : u_2 : u_3)$ gives rise to a sextic $\{\psi_6^u = 0\}$ contracted to this point and to a splitting section of Σ_2 whose pull-back this sextic is. Assuming that $u_1 \neq 0$ and normalizing the coordinates as $(1, u_2, u_3)$, we have

$$(2.8) \qquad \psi_6^u = \phi_6 + S_2^u(w, w'),$$

where S_2^u is a homogeneous polynomial of degree 2 whose coefficients s_i^u are

$$s_0^u = s_0 + (a_2 c_2 u_2 + (a_2 b_2 - a_1 b_3)u_3) + a_2 b_3 u_2 u_3 + a_2^2 u_3^2, \qquad s_i^u = (-1)^i \{s_0^u\}_i.$$

As above, the image of $\{\psi_6^u = 0\}$ in Σ_2 is the section $\{\bar{y} = -S_2^u(\bar{x})\}$.

These equations are easily obtained by changing the coordinates and placing the basepoint in question to $(1 : 0 : 0)$.

3. The Geiser involution

Now, we assume that the pencil has exactly one basepoint infinitely near to the distinguished point $(0 : 0 : 1)$. In other words, all members of (1.1) have a common tangent at $(0 : 0 : 1)$ and, hence, exactly one of them has a double point at $(0 : 0 : 1)$. We can assume that this singular member is $\{w(x) = 0\}$, thus letting $a_1 = a_2 = 0$. The resulting special case of the Bertini involution is the *Geiser involution*, see [1].

3.1. **The involution.** Most formulas in §1.1 simplify dramatically. One obviously has $A_1 = -a'_1 w(y)$ and $A_2 = -a'_2 w(y)$; hence, $\gamma_4 = w\gamma_1$, see (1.4), where

$$(3.1) \qquad \gamma_1 := -(a'_1 y_1 + a'_2 y_2)$$

is the defining polynomial of the common tangent to the members of the pencil at the distinguished basepoint $(0 : 0 : 1)$. (Here and below, w without an argument stands for $w(y)$.)

Next, there are splittings, see (1.3),

$$r'_1 = w^2 \tilde{r}'_1, \qquad r'_3 = w \tilde{r}'_3, \qquad r_i = w^3 \tilde{r}_i, \ i = 1, 2, 3,$$

with

$$\tilde{r}_1 = -a'_2\tilde{r}'_1, \qquad \tilde{r}_2 = a'_1\tilde{r}'_1, \qquad \tilde{r}_3 = a'_1 a'_2 \tilde{r}'_3,$$

$$\tilde{r}'_1 = a'^2_2 B_1 - a'_1 a'_2 B_2 + a'^2_1 B_3, \qquad \tilde{r}'_3 = a'_1 C_2 - a'_2 C_1.$$

Furthermore, one has

$$\phi_6 = w\phi_3, \qquad \psi_6 = w\psi_3, \qquad C_5 = w\tilde{C},$$

where

$$\tilde{C}(y) := -a'_2[B_1 - a'_1 b_1 y_1 y_3^2]_{y_2} + a'_1[a'_2 y_3(w - b_1 y_1^2 y_3) - B_3 y_2]_{y_1 y_2},$$

$$\phi_3(y) := -a'_1 C_2 + y_3\tilde{C},$$

$$\psi_3(y) := -a'_2 C_1 + y_3\tilde{C}.$$

Finally, after reducing the common factor w^3 in (3.1), the Geiser involution takes the form

$$z_1 = \phi_3[a'^2_2 w\phi_3 + B_3\tilde{r}'_1]_{y_1}, \quad z_2 = \psi_3[a'^2_1 w\psi_3 + B_1\tilde{r}'_1]_{y_2}, \quad z_3 = \psi_3\phi_3\tilde{C}.$$

The loci contracted to the basepoints $(1:0:0)$ and $(0:1:0)$ are the cubics $\{\psi_6 = 0\}$ and $\{\phi_6 = 0\}$, respectively, and the fixed point locus is the sextic

$$\tilde{K}(y) := \psi_3[-a'_1 wy_3 + B_1 y_1]_{y_2} - \phi_3[-a'_2 wy_3 + B_3 y_2]_{y_1} = 0.$$

One has $K = w\tilde{K}$, see (1.6). The identities of §1.2 turn into

$$y_3\tilde{r}_1 - y_1\tilde{r}_3 = a'_2\gamma_1\phi_3, \qquad y_2\tilde{r}_3 - y_3\tilde{r}_2 = a'_1\gamma_1\psi_3, \qquad y_1\tilde{r}_2 - y_2\tilde{r}_1 = -\tilde{r}'_1\gamma_1,$$

$$y_3 z_1 - y_1 z_3 = a'_2\phi_3\tilde{K}, \qquad y_2 z_3 - y_3 z_2 = a'_1\psi_3\tilde{K}.$$

3.2. The double covering $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

Let Z be the plane $\mathbb{P}^2$ blown up at the basepoints (including infinitely near) of the pencil other than $(0:0:1)$ and the infinitely near one. It is a (nodal, in general) Del Pezzo surface of degree 2, and the anti-canonical linear system maps Z to $\mathbb{P}^2$. This map is generically two-to-one, its deck translation is the Geiser involution, and its ramification locus is a quartic curve in $\mathbb{P}^2$. The anti-canonical system is the web of cubics passing through the seven points blown up; the space $H^0(Z; -K_Z)$ is generated by any of ϕ_3 or ψ_3 and by w and w'. Hence, the corresponding rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $y \mapsto \bar{z}$, is given by

$$\bar{z}_0 = \phi_3(y), \quad \bar{z}_1 = w(y), \quad \bar{z}_2 = w'(y).$$

It is straightforward that $q_0 = r_0 = s_0 = 0$. Hence, there are splittings

$$Q_4(t,t') = t\tilde{Q}_3(t,t'), \quad R_3(t,t') = t\tilde{R}_2(t,t'), \quad S_2(t,t') = t\tilde{S}_1(t,t')$$

and relations (2.2) and (2.3) turn into

$$\psi_3 = \phi_3 + \tilde{S}_1(w,w'),$$

$$\tilde{K}^2 = -4\phi_3^3 w + \phi_3^2 P_2(w,w') + \phi_3\tilde{Q}_3(w,w') + \tilde{R}_2^2(w,w').$$

Thus, the ramification locus is the quartic

$$(3.2) \qquad 4\bar{z}_0^3\bar{z}_1 = \bar{z}_0^2 P_2(\bar{z}_1, \bar{z}_2) + \bar{z}_0\tilde{Q}_3(\bar{z}_1, \bar{z}_2) + \tilde{R}_2^2(\bar{z}_1, \bar{z}_2).$$

In the coordinates chosen, the lines $\{\bar{z}_0 = 0\}$ and $\{\bar{z}_0 + \tilde{S}_1(\bar{z}_1, \bar{z}_2) = 0\}$ are double tangents (in the generalized sense) to this quartic.

Similarly, given another basepoint $(1:u_2:u_3)$, one has $S_2^u(t,t') = t\tilde{S}_1^u(t,t')$ and the cubic $\{\psi_3^u = 0\}$ singular at this point is given by (cf. (2.8))

$$\psi_3^u = \phi_3 + \tilde{S}_1^u(w,w').$$

Remark 3.3. The coefficients of the polynomials $\tilde{S}_1$, $\tilde{Q}_3$, and $\tilde{R}_2$ are the same as those of S_2, Q_4, and R_3, respectively, see §2.2, with an obvious shift by one. It is worth emphasizing that the brace operation $\{\,\cdot\,\}_i$ should be evaluated *before* the substitution $a_1 = a_2 = 0$.

3.3. **A few further observations.** Unlike the general case considered in §2, now, the fixed point locus $\{\tilde{K} = 0\}$ does pass through $(0 : 0 : 1)$. In fact, since this curve is the branch set, it follows that $\{\tilde{K} = 0\}$ is also the locus of the singular points of the singular members of the web of cubics defined by the seven non-distinguished basepoints. This curve has a double point at each of the seven basepoints. In particular, it is an anti-bicanonical curve in Z.

Under the natural identification of the web and the dual plane $(\mathbb{P}^2)\check{\,}$, the locus of the singular members themselves is the curve dual to (3.2), including the lines through the singular points of (3.2).

As another observation, note that $\{\psi_3 = 0\}$ and $\{\phi_3 = 0\}$ are special members of the web, *viz.* those singular at $(1 : 0 : 0)$ and $(0 : 1 : 0)$, respectively. As above, these cubics are contracted by the involution to the corresponding basepoints.

References

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