CONICS ON KUMMER QUARTICS

ALEX DEGTYAREV

Abstract. We classify the configurations of lines and conics in smooth Kummer quartics, assuming that all 16 Kummer divisors map to conics. We show that the number of conics on such a quartic is at most 800.

1. Introduction

All algebraic varieties considered in this paper are over $\mathbb{C}$.

Following Bauer [3], define $N_{2n}(d)$ as the maximal number of smooth rational curves of degree d that can lie on a smooth degree $2n$ $K3$-surface $X \subset \mathbb{P}^{n+1}$. The numbers $N_{2n}(1)$ are quite well understood (see [28, 25, 11, 6, 10]), whereas for $d = 2$ the only known sharp bound is $N_6(2) = 285$ (see [9]). In the most interesting case of spatial quartics, $2n = 4$, there are but three sporadic examples, with 352 (Barth, Bauer [2]), 432 (Bauer [3]), and 800 conics (see my recent paper [7], where I also motivate the conjecture that $N_4(2) = 800$).

The construction of [2, 3] embeds a Kummer $K3$-surface to $\mathbb{P}^3$ so that each of the 16 Kummer divisors is mapped to a smooth conic, and *a posteriori* (X. Roulleau, private communication), the quartic found in [7] is also of this nature. Therefore, we define a *Barth–Bauer quartic* as a smooth quartic $X \subset \mathbb{P}^3$ containing 16 pairwise disjoint smooth conics. All Barth–Bauer quartics constitute a 3-parameter family $\mathcal{B}$, and the main goal of this paper is a detailed analysis, in the spirit of [12, §6], of the configurations of lines and conics on quartics $X \subset \mathcal{B}$.

The principal results are stated in §1.2, after the necessary preparation in §1.1. We substantiate the conjecture that $N_4(2) = 800$, as well as a few other speculations motivated by [9, 12], and discover plenty of examples of smooth spatial quartics with many conics, both irreducible and reducible.

1.1. **A few basic concepts.** Throughout this paper, a *line* on a smooth quartic surface $X \subset \mathbb{P}^3$ is a smooth rational curve $l \subset X$ of projective degree 1, whereas a *conic* is a curve $c \subset X$ of projective degree 2 and arithmetic genus 0. Thus, *we do not assume a conic irreducible*: it may split into a pair of intersecting lines.

We associate with X its graphs $\operatorname{Fn}_1 X$ and $\operatorname{Fn}_2 X$ of lines and conics, respectively: their vertices are, respectively, lines and conics on X, and two vertices c_1, c_2 are connected by an edge of multiplicity $c_1 \cdot c_2 \in \mathbb{Z}$. We denote by $\operatorname{Fn}_2^* X \subset \operatorname{Fn}_2 X$ the graph of *irreducible* conics, and the union

$$\operatorname{Fn} X := \operatorname{Fn}_1 X \cup \operatorname{Fn}_2^* X,$$

1991 *Mathematics Subject Classification.* Primary: 14J28; Secondary: 14N25.
Key words and phrases. $K3$-surface, quartic surface, Kummer surface, conic.
The author was partially supported by the TÜBİTAK grant 118F413.

with the vertices further connected as above and colored according to their degree, is called the full *Fano graph* of X. The reducible conics are recovered from $\operatorname{Fn} X$ as pairs •—• of adjacent 1-vertices.

Given an abstract bi-colored graph Γ, we are interested in the *equiconical stratum*

$$\mathcal{X} := \mathcal{X}(\Gamma) := \{X \in \mathcal{B} \mid \operatorname{Fn} X \cong \Gamma\} \subset \mathcal{B},$$

provided that it is nonempty. Clearly, a necessary condition is $\Gamma \supset \Omega$, where Ω is the discrete graph of sixteen 2-vertices (here and below, a graph inclusion always refers to an induced subgraph with the induced coloring), and we actually classify the so-called *relative forms* of Γ, *i.e.*, pairs (Γ, Ω) considered up to automorphism of Γ. When considering the absolute and relative groups

- $\operatorname{Aut} \Gamma \supset \operatorname{Aut}(\Gamma, \Omega)$ of abstract graph automorphisms,
- $\operatorname{Aut}_h X \supset \operatorname{Aut}_h(X, \Omega)$ of projective automorphisms, and
- $\operatorname{Sym}_h X \supset \operatorname{Sym}_h(X, \Omega)$ of projective symplectic automorphisms,

we always assume that $X \in \mathcal{X}$ is a very general member.

Remark 1.1. According to Nikulin [22], a set Ω of 16 pairwise disjoint irreducible conics on a quartic X inherits from X a certain intrinsic structure (the 4-Kummer structure in §2.2), and *a priori* it is not obvious that this structure is preserved by the full group $\operatorname{Aut} \Gamma$. However, Corollary 1.8 states that, indeed, the 4-Kummer can be recovered solely from the graph $\Gamma = \operatorname{Fn} X$; in particular, it is safe to define the equivalence of relative forms using the full automorphism group $\operatorname{Aut} \Gamma$.

For each equiconical stratum $\mathcal{X}(\Gamma)$, one can consider the covering $\tilde{\mathcal{X}}(\Gamma) \to \mathcal{X}(\Gamma)$ consisting of pairs (X, Ω), where $X \in \mathcal{X}$ and Ω is a distinguished unordered set of 16 pairwise disjoint irreducible conics on X. There is an obvious splitting

$$\tilde{\mathcal{X}}(\Gamma) = \bigsqcup \tilde{\mathcal{X}}_i, \quad \tilde{\mathcal{X}}_i := \tilde{\mathcal{X}}(\Gamma, \Omega_i),$$

the union running over all relative forms (Γ, Ω_i) of Γ. The complex conjugation $X \mapsto \bar{X}$ induces a well-defined involution on the set $\pi_0(\mathcal{Y})$ of connected components of each (sub-)stratum $\mathcal{Y} := \mathcal{X}, \tilde{\mathcal{X}}, \tilde{\mathcal{X}}_i$, *etc.*; we denote by

$$\operatorname{rc} \mathcal{Y} := \big(r(\mathcal{Y}), c(\mathcal{Y})\big) \tag{1.2}$$

the numbers of, respectively, real components of $\mathcal{Y}$ and pairs of complex conjugate ones (*i.e.*, respectively, one- and two-element orbits of the conjugation).

1.2. **Principal results.** The main result of the paper is a complete classification, up to equiconical deformation, of all Barth–Bauer quartics $X \in \mathcal{B}$ and pairs (X, Ω), where Ω is a distinguished unordered set of 16 pairwise disjoint irreducible conics on X. The findings are presented in Tables 3, 5, and 6 (see [8] for the full version). Let n and $\tilde{n}$ be the numbers of isomorphism classes of Fano graphs and relative forms, respectively, and let (r, c) and $(\tilde{r}, \tilde{c})$ be the corresponding total component counts, see (1.2). Itemizing by the codimension of the strata in $\mathcal{B}$, we have:

- $\operatorname{codim} = 0$: $(n; r, c) = (\tilde{n}; \tilde{r}, \tilde{c}) = (1; 1, 0)$, see Table 3;
- $\operatorname{codim} = 1$: $(n; r, c) = (\tilde{n}; \tilde{r}, \tilde{c}) = (7; 7, 0)$, see Table 3;
- $\operatorname{codim} = 2$: $(n; r, c) = (43; 43, 0)$, $(\tilde{n}; \tilde{r}, \tilde{c}) = (47; 49, 0)$, see Table 5;
- $\operatorname{codim} = 3$: $(n; r, c) = (211; 208, 189)$, $(\tilde{n}; \tilde{r}, \tilde{c}) = (285; 231, 682)$, see [8].

Apart from a plethora of examples of smooth quartics with many conics, these data are of rather technical nature. For this reason, we state also a few immediate consequences that are of a more general interest. Proofs, if any, are found in §7.

Theorem 1.3 (see §7.1)**.** *The maximal number of conics on a Barth–Bauer quartic is* 800*. Up to projective transformation, there is a unique Barth–Bauer quartic with* 800 *conics*: *it is the Mukai quartic*

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 + 12z_0z_1z_2z_3 = 0$$

admitting a faithful symplectic action of the Mukai group M_{20}*, see* [21].

A smooth quartic with 800 irreducible conics was first discovered in [7]; then, X. Roulleau observed (with a reference to [4]) that this quartic must be given by Mukai's polynomial above, upon which B. Naskręcki found explicit equations of all 800 conics. Together with [7], Theorem 1.3 substantiates the conjecture that 800 is the sharp upper bound on the number of conics on a smooth quartic surface.

Theorem 1.4 (see Tables 3, 5, and 6)**.** *The maximal number of lines on a Barth–Bauer quartic is* 48*. Up to projective transformation, there are four Barth–Bauer quartics* (*two real and two complex conjugate*) *with* 48 *lines, see Table 6.* ◁

Recall that the maximal number of lines on a smooth quartic is 64 (see [28, 25]) and, according to [11], the only quartic with 64 lines is Schur's quartic [27]

$$z_0(z_1^3 - z_0^3) = z_2(z_3^3 - z_2^3).$$

Remarkably, this quartic, denoted $\mathbf{X}_{64}$ in [11], is also Kummer, but its Kummer divisors map to lines rather than conics.

The next statement illustrates the speculation (see [9]) that reducible conics do not affect the upper bounds, as a quartic with many conics has no lines. (A similar phenomenon is observed in [12], where we study lines on octic $K3$-surfaces that are allowed to have singularities: the presence of exceptional divisors does reduce the upper bound on the number of lines.)

Theorem 1.5 (see Tables 3, 5, and 6)**.** *Let* X *be a Barth–Bauer quartic. Then*:

(1) *if* $|\mathrm{Fn}_2 X| > 560$*, then* X *is a singular* $K3$*-surface*;
(2) *if* $|\mathrm{Fn}_2 X| > 576$*, then* X *has no lines or reducible conics*;
(3) *if* $|\mathrm{Fn}_2^* X| > 544$*, then* X *has no lines or reducible conics.* ◁

Following the tradition, we also consider the problem over $\mathbb{R}$, *i.e.*, try to estimate the number of *real* conics on a *real* Barth–Bauer quartic. Recall that a *real structure* on a complex surface X is an anti-holomorphic involution $\tau\colon X \to X$, and a curve $c \subset X$ is called *real* (with respect to τ) if $\tau(c) = c$. In the next statement, we do *not* assume in advance that the real structure should preserve (any) collection of 16 Kummer conics, even though it is the case for the maximizing family.

Theorem 1.6 (see §7.3)**.** *The maximal number of* real *conics on a* real *Barth–Bauer quartic is* 560*. There is a unique* 1*-parameter equiconical family* (*the row marked with a* $*$ *in Table 5*) *of real Barth–Bauer quartics with* 560 *real conics*; *all conics are irreducible and have real points.*

The closure of the 1-parameter family given by Theorem 1.6 contains a single (up to projective transformation) singular quartic (see $*$ in Table 6) with the same collection of 560 real conics and 24 pairs of complex conjugate ones.

The last two theorems should extend to all smooth quartics; however, unlike Theorem 1.3, the bounds given by Theorems 1.5 and 1.6 would need to be corrected. Some counterexamples, with many reducible conics, can be found in [11]. Thus,

- $\mathbf{X}_{64}$ has 576 reducible $+$ 144 irreducible $= 720$ conics, *cf.* Theorem 1.5(2).

In fact, 576 is the maximal number of reducible conics in a smooth quartic and 720 appears to be the correct bound for Theorem 1.5(2). As another example,

- the quartic $\mathbf{Y}_{56}$ with 56 (the maximal number, see [11]) real lines given by
$$3z_0^2z_1z_2 + 3z_1z_2z_3^2 - z_1^3z_2 - z_1z_2^3 + 2\epsilon z_0^3z_3 - 2\epsilon z_0z_3^3 = 0, \quad \epsilon^2 = 2.$$
has 448 reducible $+$ 188 irreducible $= 636$ conics, all real, *cf.* Theorem 1.6.

It appears that Barth–Bauer quartics are not good candidates for maximizing the number of real conics; this fact is discussed in §7.2. I would conjecture the existence of a smooth real quartic with more than 720 irreducible real conics.

Proposition 1.7 (see §7.4)**.** *Let $X \in \mathcal{X}$ be a very general member of an equiconical stratum of Barth–Bauer quartics. Then the lattice $NS(X)$ is generated over $\mathbb{Z}$ by the classes of lines and conics on X. In particular, $O_h(NS(X)) = \operatorname{Aut}(\operatorname{Fn} X)$.*

Corollary 1.8 (see §7.4)**.** *Let $\Gamma := \operatorname{Fn} X$ be the full Fano graph of a Barth–Bauer quartic. Then any discrete subgraph $\Omega \subset \Gamma$ with 16 2-vertices inherits from Γ a canonical 4-Kummer structure (see §2.2).*

1.3. **Contents of the paper.** In §2, we describe the homological properties of the Barth–Bauer quartics, mainly using Nikulin [22]. In §3, we discuss the equiconical strata and their connectedness (following Nikulin [23] and Dolgachev [13]); then, we describe the construction by means of extensions *via* extra lines and conics.

The results of the computation are presented in §4–§6, where we treat the strata of codimension 1 to 3, respectively. §7 fills in a few missing details in the proofs.

1.4. **Common notation.** Unless stated otherwise, all lattices considered in the paper are *even*: $x^2 = 0 \bmod 2$ for all $x \in L$. We use $\oplus$ for *orthogonal* direct sums of lattices, as opposed to $+$ for direct, but not necessarily orthogonal sums. For a lattice L, we use the following notation:

- $\det L$ is the determinant of the Gram matrix of L in any integral basis;
- L is *nondegenerate* (resp. *unimodular*) if $\det L \neq 0$ (resp. $\det L = \pm 1$);
- $\sigma_\pm(L)$ are the inertia indices of the quadratic space $L \otimes \mathbb{R}$;
- $O(L)$ is the orthogonal group of L;
- $O^+(L) \subset O(L)$ is the subgroup preserving the *positive sign structure*, *i.e.*, coherent orientation of maximal positive definite subspaces of $L \otimes \mathbb{R}$;
- $L^\vee := \{x \in L \otimes \mathbb{Q} \,|\, x \cdot y \in \mathbb{Z} \text{ for all } y \in L\}$ is the *dual group*;
- $\operatorname{discr} L := L^\vee/L$ is the *discriminant form* of L (see q_L in [23]);
- $\operatorname{discr}_p L := (\operatorname{discr} L) \otimes \mathbb{Z}_p$ is the p-primary part for a prime p;
- $\operatorname{Aut}(\operatorname{discr} L)$ *etc.* are the groups of auto-isometries of discriminant forms;
- $\mathrm{d}_L \colon O(L) \to \operatorname{Aut}(\operatorname{discr} L)$ is the canonical homomorphism;
- $nL := L^{\oplus n}$ for an integer $n \geqslant 1$;
- $L(q)$, $q \in \mathbb{Q}$, is the same abelian group with the form $x \otimes y \mapsto q(x \cdot y)$.

We denote by $[a] := \mathbb{Z}u$, $u^2 = a$, a rank 1 lattice and use the inline notation

(1.9) $[a, b, c]$ stands for the lattice $\mathbb{Z}u + \mathbb{Z}v$, $u^2 = a$, $v^2 = c$, $u \cdot v = b$.

to describe lattices of rank 2. We fix a lattice $\mathbf{L}$ isomorphic to the second homology

$$\mathbf{L} := H_2(X; \mathbb{Z}) \cong 2\mathbf{E}_8 \oplus 3\mathbf{U} \tag{1.10}$$

of a $K3$-surface; here, $\mathbf{E}_8$ and $\mathbf{U} \cong [0, 1, 0]$ are the unique unimodular even lattices of signature $(0, 8)$ and $(1, 1)$, respectively.

Given a lattice $\mathbb{Z}\Omega$ with a distinguished basis Ω, we identify subsets $\mathfrak{s} \subset \Omega$ with the vectors $\mathfrak{s} := \sum e \in \mathbb{Z}\Omega$, the summation running over $e \in \mathfrak{s}$. We will also use the shorthand notation

$$\hbar := \tfrac{1}{2}h \in \mathbb{Q}h, \qquad \|\mathfrak{s}/\mathfrak{r}\| := \tfrac{1}{2}(\mathfrak{s} \cap \mathfrak{r}) - \tfrac{1}{2}(\mathfrak{s} \smallsetminus \mathfrak{r}) \in \mathbb{Q}\Omega, \quad \mathfrak{r}, \mathfrak{s} \subset \Omega,$$

where h will commonly stand for the class of the hyperplane section. We use the deprecated notation $\bar{\mathfrak{s}} := \Omega \smallsetminus \mathfrak{s}$ for the complement of a subset $\mathfrak{s} \subset \Omega$.

On a $K3$-surface X, we identify (-2)-curves $c \subset X$ (irreducible or reducible) with their classes in $NS(X)$. The *transcendental lattice* $NS(X)^\perp \subset H_2(X;\mathbb{Z}) \cong \mathbf{L}$ is denoted by $T(X)$. A $K3$-surface X is *singular* if $\operatorname{rk} NS(X) = 20$.

1.5. **Acknowledgements.** I would like to express my gratitude to Dino Festi, Bartosz Naskręcki, Sławomir Rams, Xavier Roulleau, and Matthias Schütt for a number of fruitful discussions.

2. The Barth–Bauer family

In this section, we give a detailed description of the homological properties of an abstract Kummer surface and then, of a generic Barth–Bauer quartic.

2.1. **Abstract Kummer structures** (see [22]). Consider the set $\Omega := \{1, \ldots, 16\}$, denote $\mathcal{O}_0 := \{\varnothing\}$ and $\mathcal{O}_{16} := \{\Omega\}$, and define an *abstract Kummer structure* as a collection $\mathcal{O}_8$ of 30 eight-element subsets $\mathfrak{o} \subset \Omega$ such that $\mathcal{O}_* := \bigcup_n \mathcal{O}_n$ is closed under the symmetric difference $\triangle$. (We extend $\mathcal{O}_n$ and other similar notations *via* $\mathcal{O}_n := \varnothing$ unless the index $n \in \mathbb{Z}$ has been mentioned explicitly.)

An abstract Kummer structure gives rise to the sets

$$\mathcal{C}_* := \bigl\{\mathfrak{s} \subset \Omega \bigm| |\mathfrak{s} \cap \mathfrak{o}| = 0 \bmod 2 \text{ for all } \mathfrak{o} \in \mathcal{O}_*\bigr\}, \quad \mathcal{C}_n := \bigl\{\mathfrak{s} \in \mathcal{C}_* \bigm| |\mathfrak{s}| = n\bigr\}$$

and equivalence relation

$$\mathfrak{r} \sim \mathfrak{s} \text{ iff } \mathfrak{r} \triangle \mathfrak{s} \in \mathcal{O}_* \text{ and equivalence classes } [\mathfrak{s}] \text{ and } [\mathfrak{s}]_n := [\mathfrak{s}] \cap \mathcal{C}_n,\ n \in \mathbb{Z}.$$

According to Nikulin [22], an abstract Kummer structure exists and is unique up to permutation. Consider the setwise stabilizer $\operatorname{stab} \mathcal{O}_* \subset \mathbb{S}(\Omega)$ and its subgroup $\mathfrak{G}_\omega$ acting identically on $\mathcal{C}_*/{\sim}$. Then, the group $\mathfrak{G}_\omega \cong (\mathbb{Z}/2)^4$ (see #21 in Table 2) acts simply transitively on Ω, making it a $\mathfrak{G}_\omega$-torsor or, equivalently, a dimension 4 affine space over $\mathbb{F}_2$, and $\mathcal{O}_8$ is the set of all affine hyperplanes. In this language, $\mathfrak{G}_\omega \subset \operatorname{stab} \mathcal{O}_*$ are, respectively, the group of translations and that of all affine linear transformations of Ω.

For an alternative description, consider the extended binary Golay code (see, *e.g.*, [5]) $\mathcal{G}_*$ on the set $\{1, \ldots, 24\}$ and take for Ω a codeword of length 16. Then,

$$\mathcal{O}_8 = \bigl\{\mathfrak{o} \bigm| \mathfrak{o} \subset \Omega, \mathfrak{o} \in \mathcal{G}_8\bigr\}, \qquad \mathcal{C}_* = \bigl\{\mathfrak{o} \cap \Omega \bigm| \mathfrak{o} \in \mathcal{G}_*\bigr\}.$$

(Note that $\mathcal{C}_n = \varnothing$ if n is odd or $n = 2, 14$.) The group $\operatorname{stab} \mathcal{O}_*$ is the restriction to Ω of its setwise stabilizer in the Mathieu group $M_{24} = \operatorname{stab} \mathcal{G}_* \subset \mathbb{S}_{24}$.

Consider the orthogonal direct sum $\mathbb{Z}\Omega := \bigoplus \mathbb{Z}e$, $e \in \Omega$, $e^2 = -2$. An abstract Kummer structure on Ω defines a finite index extension $\mathbf{S}(\mathcal{O}_*) \supset \mathbb{Z}\Omega$, *viz.* the one generated over $\mathbb{Z}$ by the vectors $e \in \Omega$ and $\frac{1}{2}\mathfrak{o} \in \mathbb{Q}\Omega$, $\mathfrak{o} \in \mathcal{O}_*$.

Lemma 2.1 (see Nikulin [22])**.** *The lattice $\mathbf{S} := \mathbf{S}(\mathcal{O}_*)$ is the only, up to the action of the orthogonal group $O(\mathbb{Z}\Omega)$, finite index extension of $\mathbb{Z}\Omega$ such that*

(1) $\mathbf{S}$ *has no vectors of square (-2) other than $\pm e$, $e \in \Omega$, and*
(2) $\mathbf{S}$ *admits a primitive isometric embedding to the lattice $\mathbf{L}$ given by* (1.10).

The original abstract Kummer structure $\mathcal{O}_*$ *is recovered from an overlattice* $\mathbf{S}\supset\mathbb{Z}\Omega$ *satisfying the two conditions above* via $\mathcal{O}_*=\{\mathfrak{o}\subset\Omega\,|\,\frac{1}{2}\mathfrak{o}\in\mathbf{S}\}$. ◁

The primitive isometric embedding $\mathbf{S}\hookrightarrow\mathbf{L}$ is unique up to isomorphism, and the orthogonal complement is

$$\mathbf{T}:=\mathbf{S}^\perp_{\mathbf{L}}\cong 3\mathbf{U}(2).$$

In particular, one has

$$u^2=0 \bmod 4,\quad u\cdot v=0 \bmod 2\quad \text{for any } u,v\in\mathbf{T}. \tag{2.2}$$

The next two statements are immediate consequences of [22] and the techniques developed in [23] (where the notation q_L is used for the discriminant $\operatorname{discr} L$).

Lemma 2.3. *Assume an abstract Kummer structure* $\mathcal{O}_*$ *and a primitive isometric embedding* $\mathbf{S}:=\mathbf{S}(\mathcal{O}_*)\hookrightarrow\mathbf{L}$ *fixed. Then, there are canonical bijective isometries*

$$\kappa\colon \mathbf{T}/2\mathbf{T}\to\operatorname{discr}\mathbf{T}\to\operatorname{discr}\mathbf{S}\to\mathbf{S}/2\mathbf{S}\to\mathcal{C}_*/{\sim},$$

where the quadratic forms on $\mathbf{T}/2\mathbf{T}$ *and* $\mathbf{S}/2\mathbf{S}$ *are* $s\mapsto\frac{1}{4}s^2 \bmod 2\mathbb{Z}$ *and* $\mathcal{C}_*/{\sim}$ *is endowed with the group law induced by* $\triangle$ *and the form* $[\mathfrak{s}]\mapsto\frac{1}{2}|\mathfrak{s}| \bmod 2\mathbb{Z}$. ◁

Lemma 2.4. *Under the assumptions of Lemma 2.3, consider another overlattice* $\mathbf{S}\subset N\subset\mathbf{L}$ *primitive in* $\mathbf{L}$ *and denote by* $\mathbf{S}^\perp:=N\cap\mathbf{T}$ *the orthogonal complement of* $\mathbf{S}$ *in* N*. Then* $\mathbf{S}^\perp$ *is primitive in* $\mathbf{T}$ *and* $N\subset(\mathbf{S}\oplus\mathbf{S}^\perp)\otimes\mathbb{Q}$ *is generated over* $\mathbb{Z}$ *by* $\mathbf{S}\oplus\mathbf{S}^\perp$ *and all vectors*

$$\tfrac{1}{2}(\mathfrak{s}+s),$$

where $s\in\mathbf{S}^\perp$ *and* $\mathfrak{s}\in\kappa(s \bmod 2\mathbf{S}^\perp)$ *is a representative. In fact, it suffices to fix a basis for* $\mathbf{S}^\perp/2\mathbf{S}^\perp$ *and take for* s *one representative of each basis element.* ◁

2.2. **Polarized Kummer surfaces** (see [22]). Let X be the Kummer surface of a complex torus A. The set Ω of the 16 Kummer divisors on X is in a bijection with the set of order 2 points in A, which is a subgroup $(\mathbb{Z}/2)^4\subset A$; thus, Ω acquires from X a natural abstract Kummer structure. Alternatively, the primitive hull of $\mathbb{Z}\Omega$ in $NS(X)$ must be a lattice $\mathbf{S}$ as in Lemma 2.1; by the lemma, $\mathbf{S}$ gives rise to an abstract Kummer structure on Ω.

Now, assume that X is embedded, as a smooth surface, to the projective space $\mathbb{P}^{d+1}$ so that each Kummer divisor $e\in\Omega$ is mapped to an *irreducible* conic. Denote by $h\in NS(X)$ the class of a hyperplane section and assume that it is not divisible by 2 (which is always the case if $h^2\ne 0 \bmod 8$ or $h^2<16$, see [26]). The lattice

$$\mathbb{Z}\Omega+\mathbb{Z}h\subset NS(Z),\qquad h^2=2d,\quad h\cdot e=2 \text{ for all } e\in\Omega,$$

splits as

$$\mathbb{Z}\Omega\oplus\mathbb{Z}\tilde{h},\qquad \tilde{h}:=h+\Omega,\quad \tilde{h}^2=h^2+32, \tag{2.5}$$

and from (2.2) we conclude that $h^2=0 \bmod 4$. Then, by Lemma 2.4, the primitive hull N of $\mathbb{Z}\Omega+\mathbb{Z}h$ in $NS(X)$ contains the overlattice

$$\mathbf{S}_h:=\mathbf{S}_h(\mathcal{K}_*)\supset\mathbb{Z}\Omega+\mathbb{Z}h \text{ spanned by } \mathbf{S}(\mathcal{O}_*),\ h, \text{ and } \tfrac{1}{2}(h+\mathfrak{k}),\ \mathfrak{k}\in\mathcal{K}_*, \tag{2.6}$$

where $\mathcal{K}_*\subset\mathcal{C}_*$ is a certain equivalence class, $\mathcal{K}_*\ne\mathcal{O}_*$, such that $2|\mathfrak{k}|=h^2 \bmod 8$ for $\mathfrak{k}\in\mathcal{K}_*$. A choice of this extra class, which is determined by the smooth embedding $X\hookrightarrow\mathbb{P}^{d+1}$, is called an h^2-*Kummer structure* on Ω. This choice defines a coarser equivalence relation

$$\mathfrak{r}\approx\mathfrak{s} \text{ iff } \mathfrak{r}\triangle\mathfrak{s}\in\mathcal{O}_*\cup\mathcal{K}_* \text{ and equivalence classes } [\![\mathfrak{s}]\!] \text{ and } [\![\mathfrak{s}]\!]_n:=[\![\mathfrak{s}]\!]\cap\mathcal{C}_n,\ n\in\mathbb{Z}.$$

If $h^2 = 4$, a 4-*Kummer structure* $\mathcal{K}_* = \mathcal{K}_6 \cup \mathcal{K}_{10}$ has 16 sets of size 6 and 16 sets of size 10; it is unique up to $\operatorname{stab}\mathcal{O}_*$. The discriminant of the respective lattice $\mathbf{S}_h$ has 2- and 3-torsion:

$$\operatorname{discr}_2 \mathbf{S}_h \cong \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \oplus \left[\tfrac{1}{4}\right], \qquad \operatorname{discr}_3 \mathbf{S}_h \cong \left[\tfrac{4}{9}\right]. \tag{2.7}$$

Note that $\mathcal{O}_* = \{\mathfrak{k}_1 \vartriangle \mathfrak{k}_2 \,|\, \mathfrak{k}_1, \mathfrak{k}_2 \in \mathcal{K}_*\}$ is determined by $\mathcal{K}_*$, provided that the set obtained is an abstract Kummer structure (which is a restriction on $\mathcal{K}_*$).

2.3. **Barth–Bauer quartics.** As explained in the previous section, the Néron–Severi lattice of a Barth–Bauer quartic (X, Ω) with a distinguished set Ω of Kummer conics must contain the overlattice $\mathbf{S}_h$ as in (2.6) defined by a certain 4-Kummer structure on Ω. To complete the description of the generic Néron–Severi lattice, we need a realizability criterion.

Definition 2.8 (*cf.* Saint-Donat [26])**.** A hyperbolic overlattice $N \supset \mathbf{S}_h$, see (2.6), is called *admissible* if there is no vector $r \in N$ such that either

1. $r^2 = -2$ and $r \cdot h = 0$ (*exceptional divisor*), or
2. $r^2 = 0$ and $r \cdot h = \pm 2$ (2-*isotropic vector*), or
3. $r^2 = -2$, $r \cdot h = 1$, and $r \cdot e < 0$ for some $e \in \Omega$ (*missing conic*).

(The last condition is due to the fact that we insist that each conic $e \in \Omega$ should be irreducible.) An admissible lattice N is called *geometric* if it admits a primitive isometric embedding $N \hookrightarrow \mathbf{L}$, see (1.10).

The next statement is well known. It follows from the surjectivity of the period map for $K3$-surfaces (Kulikov [19]) and the results of Saint-Donat [26]; *cf.* also [9] or [12] for an accurate restatement of [26] in the homological language.

Proposition 2.9. *Consider an overlattice $N \supset \mathbf{S}_h \supset \mathbb{Z}\Omega + \mathbb{Z}h$, $h^2 = 4$, as in* (2.6)*. Then, there exists a Barth–Bauer quartic $(X, \Omega) \subset \mathbb{P}^3$ such that*

$$\bigl(NS(X); \Omega, h\bigr) \cong (N; \Omega, h)$$

if and only if N is geometric. ◁

Corollary 2.10. *Given a Barth–Bauer quartic $(X, \Omega) \subset \mathbb{P}^3$, the primitive hull N of $\mathbb{Z}\Omega + \mathbb{Z}h$ in $NS(X)$ is the lattice $\mathbf{S}_h$ given by* (2.6)*.*

Proof. According to Lemma 2.4, the only other option is that the sublattice $\mathbb{Z}\tilde{h}$, $\tilde{h} := h + \Omega$, is not primitive in $\mathbf{T}$, *i.e.*, $\tilde{h}$ is divisible by 3. (Recall that $9 \mid 36 = \tilde{h}^2$.) But then, for any $\mathfrak{k} \in \mathcal{K}_{10}$, the vector $\frac{1}{3}\tilde{h} - \frac{1}{2}(h + \mathfrak{k}) \in N$ would be an exceptional divisor, see Definition 2.8(1). □

Using the description of the nef cone of a $K3$-surface (see, *e.g.*, [18, §8.1]), one can easily compute the sets of lines and conics on a smooth quartic X in terms of the polarized lattice $N := NS(X) \ni h$ (*cf.* [9]):

$$\begin{aligned} \operatorname{Fn}_n(N, h) &= \bigl\{u \in N \bigm| u^2 = -2,\ u \cdot h = n\bigr\}, \quad n = 1, 2, \\ \operatorname{Fn}_2^*(N, h) &= \bigl\{u \in \operatorname{Fn}_2(N, h) \bigm| u \cdot v \geqslant 0 \text{ for all } v \in \operatorname{Fn}_1(N, h)\bigr\}. \end{aligned} \tag{2.11}$$

For a very general Barth–Bauer quartic $X \in \mathcal{B}$, using the lattice $NS(X) = \mathbf{S}_h$ given by Corollary 2.10, this computation shows that X has no lines (as $h \cdot u = 0 \bmod 2$ for all $u \in \mathbf{S}_h$) and it has 352 conics, all irreducible, *viz.*

- the 16 original Kummer conics $e \in \Omega$,

TABLE 1. $\mathfrak{G}$-orbits on $\mathcal{C}_n$

n	even	odd	$\mathcal{O}_* \cup \mathcal{K}_*$
0			1×1
4	15×4	20×4	
6	15×16	12×16	1×16
8	15×24	20×24	1×30
10	15×16	12×16	1×16
12	15×4	20×4	
16			1×1

TABLE 2. Symplectic groups G_ω

#	$\lvert G_\omega \rvert$	index	G_ω
21	16	14	C_4^2
39	32	27	$2^4 C_2$
49	48	50	$2^4 C_3$
56	64	138	$\Gamma_{25} a_1$
65	96	227	$2^4 D_6$
75	192	1023	$4^2 \mathfrak{A}_4$
81	960	11357	M_{20}

- 16 *dual Kummer conics* $e^* := h - e$, $e \in \Omega$, and
- 320 *Barth–Bauer*, or *B*2*-conics*

$$\hbar + \|\mathfrak{k}/\mathfrak{s}\|, \qquad \mathfrak{s} \subset \mathfrak{k} \in \mathcal{K}_6, \ |\mathfrak{s}| = 3. \tag{2.12}$$

This computation agrees with [2] and leads us to the following corollary.

Corollary 2.13. *The Néron–Severi lattice* $NS(X) = \mathbf{S}_h$ *of a very general Barth–Bauer quartic* (X, Ω) *is generated over* $\mathbb{Z}$ *by the conics on* X*. Hence, one has*

$$O_h\big(NS(X)\big) = O_h(\mathbf{S}_h) = \operatorname{Aut}(\operatorname{Fn} X)$$

and the 4*-Kummer structure on* $\Omega \subset \operatorname{Fn} X$ *is recovered from* $\operatorname{Fn} X$*.* ◁

2.4. **Symmetries.** It is immediate that, for a very general Barth–Bauer quartic (X, Ω) as in Corollary 2.13, one has

$$\operatorname{Aut}(\operatorname{Fn} X, \Omega) = \mathfrak{G} := \operatorname{stab}(\mathcal{K}_* \cup \mathcal{O}_*), \qquad \operatorname{Aut}(\operatorname{Fn} X) = \mathfrak{G} \times (\mathbb{Z}/2),$$

where $\mathfrak{G} \subset \mathbb{S}(\Omega)$ is the setwise stabilizer of the 4-Kummer structure and the second factor $\mathbb{Z}/2$ is generated by the duality involution $c \mapsto c^* := h - c$. (Geometrically, $c \cup c^*$ is the quartic curve cut by X on the plane spanned by c.)

Given a 4-Kummer structure $\mathcal{K}_*$, we subdivide the sets $\mathfrak{s} \in \mathcal{C}_*$ into *even* and *odd*, according to the parity $|\mathfrak{s} \cap \mathfrak{k}| \bmod 2$ of the intersection with some (any) $\mathfrak{k} \in \mathcal{K}_*$. The $\mathfrak{G}$-action on $\mathcal{C}_*$ preserves parity and both $\sim$- and $\approx$-equivalence. Each nonempty set $\mathcal{C}_n \smallsetminus (\mathcal{O}_* \cup \mathcal{K}_*)$, $n \in \mathbb{Z}$, consists of exactly two $\mathfrak{G}$-orbits: one even and one odd, see Table 1 where the counts are given in the form

$$(\text{number of } \sim\text{-classes}) \times (\text{size of a class}).$$

In view of Lemma 2.3 and the global Torelli theorem for $K3$-surfaces [24], the subgroup $\mathfrak{G}_\omega \subset \mathfrak{G}$ acting identically on $\mathcal{C}_*/\!\sim$ (see §2.1 and #21 in Table 2) is the group of symplectic projective automorphisms of a very general Barth–Bauer quartic X; hence, it acts symplectically on any Barth–Bauer quartics. For further references, all groups appearing in this way (for quartics generic in their respective equiconical strata) are listed in Table 2, where # is a reference to the list found in Xiao [29], "index" is the index in the GAP's small group library, so that

$$G_\omega = \texttt{SmallGroup}(|G_\omega|, \mathit{index}),$$

and the last column is a description of the group in the notation of [29].

3. Equiconical strata

Below, we discuss the connected components of the equiconical strata (see §3.1) and their construction by means of extensions of the lattice $\mathbf{S}_h$ (see §3.2). In §3.3, we outline the algorithms used in the computation.

3.1. **Connected components** (see [13, 23]). Fix a bi-colored graph $\Gamma \supset \Omega$ and let $\mathcal{F}(\Gamma) := (\mathbb{Z}\Gamma + \mathbb{Z}h)/\ker$, where $\mathbb{Z}\Gamma$ is freely generated by the vertices $v \in \Gamma$ and

$$h^2 = 4, \quad v^2 = -2, \quad v \cdot h = \operatorname{color}(v), \quad u \cdot v = \#\operatorname{edges}(u, v), \quad u, v \in \Gamma.$$

According to Proposition 2.9 and (2.11), the original graph Γ is the Fano graph of a Barth–Bauer quartic if and only if there is an extension

$$N \supset \mathcal{F}(\Gamma)\colon \quad N \text{ is geometric, } [N : \mathcal{F}(\Gamma)] < \infty, \text{ and } \operatorname{Fn}(N, h) = \Gamma. \tag{3.1}$$

Furthermore,

$$\mathcal{X}(\Gamma) = \bigsqcup \mathcal{X}(N),$$

the union running over the set of all such extensions, regarded up to abstract lattice isomorphism preserving h.

Thus, we assume that there is a primitive isometry $\iota\colon N \hookrightarrow \mathbf{L}$. Given a subgroup $G \subset O^+(N)$, introduce the following definitions:

- two primitive isometries $\iota_i\colon N \hookrightarrow \mathbf{L}$, $i = 1, 2$, are *G-equivalent* if there exists a pair $\alpha \in G$, $\beta \in O^+(\mathbf{L})$ such that $\iota_2 \circ \alpha = \beta \circ \iota_1$;
- two (anti-)isometries $\psi_i\colon \operatorname{discr} N \to \operatorname{discr} T$ are G-equivalent if there exists a pair $\alpha \in G$, $\beta \in O^+(T)$ such that $\psi_2 \circ \mathrm{d}_N(\alpha) = \mathrm{d}_T(\beta) \circ \psi_1$.

Note the usage of O^+ rather than the more conventional O: this approach excludes anti-isomorphisms of $K3$-surfaces, *cf.* Proposition 3.4(3) below.

Proposition 3.2 (*cf.* Nikulin [23])**.** *A primitive isometry $\iota\colon N \to \mathbf{L}$ gives rise to a bijective anti-isometry $\psi : \operatorname{discr} N \to \operatorname{discr} N^\perp$; as a consequence, the genus of the lattice $N^\perp$ is independent of ι, depending on N only.*

The G-equivalence classes of isometries $\iota\colon N \hookrightarrow \mathbf{L}$ are in a canonical bijection with the following sets of data:

(1) *a lattice* (*isomorphism class*) $T \in \operatorname{genus}(N^\perp)$, *and*
(2) *a G-equivalence class of anti-isometries $\psi\colon \operatorname{discr} N \to \operatorname{discr} T$.* ◁

For the isometry $\iota\colon N \hookrightarrow \mathbf{L}$ corresponding to a pair (T, ψ) as in Proposition 3.2, we have

$$\{g \in O(\mathbf{L}) \mid g(N) = N,\ g|_N \in G_N,\ g|_T \in G_T\} = G_N \times_{\operatorname{Aut}(\operatorname{discr} T)} G_T, \tag{3.3}$$

where $G_N \subset O(N)$, $G_T \subset O(T)$ is a given pair of subgroups and the amalgamated product is over the diagram

$$G_N \to O(N) \xrightarrow{\psi \circ \mathrm{d}_N} \operatorname{Aut}(\operatorname{discr} T) \xleftarrow{\mathrm{d}_T} O(T) \leftarrow G_T.$$

(In more formal notation, G_N and G_T should first be replaced with the respective pull-backs of the intersection of their images in $\operatorname{Aut}(\operatorname{discr} T)$.)

An *N-polarized* (more precisely, (N,h)-polarized) $K3$-surface is defined as a pair (X,φ), where X is a $K3$-surface and $\varphi\colon N \hookrightarrow NS(X)$ is a primitive isometry such that $\varphi(h)$ is ample. Two N-polarized $K3$-surfaces (X_i,φ_i), $i=1,2$, are *isomorphic* if there exists an isomorphism $f\colon X_1 \to X_2$ such that $\varphi_2 = f_*\circ\varphi_1$. An N-polarized $K3$-surface (X,φ) defines a $\{1\}$-equivalence class $\varphi\colon N \hookrightarrow NS(X) \hookrightarrow \mathbf{L}$ of primitive isometries and, according to Dolgachev [13], assuming that N satisfies condition (1) in Definition 2.8 (so that h has a chance to be ample), each such class gives rise to a coarse moduli space

$$\mathcal{Z}(N \hookrightarrow \mathbf{L}),$$

which is a connected quasi-projective variety of dimension $20 - \operatorname{rk} N$. If N is also admissible, the subspace

$$\mathcal{Z}^\circ(N \hookrightarrow \mathbf{L}) := \bigl\{X \in \mathcal{Z}(N \hookrightarrow \mathbf{L}) \bigm| X \subset \mathbb{P}^3 \text{ is smooth, } \operatorname{Fn}(X) = \operatorname{Fn}(N,h)\bigr\}$$

is the complement of the countable collection of divisors

$$\bigl\{X \in \mathcal{Z}(N \hookrightarrow \mathbf{L}) \bigm| NS(X) \ni r\bigr\}, \qquad r \in \mathbf{L} \smallsetminus N,$$

where $r^2 = -2$ and $|r\cdot h| \leqslant 2$ or $r^2 = 0$ and $r\cdot h = 2$, *cf.* (2.11) and Definition 2.8; hence, it is still connected. Finally, getting rid of the N-marking φ, we have

$$\mathcal{X}(N)/PGL(\mathbb{C},4) = \Bigl(\bigsqcup \mathcal{Z}(N \hookrightarrow \mathbf{L})\Bigr)/O_h(N),$$

where the union runs over all $\{1\}$-isomorphism classes of isometries. Since $PGL(\mathbb{C},4)$ is connected, we arrive at the following well known statement.

Proposition 3.4. *Fix an extension $N \supset \mathcal{F}(\Gamma)$ as in (3.1), regarded up to abstract lattice isomorphism preserving h, and let $G := O_h(N)$. Then:*

1. *the stratum $\mathcal{X}(N) \subset \mathcal{B}$ is of pure codimension $\operatorname{rk} N - 17$;*
2. *the connected components of $\mathcal{X}(N)$ are in a natural bijection with the G-equivalence classes of primitive isometries $N \hookrightarrow \mathbf{L}$,* cf. *Proposition 3.2;*
3. *a component is real if and only if, in Proposition 3.2, there is an isometry $\beta \in O(T) \smallsetminus O^+(T)$ such that $\mathrm{d}_T(\beta) \in \psi\circ\mathrm{d}_N(G)$.* ◁

A similar argument shows that

$$\tilde{\mathcal{X}}(\Gamma,\Omega) = \bigsqcup \tilde{\mathcal{X}}(N,\Omega),$$

where the union runs over all finite index extensions $N \supset \mathcal{F}(\Gamma)$ as in (3.1) regarded up to lattice isomorphism preserving h and Ω as a set, and then the connected components of each $\tilde{\mathcal{X}}(N,\Omega)$ are described by an analogue of Proposition 3.4, with N replaced with a pair (N,Ω) and, respectively, G changed to $O_h(N,\Omega)$.

3.2. **Extensions of $\mathbf{S}_h$.** Paraphrasing (3.1), strata of codimension r are described by geometric extension, necessarily primitive, $N \supset \mathbf{S}_h$ of corank r that are rationally generated by lines and conics, *i.e.*, finite index extensions of lattices of the form

$$\mathbf{S}_h[u_1,\ldots,u_r] := \mathbf{S}_h + \mathbb{Z}u_1 + \ldots + \mathbb{Z}u_r, \tag{3.5}$$

where $u_1,\ldots,u_r$ are extra lines or conics linearly independent over $\mathbf{S}_h$. To complete the description of (3.5), we need to specify, for $i,j = 1,\ldots,r$,

1. the intersections $u_i\cdot h = 1$ or 2 if u_i is a line or conic, respectively,
2. the intersections $u_i\cdot e$ for all $e \in \Omega$, and

(3) the Gram matrix $[u_i \cdot u_j]$, where $u_i^2 = -2$ for all i.

Geometrically, it is obvious that, given two lines l_1, l_2, a conic c, and an *irreducible* conic e, one has

$$l_1 \cdot l_2 \in \{0, 1\}, \qquad l_1 \cdot e \in \{0, 1, 2\}, \qquad c \cdot e \in \{0, 1, 2, 4\}$$

and $c \cdot e = 4$ if and only if $c + e = h$, in which case c and e are dependent over $\mathbf{S}_h$. It follows that item (2) in the above list is determined by the *supports*

$$\operatorname{supp}_n u_i := \bigl\{ e \in \Omega \bigm| e \cdot u_i = n \bigr\} \subset \Omega, \quad n = 1, 2, \tag{3.6}$$

which, for each u_i, are two disjoint subsets of Ω. Denoting, for short, $\mathfrak{u}_i := \operatorname{supp}_1 u_i$ and $\mathfrak{u}'_i := \operatorname{supp}_2 u_i$, we have

$$\mathfrak{u}_i \in \mathcal{C}_* \text{ is even if } u_i \text{ is a conic}, \qquad \mathfrak{u}_i \in \mathcal{C}_* \text{ is odd if } u_i \text{ is a line} \tag{3.7}$$

(to make sure that $u_i \cdot v \in \mathbb{Z}$ for each $v \in \mathbf{S}_h$) and, if u_i is a conic, the dual conic $u_i^* := h - u_i$ has the supports

$$\operatorname{supp}_1 u_i^* = \mathfrak{u}_i, \qquad \operatorname{supp}_2 u_i^* = \overline{\mathfrak{u}_i \cup \mathfrak{u}'_i}. \tag{3.8}$$

Convention 3.9. We will construct the lattice $\mathbf{S}_h[u_1, \ldots]$ inductively, adding one extra line/conic at a time. To reduce the overcounting, we will always assume that

- all extra lines are added first, followed by extra conics, and
- the number of lines in the generating set is maximal possible.

Lemma 3.10. *Under Convention 3.9, the following further restrictions hold*:

(1) *all extra conics are irreducible, and*
(2) *for each pair $u_i \neq u_j$ of distinct extra generators, one has*

$$u_i \cdot u_j \in \{0, 1, 2\} \textit{ if both } u_i,\ u_j \textit{ are conics}, \qquad u_i \cdot u_j \in \{0, 1\} \textit{ otherwise.}$$

Proof. If a conic $u = l_1 + l_2$ is reducible in (a finite index extension of) $N + \mathbb{Z}u$, then at least one of the lines l_1, l_2 is not in N and can replace u in the generating set. For the second statement, in view of the first one, the only value that needs to be ruled out is $l \cdot c = 2$ for a line l and conic c. In this case, $l' := h - c - l$ is also a line and l, l' generate the same extension as l, c. □

3.3. **Technical remarks.** We conclude this section with a discussion of a few tricks used in the computation.

On a case-by-case basis, we observe that any extension N as in (3.1) is necessarily trivial, *i.e.*, $N = \mathcal{F}\bigl(\operatorname{Fn}(N, h)\bigr)$ (*cf.* Proposition 1.7). More precisely, when analyzing a corank 1 extension

$$S_r := \mathbf{S}_h[\ldots, u_r] \supset S_{r-1} := \mathbf{S}_h[\ldots],$$

we observe that any proper finite index extension $N \supset S_r$ either fails to be admissible or has a line or conic $u'_r \notin S_r$ (sometimes violating Convention 3.9), so that we can replace S_r with $\mathbf{S}_h[\ldots, u'_r]$. Note also that, by induction, we can assume S_{r-1} primitive in N; therefore, it suffices to consider further extensions generated by isotropic vectors of the form

$$(\alpha u_r + v) \bmod S_r, \qquad \alpha \in \mathbb{Q} \smallsetminus \mathbb{Z},\ v \in S_{r-1};$$

this reduces the amount of computation.

TABLE 3. Strata of codimension $\leqslant 1$ (see §4.4)

Name	Patterns	δ_2^2	δ_3	Lines	Conics	$\lvert G\rvert$	i_Ω	G_ω	$\lvert\det\rvert$	(r,c)
open					352	23040	2	21	576	$(1,0)$
$1^\star$	$4_0^\star, 12_0^\star$	$1/4$	± 1	8	$16+336$	1152	2	21	400	$(1,0)$
$2^\star$	$6_0^\star, 10_0^\star$	$5/4$	± 4	32	$160+192$	1920	2	21	208	$(1,0)$
3	$4_0, 10_0$	1	± 3		392	3072	4	21	576	$(1,0)$
4	$6_0, 8_0$	1	± 2		432	3072	4	21^2	448	$(1,0)$
5	12_0	1	± 1		360	1536	2	21	832	$(1,0)$
6	12_1	0	± 4		384	3072	4	21	640	$(1,0)$
7	12_2	1	0		400	3072	2	21^2	576	$(1,0)$

This fact has multiple consequences. First, in Proposition 3.4 (and its analogue for relative forms) we can take for N the original lattice $\mathbf{S}_h[\ldots, u_r]$. Second, there are natural isomorphisms

$$O_h(N) = \operatorname{Aut}\operatorname{Fn}(N,h), \qquad O_h(N,\Omega) = \operatorname{Aut}\bigl(\operatorname{Fn}(N,h),\Omega\bigr),$$

so that these orthogonal groups used in Proposition 3.2 can easily be computed by the `Digraphs` package in `GAP` [14]. Finally, the unevoidable overcounting is easily eliminated by identifying the strata with isomorphic graphs Γ or pairs (Γ,Ω).

When computing the number of connected components, Proposition 3.2 is used literally if $\operatorname{rk} N = 20$, *i.e.*, the stratum in question has codimension 3 and consists of singular $K3$-surfaces: in this case, T is positive definite of rank 2 and the genera of such forms and their orthogonal groups are easily computed, see, *e.g.*, Gauss [15]. In all other cases, with T indefinite and $O(T)$ typically infinite, we use the results of Miranda–Morrison [20], which combine both steps (1) and (2) in Proposition 3.2 into a single homomorphism

$$d^\perp\colon O(N) \to E_+(T)$$

to a certain elementary abelian 2-group $E_+(T)$ computed in terms of $\operatorname{genus}(T)$, so that the G-equivalence classes of primitive isometries $N \hookrightarrow \mathbf{L}$ are in a canonical bijection with $E_+(T)/\operatorname{Im} d^\perp$. We omit the details, which are purely algorithmic.

4. Strata of codimension 1

The goal of this section is compiling Table 3 listing the seven equiconical strata of codimension 1. (The first row shows the open stratum of codimension 0.) The notation is explained in §4.4 below. The maximal number of conics is 432, as in [3].

4.1. **Patterns.** As explained in §3.2, a codimension 1 stratum in $\mathcal{B}$ is defined by a finite index extension $N \supset \mathbf{S}_h[u]$, see (3.5), where $u \notin \mathbf{S}_h$ is an extra line or conic. (We do not use the irreducibility of u.) If $\lvert\operatorname{supp}_1 u\rvert = p$ and $\lvert\operatorname{supp}_2 u\rvert = q$, we say that

$$u \text{ has } \textit{pattern } p_q \text{ (if } u \text{ is a conic) or } p_q^\star \text{ (if } u \text{ is a line).}$$

By the Hodge index theorem, $\mathbf{S}_h[u] \subset NS(X)$ must be hyperbolic. Denoting by $\tilde{u}$ the orthogonal projection of u to $\mathbf{S}_h^\perp \otimes \mathbb{Q}$ and using (2.5), we arrive at

$$36\tilde{u}^2 = -(p+2q+\varepsilon)^2 + 18p + 72q - 72,$$

TABLE 4. Sylvester test for conics (left) and lines (right)

	0		2		4		6		8		10		12		14		16
0	○	○	·	·	·	·	·	·	·	·	·	·	·	·	·	○	○
2	×	·	·	·	·	·	·	·	·	·	·	·	·	·	×		
4	●	·	·	·	·	·	·	·	·	·	·	·	●				
6	●	·	·	·	·	·	·	·	·	·	●						
8	●	·	·	·	·	·	·	·	●								
10	●	○	○	·	○	○	●										
12	●	●	●	●	●												
14	×	×	×														
16	○																

	0		2		4		6		8		10		12		14		16
0	×	×	·	·	·	·	·	·	·	·	·	·	·	·	·	·	×
2	×	·	·	·	·	·	·	·	·	·	·	·	·	·	·		
4	●	·	·	·	·	·	·	·	·	·	·	·	·				
6	●	·	·	·	·	·	·	·	·	·	·						
8	○	·	·	·	·	·	·	·	·								
10	●	·	·	·	·	·	·										
12	●	○	○	○	○												
14	×	×	×														
16	×																

where p, q are as above and $\varepsilon := u \cdot h \in \{1, 2\}$. The requirement that $\tilde{u}^2 < 0$ results in Table 4, left (for conics) or right (for lines). In the table,

- a dot $\cdot$ marks the pairs (p, q) for which $\tilde{u}^2 \geqslant 0$,
- a cross $\times$ marks the remaining pairs ruled out by the parity condition (3.7); it is due to (3.7) that we list only even values of p, and
- a $\circ$ marks the pairs further ruled out by one of the lemmas below.

The next two lemmas state, in particular, that the lattice $\mathbf{S}_h[u]$ depends on the *size* $|\operatorname{supp}_2 u|$ and *equivalence class* $[\operatorname{supp}_1 u]$ only.

Lemma 4.1. *Let $u \notin \mathbf{S}_h$ be an extra conic (line), $\mathfrak{u} := \operatorname{supp}_1 u$, and $\mathfrak{u}' := \operatorname{supp}_2 u$. Then, for any set $\mathfrak{v}' \subset \bar{\mathfrak{u}}$, $|\mathfrak{v}'| = |\mathfrak{u}'|$, there is a conic (respectively, line) $v \in \mathbf{S}_h[u]$ such that $\operatorname{supp}_1 v = \mathfrak{u}$ and $\operatorname{supp}_2 v = \mathfrak{v}'$.*

Proof. The vector in question is $u + (\mathfrak{s} \smallsetminus \mathfrak{u}') - (\mathfrak{u}' \smallsetminus \mathfrak{s})$. □

Lemma 4.2. *Let $u \notin \mathbf{S}_h$ be an extra conic (line), $\mathfrak{u} := \operatorname{supp}_1 u$, and $p := |\mathfrak{u}|$. Let, further, $\mathfrak{v}$ be any of the following sets:*

1. *$\mathfrak{v} \in [\mathfrak{u}]_p$, or*
2. *$\mathfrak{v} \in [\![\mathfrak{u}]\!]_{14-p}$, if u is a conic and $\operatorname{supp}_2 u = \varnothing$, or*
3. *$\mathfrak{v} \in [\![\mathfrak{u}]\!]_{16-p} \smallsetminus [\mathfrak{u}]$, if u is a line and $\operatorname{supp}_2 u = \varnothing$.*

Then, there is a conic (respectively, line) $v \in \mathbf{S}_h[u]$ such that $\operatorname{supp}_1 v = \mathfrak{v}$.

In items (2), (3), the assumption $\operatorname{supp}_2 u = \varnothing$ is for the sake of simplicity, as the other cases are redundant, see Lemmas 4.5 and 4.6 below.

Proof of Lemma 4.2. Any set $\mathfrak{v} \neq \mathfrak{u}$ as in item (1) is of the form $\mathfrak{u} \mathbin{\triangle} \mathfrak{o}$ for some set $\mathfrak{o} \in \mathcal{O}_8$, $|\mathfrak{u} \cap \mathfrak{o}| = 4$. Let $\mathfrak{o}_+ := \mathfrak{o} \cap \operatorname{supp}_2 u$ and pick a subset $\mathfrak{o}_- \subset \mathfrak{o} \cap \mathfrak{u}$ such that $|\mathfrak{o}_-| = |\mathfrak{o}_+|$. Then, $v := u + \|\mathfrak{u}/\mathfrak{o}\| + \mathfrak{o}_+ - \mathfrak{o}_-$ is as in the statement.

A set $\mathfrak{v}$ as in item (2) is of the form $\overline{\mathfrak{u} \mathbin{\triangle} \mathfrak{o}}$ for some $\mathfrak{o} \in \mathcal{K}_6$ such that $|\mathfrak{o} \cap \mathfrak{u}| = 2$, and the conic $v := u + \hbar + \|\mathfrak{u}/\mathfrak{o}\|$ is as required. For item (3), consider the $B2$-conic

$$w := \hbar - \|\mathfrak{o}/\mathfrak{u}\|, \quad \mathfrak{o} \in \mathcal{K}_6,\ |\mathfrak{o} \cap \mathfrak{u}| = 3; \tag{4.3}$$

one has $w \cdot u = -1$ and, hence, $v := w - u$ is a line, $\operatorname{supp}_1 v = \overline{\mathfrak{u} \mathbin{\triangle} \mathfrak{o}}$. □

4.2. **Extra conics.** Below are a few further restrictions on the supports of an extra conic $u \notin \mathbf{S}_h$. By (3.7), the 1-support $\operatorname{supp}_1 u \in \mathcal{C}_*$ is an even set.

Lemma 4.4. *If $u \notin \mathbf{S}_h$ is an extra conic and $\mathfrak{u} := \operatorname{supp}_1 u \in \mathcal{O}_* \cup \mathcal{K}_*$, then the lattice $\mathbf{S}_h[u]$ has no geometric extensions.*

Proof. Assuming the contrary, let $\mathfrak{u}' := \operatorname{supp}_2 u$ and consider the vector

$$\hat{u} := \begin{cases} u - \|\mathfrak{u}/\varnothing\| + \mathfrak{u}', & \text{if } \mathfrak{u} \in \mathcal{O}_*, \\ \hbar - u + \|\bar{\mathfrak{u}}/\mathfrak{u}'\|, & \text{if } \mathfrak{u} \in \mathcal{K}_*. \end{cases}$$

We have $\hat{u} \in \mathbf{T}$ (see §2.1) and, respectively,

$$\hat{u}^2 = \tfrac{1}{2}|\mathfrak{u}| + 2|\mathfrak{u}'| - 2, \quad \hat{u}\cdot h = 2 + |\mathfrak{u}| + 2|\mathfrak{u}'| \quad \text{if } \mathfrak{u} \in \mathcal{O}_*,$$
$$\hat{u}^2 = -\tfrac{1}{2}|\mathfrak{u}| + 5, \quad \hat{u}\cdot h = 16 - |\mathfrak{u}| - 2|\mathfrak{u}'| \quad \text{if } \mathfrak{u} \in \mathcal{K}_*.$$

In view of (2.2), the existence of $\hat{u} \in \mathbf{T}$ rules out patterns p_0, $p = 0, 6, 8, 16$. The few remaining cases are considered below; we take into account Table 4 and use the duality (3.8) to assume that $|\mathfrak{u}| + 2|\mathfrak{u}'| \leqslant 16$. In all cases, $\hat{u}^2 = 0$.

The pattern 10_2: we have $\hat{u}\cdot h = 2$, *i.e.*, $\hat{u}$ is 2-isotropic, see Definition 2.8(2).

The patterns 0_1 *and* 10_q, $q = 0, 1$: by Lemma 2.4, a geometric extension of $\mathbf{S}_h[u]$ must contain $v := -\frac{1}{2}\hat{u} - \|\mathfrak{s}/\varnothing\|$ for some $\mathfrak{s} \in \mathcal{C}_4$. We have $\hat{u}\cdot h = 6$ or 4; hence,

$$v^2 = -2, \quad v\cdot h = 1 \text{ or } 2, \quad v\cdot e = -1 \text{ for each } e \in \mathfrak{s},$$

resulting in a missing conic, see Definition 2.8(3), or exceptional divisor $v - e$, see Definition 2.8(1), respectively. □

Lemma 4.5. *Let* $u \notin \mathbf{S}_h$ *be an extra conic,* $\mathfrak{u} := \operatorname{supp}_1 u \in \mathcal{C}_{10}$*, and assume that* $\mathfrak{u}' := \operatorname{supp}_2 u \neq \varnothing$*. Then the lattice* $\mathbf{S}_h[u]$ *has no geometric extensions.*

Proof. Using the duality (3.8), assume that $|\mathfrak{u}'| \leqslant 2$. Since $\mathfrak{u} \notin \mathcal{K}_{10}$ (see Lemma 4.4), there is a set $\mathfrak{s} \in \mathcal{K}_6$ such that $|\mathfrak{u} \cap \mathfrak{s}| = 2$; by Lemma 4.1, we can also assume that $\mathfrak{u}' \subset \mathfrak{s}$. If $|\mathfrak{u}'| = 2$, there is a 3-element subset $\mathfrak{r} \subset \mathfrak{s} \smallsetminus \mathfrak{u}'$ such that $|\mathfrak{u} \cap \mathfrak{r}| = 1$; if $|\mathfrak{u}'| = 1$, take $\mathfrak{r} := \mathfrak{s} \smallsetminus (\mathfrak{u} \cup \mathfrak{u}')$. In both cases, we have $u \cdot v = -1$ for the $B2$-conic

$$v := \hbar + \|\mathfrak{s}/\mathfrak{r}\| \in \mathbf{S}_h[u],$$

cf. (2.12); hence, $u - v$ is an exceptional divisor, see Definition 2.8(1). □

4.3. **Extra lines.** The remaining restrictions on an extra line $u \notin \mathbf{S}_h$ are given by the following lemma. By (3.7), $\operatorname{supp}_1 u \in \mathcal{C}_*$ is an odd set.

Lemma 4.6. *Let* $u \notin \mathbf{S}_h$ *be an extra line, and let* $\mathfrak{u} := \operatorname{supp}_1 u$*,* $\mathfrak{u}' := \operatorname{supp}_2 \mathfrak{u}$*. If* $|\mathfrak{u}| = 8$ *or* $|\mathfrak{u}| = 12$ *and* $\mathfrak{u}' \neq \varnothing$*, then the lattice* $\mathbf{S}_h[u]$ *is not admissible.*

Proof. If $|\mathfrak{u}| = 8$ (and necessarily $\mathfrak{u}' = \varnothing$, see Table 4), then $v := 2u + \|\Omega/\mathfrak{u}\|$ is a 2-isotropic vector, see Definition 2.8(2). Thus, assume that $|\mathfrak{u}| = 12$ and, as in the proof of Lemma 4.2, let $\mathfrak{o} \in \mathcal{K}_6$ be such that $|\mathfrak{o} \cap \mathfrak{u}| = 3$. Using Lemma 4.1, we can assume that $\mathfrak{o}_+ := \mathfrak{o} \cap \operatorname{supp}_2 u \neq \varnothing$, and pick $\mathfrak{o}_- \subset \mathfrak{o} \cap \mathfrak{u}$ so that $|\mathfrak{o}-| = |\mathfrak{o}_+|$. It is immediate that

$$v := \hbar - \|\mathfrak{o}/\mathfrak{u}\| - \mathfrak{o}_+ + \mathfrak{o}_- - u$$

is a line, but $v \cdot e = -1$ for each $e \in \mathfrak{o}_- \neq \varnothing$, *cf.* Definition 2.8(3). □

4.4. **The list of strata.** In view of Table 4 and Lemmas 4.4, 4.5, and 4.6, we are left with the following eleven patterns for an extra conic or line:

$$4_0, 10_0; 6_0, 8_0; 12_0; 12_1; 12_2 \quad \text{or} \quad 4_0^\star, 12_0^\star; 6_0^\star, 10_0^\star.$$

By Lemma 4.2(2), (3), each pair of patterns separated by a comma, *e.g.*, $4_0, 10_0$, results in the same lattice $\mathbf{S}_h[u]$: in the example, if u has pattern 4_0, there exists $v \in \mathbf{S}_h[u]$ with pattern 10_0 such that $\mathbf{S}_h[u] = \mathbf{S}_h[v]$, and *vice versa*.

Lemmas 4.1 and 4.2(1) state that the lattice $\mathbf{S}_h[u]$ depends only on the pattern of u and the equivalence class $[\mathfrak{u}]$ of its 1-support $\mathfrak{u} := \operatorname{supp}_1 u$. Up to isomorphism preserving Ω, the corank 1 extension $\mathbf{S}_h[u] \supset \mathbf{S}_h$ is determined by the $\mathfrak{G}$-orbit of $\mathfrak{u}$; due to Table 1, parity condition (3.7), and Lemma 4.4 eliminating the exceptional orbit $\mathcal{O}_* \cup \mathcal{K}_*$, we arrive at the seven classes collected in Table 3. Letting $X \subset \mathcal{X}$ be a very general member of the respective family, listed in the table are

- the notation for the codimension 1 stratum $\mathcal{X} \subset \mathcal{B}$ (for future references),
- the patterns of the lines and conics on X,
- the projections δ_2, δ_3 to $\operatorname{discr} \mathbf{S}_h$ (see §4.5 below).

The rest of the data is common for Tables 3, 5, and 6:

- the numbers of lines and conics on X; the latter is given by a single count if all conics are irreducible or as (reducible) + (irreducible) otherwise,
- the size of the group $G := \operatorname{Aut}(\operatorname{Fn} X)$ of abstract graph automorphisms of the bi-colored Fano graph $\operatorname{Fn} X$,
- the index $i_\Omega := [G : G_\Omega]$ of the subgroup $G_\Omega \subset G$ preserving Ω as a set,
- the group (reference to Table 2) G_ω of symplectic automorphisms of X and the index $[\operatorname{Aut}_h X : G_\omega]$ (if greater than 1) as a superscript,
- the determinant $|\det T(X)|$ or, in Table 6, the lattice $T(X)$ itself,
- the numbers $(r, c) := \operatorname{rc} \mathcal{X}$, see (1.2), and $(\tilde{r}_i, \tilde{c}_i) := \operatorname{rc} \tilde{\mathcal{X}}_i$ (separately for each relative form, *cf.* Remark 6.1 below), in the form

$$\begin{array}{lll} \text{both values } (r,c) \to (\tilde{r}_i, \tilde{c}_i), & \text{if } (r,c) \neq (\tilde{r}_i, \tilde{c}_i), \\ \text{the common value } (r,c), & \text{if } (r,c) = (\tilde{r}_i, \tilde{c}_i). \end{array} \tag{4.7}$$

A straightforward computation using `GAP` [14] yields the following statement.

Proposition 4.8. *For each of the (pairs of) patterns in Table 3, the corank* 1 *extension $\mathbf{S}_h[u]$ is geometric and has no proper geometric finite index extensions. Furthermore, the lines and conics in the complement $\mathbf{S}_h[u] \smallsetminus \mathbf{S}_h$ are precisely those given by Lemmas 4.1 and 4.2 and duality* (3.8). ◁

Concerning the last statement, in all but one cases each pair $\mathfrak{u}$, $\mathfrak{u}'$ given by the lemmas and (3.8) is represented by a single line/conic. The last case 7, with the pattern 12_2, is special, as Lemma 4.1 and (3.8) give us two conics for each pair.

In the presence of lines (rows $1^\star$ and $2^\star$ of the table), some of the $B2$-conics, *viz.* those given by (4.3), become reducible. However, the lattice contains no new conics, reducible or not: the total conic count is still 352.

4.5. **Clusters.** Given a geometric extension $N \supset \mathbf{S}_h$, we define a *cluster* as the set of all lines and conics (irreducible or reducible) in $N \smallsetminus \mathbf{S}_h$ that project to the same point in the projectivization $\mathbb{P}\big((N/\mathbf{S}_h) \otimes \mathbb{Q}\big)$. According to Proposition 4.8, there are seven isomorphism classes (henceforth referred to as the *types*) of clusters listed in Table 3 and, for each cluster C,

$$\text{the primitive hull of } \mathbf{S}_h \cup C \text{ in } N \text{ is } \mathbf{S}_h[u] \text{ for any } u \in C.$$

All vectors $u \in C$ share the same (pair of) patterns and class $[\![\operatorname{supp}_1 u]\!]$.

For an alternative description of a cluster C, observe that a vector $u \in N$ defines a linear functional $\mathbf{S}_h \to \mathbb{Z}$, $x \mapsto x \cdot u$, and, thus, a class in $\operatorname{discr} \mathbf{S}_h$. Since any two vectors $u, v \in C$ are related *via* $u \pm v \in \mathbf{S}_h$ (*cf.* Lemmas 4.1 and 4.2), we conclude that C projects to a pair of opposite vectors $\pm\delta(C) \in \operatorname{discr} \mathbf{S}_h$. The components

$$\delta_p := \delta_p(C) \in \operatorname{discr}_p \mathbf{S}_h, \quad p = 2, 3,$$

of $\delta(C)$ are characterized in Table 3 by means of the value $\delta_2^2 \bmod 2\mathbb{Z}$ and element δ_3 of the group $\operatorname{discr}_3 \mathbf{S}_h \cong \mathbb{Z}/9$ generated by $\frac{1}{9}(h+\Omega)$. Note that

$$\delta_2(C) \text{ has order 2 or 4 if } C \text{ consists of conics or lines, respectively, and } \delta_2 \neq 0 \text{ or } \hbar \bmod \mathbf{S}_h \text{ (see Lemma 4.4).} \tag{4.9}$$

Comparing the counts, *cf.* (2.7) and Table 1, we arrive at the following statement.

Proposition 4.10. *The common support* $[\![\operatorname{supp}_1 u]\!]$, $u \in C$, *of any cluster* $C \subset N$ *and, hence, the sublattice* $\mathbf{S}_h[u] \subset N$ *is determined by the image* $\delta_2(C)$ *and, in the case of types 3, 4, 5, 7, by any one of the following*:

- *the type specification* (3, 4, 5, *or* 7), *or, equivalently,*
- *the image* $\delta_3(C) \in \operatorname{discr}_3 \mathbf{S}_h$. ◁

In the next section (see Corollary 5.1) we observe that distinct clusters C_1, C_2 in any *geometric* extension $N \supset \mathbf{S}_h$ have distinct images $\pm\delta_2(C_1) \neq \pm\delta_2(C_2)$.

5. Strata of codimension 2

Next step is the analysis of the double (self-)intersections of the seven strata of codimension 1 described in Table 3. The resulting 47 relative forms of the 43 equiconical families are shown in Table 5. The maximal number of conics is 560; it is in the row marked with a $*$ (see also Theorem 1.6).

To compile the table, we analyze all corank 2 extensions

$$\mathbf{S}_h[u_1, u_2], \quad u_i \in C_i,\ i = 1, 2,$$

where $C_1 \neq C_2$ is a pair of distinct clusters (see §4.5), which are represented by their classes $\alpha_i := \delta_2(C_i) \in \operatorname{discr}_2 \mathbf{S}_h$, $i = 1, 2$. The candidates are found as follows:

- pick one of the 27 $\mathfrak{G}$-orbits of pairs (α_1, α_2) satisfying (4.9),
- for each α_i with $\alpha_i^2 = 1 \bmod 2$, pick a type, see Proposition 4.10, and
- specify the product $u_1 \cdot u_2$, see item (3) in §3.2.

For the latter, and for the construction of $\mathbf{S}_h[u_1, u_2]$, we need to select a pair of representatives $u_i \in C_i$. The values for $u_1 \cdot u_2$ are limited by Lemma 3.10 and the Hodge index theorem: the orthogonal projection of $\mathbb{Z}u_1 + \mathbb{Z}u_2$ to $\mathbf{S}_h^\perp \otimes \mathbb{Q}$ must be negative definite.

The resulting list is analyzed as explained in §3.3; we omit the details.

5.1. **Notation in Table 5.** We use the same notation as in Table 3 (see §4.4), but instead of the first four columns we list the types (as references to Table 3) of the clusters C_i found in N and the products $\circ$ of the images $\alpha_i := \delta_2(C_i) \in \operatorname{discr}_2 \mathbf{S}_h$. Each product is $n/4 \bmod \mathbb{Z}$, where n is the number given in the table. We show:

- for three *order* 2 elements $\alpha_1, \alpha_2, \alpha_3$, the common product $\alpha_i \cdot \alpha_j$, $i \neq j$;
- in all other cases, the products $\alpha_1 \cdot \alpha_i$, $i \geqslant 2$.

In view of Corollary 5.2 below, these data determine the collection of elements up to the induced action of $\mathfrak{G}$.

For strata of codimension 2, an abstract Fano graph Γ may admit several relative forms. They are listed in separate rows and prefixed with equal superscripts: *e.g.*, the two rows prefixed with [1] represent the same abstract Fano graph.

The row prefixed with a $*$ contains the family maximizing the number of real conics in a real Barth–Bauer quartic (see Theorem 1.6).

TABLE 5. Strata of codimension 2 (see §5.1 and §4.4)

Clusters	$\circ$	Lines	Conics	$\lvert G\rvert$	i_Ω	G_ω	$\lvert\det\rvert$	(r,c)
$1^\star, 1^\star, 4$	$1, 2$	16	$48+384$	512	4	21^2	224	$(1,0)$
$1^\star, 1^\star, 7$	$3, 2$	16	$48+352$	256	2	21^2	256	$(1,0)$
$1^\star, 2^\star, 3, 6$	$3, 0, 2$	40	$240+184$	192	2	21	144	$(1,0)$
$1^\star, 3$	0	8	$16+376$	384	4	21	336	$(1,0)$
$1^\star, 3$	2	8	$16+376$	256	4	21	384	$(1,0)$
$1^\star, 4$	0	8	$16+416$	384	4	21^2	304	$(1,0)$
$1^\star, 5$	0	8	$16+344$	192	2	21	464	$(1,0)$
$1^\star, 5$	0	8	$16+344$	192	2	21	400	$(1,0)$
$1^\star, 5$	2	8	$16+344$	128	2	21	576	$(1,0)$
$1^\star, 6$	2	8	$16+368$	384	4	21	400	$(1,0)\to(2,0)$
$1^\star, 6$	0	8	$16+368$	256	4	21	416	$(1,0)$
$1^\star, 7$	0	8	$16+384$	384	2	21^2	400	$(1,0)$
$2^\star, 3$	2	32	$160+232$	768	4	21	192	$(1,0)$
$2^\star, 4$	2	32	$160+272$	768	4	21^2	160	$(1,0)$
$2^\star, 5$	0	32	$160+200$	192	2	21	272	$(1,0)$
$2^\star, 5$	2	32	$160+200$	384	2	21	256	$(1,0)$
$2^\star, 6$	0	32	$160+224$	768	4	21	224	$(1,0)$
$2^\star, 7$	0	32	$160+240$	384	2	21^2	208	$(1,0)$
[3] $3, 3$	0		432	512	2	21	576	$(1,0)$
$3, 3, 3$	2		472	4608	8	21	432	$(1,0)\to(2,0)$
[1] $3, 3, 7$	2		480	384	2	21^2	432	$(1,0)$
[1] $3, 4, 5$	2		480	384	4	21^2	432	$(1,0)$
$3, 4, 6$	0		504	1024	8	21^2	384	$(1,0)$
$3, 5$	0		400	512	4	21	768	$(1,0)$
$3, 5$	2		400	384	4	21	816	$(1,0)$
[3] $3, 5, 6$	0		432	512	4	21	576	$(1,0)$
[4] $3, 6$	2		424	384	4	21	624	$(1,0)$
$3, 7$	0		440	1024	4	21^2	576	$(1,0)$
$4, 4, 6$	0		544	2048	8	39^2	320	$(1,0)$
$*$ $4, 4, 7$	2		560	1152	6	49^2	304	$(1,0)$
$4, 5$	0		440	512	4	21^2	640	$(1,0)$
$4, 5, 5$	2		448	768	4	21^2	560	$(1,0)$
[2] $4, 6$	2		464	384	4	21^2	496	$(1,0)$
$4, 7$	0		480	1024	4	39^2	448	$(1,0)$
$5, 5$	0		368	512	2	21	1088	$(1,0)$
$5, 5$	0		368	512	2	21	1024	$(1,0)$
$5, 5$	2		368	384	2	21	1200	$(1,0)$
$5, 5, 7$	2		416	384	2	21^2	688	$(1,0)$
$5, 6$	0		392	512	4	21	896	$(1,0)$
$5, 6$	2		392	384	4	21	880	$(1,0)$
[4] $5, 6, 6$	2		424	384	2	21	624	$(1,0)$
$5, 7$	0		408	512	2	21^2	832	$(1,0)$
$6, 6$	0		416	1536	6	21	704	$(1,0)$
[2] $6, 6, 7$	2		464	384	2	21^2	496	$(1,0)$
$6, 7$	0		432	1024	4	21^2	640	$(1,0)$
$7, 7$	0		448	2048	2	39^2	576	$(1,0)$
$7, 7, 7$	2		496	2304	2	49^2	432	$(1,0)$

5.2. **A few consequences of Table 5.** In spite of the large number of classes, one can still observe a few common properties. We omit their proofs, as they mainly consist of ruling out a large number of simple cases similar to §4.

Corollary 5.1. *In any* geometric *extension* $N \supset \mathbf{S}_h$*, distinct clusters* $C_1, C_2 \subset N$ *project to distinct pairs* $\{\pm\delta_2(C_1)\} \neq \{\pm\delta_2(C_2)\} \subset \operatorname{discr}_2 \mathbf{S}_h$. ◁

Corollary 5.2. *Given a geometric corank* 2 *extension* $N \supset \mathbf{S}_h$ *generated over* $\mathbb{Q}$ *by lines and conics, the images* $\delta_2(C)$ *of all clusters* $C \subset N$ *generate a subgroup*

$$\mathbb{Z}/2 \times \mathbb{Z}/2 \not\ni \hbar \bmod \mathbf{S}_h \quad \textit{or} \quad \mathbb{Z}/4 \times \mathbb{Z}/2$$

in $\operatorname{discr}_2 \mathbf{S}_h$. ◁

Corollary 5.3. *A geometric corank* 2 *extension* $N \supset \mathbf{S}_h$ *that is generated over* $\mathbb{Q}$ *by lines and conics has two to four clusters. Given any two clusters* $C_1 \neq C_2 \subset N$ (*except* 3, 6 *in the third row of Table 5*) *and any pair of representatives* $u_i \in C_i$, $i = 1, 2$*, one has* $N = \mathbf{S}_h[u_1, u_2]$. ◁

Immediately by the definition of clusters, the excessive (*i.e.*, beyond the common 352 conics) line/conic counts of N are the sums of those over all clusters $C \subset N$. As in the case of codimension 1, in the presence of lines (clusters $1^\star$ and $2^\star$), some of the $B2$-conics become reducible. However, starting from codimension 2, some of the extra conics may also be reducible; for example, in the third row of Table 5, *each* of the 32 conics in the last cluster 6 is reducible. Since each reducible conic lies in a corank 2 extension, we can state the following general count:

$$\begin{aligned}\#\{\text{reducible conics}\} &= 16\ B2\text{-conics for each cluster } 1^\star \\ &+ 160\ B2\text{-conics for each cluster } 2^\star \\ &+ 16 \text{ type } 4 \text{ or } 7 \text{ conics for each pair } 1^\star, 1^\star \\ &+ 32 \text{ type } 3 + 32 \text{ type } 6 \text{ conics for each pair } 1^\star, 2^\star.\end{aligned} \tag{5.4}$$

6. Rigid quartics (codimension 3)

There remains to analyze the "triple intersections" of the strata, *i.e.*, projectively rigid (*aka* singular) quartics. The computation results in 285 relative forms of 211 abstract isomorphism classes of Fano graphs, most represented by several projective equivalence classes: up to projective equivalence, there are

- 208 real and 189 pairs of complex conjugate quartics X, and
- 231 real and 682 pairs of complex conjugate pairs (X, Ω).

The converse also holds: a singular $K3$-surface may admit several embeddings to $\mathbb{P}^3$ as a Barth–Bauer quartic, sometimes with non-isomorphic Fano graphs (see, *e.g.*, the two occurrences of the transcendental lattice $T = [4, 0, 40]$ in Table 6).

The relative forms with more than 536 conics are shown in Table 6, sorted by the decreasing of the total conic count; a complete list is found in [8].

The computation is similar to §5:

- pick one of the 134 $\mathfrak{G}$-orbits of triples $(\alpha_1, \alpha_2, \alpha_3)$ such that each pair is as in Corollary 5.2; in particular, all α_i are distinct and satisfy (4.9),
- for each α_i with $\alpha_i^2 = 1 \bmod 2$, pick a type, see Proposition 4.10, and
- specify the product $u_i \cdot u_j$, $i \neq j$, see item (3) in §3.2.

TABLE 6. Rigid quartic with many conics (see §6.1 and §4.4)

	Clusters	Lines	Conics	$\|G\|$	i_Ω	G_ω	T	(c, r)
	$4,4,4,4,6,7,7$		800	15360	60	81^2	$[4,0,40]$	$(1,0)$
	$4,4,4,7,7,7$		736	3072	16	75^2	$[12,0,16]$	$(1,0)\to(0,1)$
	$4,4,4,6,6,7$		704	768	12	65^2	$[8,4,28]$	$(0,1)$
	$3,3,4,4,5,6,7$		680	768	12	49^2	$[12,4,20]$	$(0,1)\to(0,2)$
	$3,4,4,6,6,7$		664	768	12	49^2	$[4,0,60]$	$(1,0)\to(2,0)$
	$7,7,7,7,7,7$		640	3072	2	75^2	$[4,0,72]$	$(1,0)$
1	$3,3,4,6,6,7$		624	256	4	39^2	$[12,4,24]$	$(0,2)$
1	$3,4,4,5,6,6$		624	256	8	39^2	$[12,4,24]$	$(0,2)\to(0,4)$
2	$3,3,3,7,7,7$		616	384	2	49^2	$[4,0,72]$	$(2,0)$
2	$3,4,4,5,5,7$		616	384	6	49^2	$[4,0,72]$	$(2,0)$
$*$	$4,4,7,7$		608	768	6	65^2	$[4,0,76]$	$(1,0)$
							$[16,4,20]$	$(0,1)$
3	$3,4,4,5,5,5,6$		608	512	8	39^2	$[8,0,36]$	$(1,0)\to(2,0)$
3	$3,3,3,3,7,7$		608	512	2	39^2	$[8,0,36]$	$(1,0)$
4	$3,3,3,4,5,5,6$		600	512	8	21^2	$[12,0,24]$	$(1,0)\to(4,0)$
4	$3,3,3,3,3,7$		600	512	4	21^2	$[12,0,24]$	$(1,0)\to(0,1)$
5	$4,4,6,7$		592	384	6	49^2	$[12,0,28]$	$(0,2)$
5	$6,6,6,7,7,7$		592	384	2	49^2	$[12,0,28]$	$(0,2)$
	$4,4,5,5,5,7$		584	384	6	49^2	$[8,0,44]$	$(0,1)$
	$4,7,7,7$		576	1536	4	65^2	$[12,0,28]$	$(2,0)$
	$3,3,4,5,5,7$		576	512	8	39^2	$[12,4,28]$	$(0,1)\to(0,2)$
6	$4,4,6,6$		576	512	8	39^2	$[4,0,88]$	$(1,0)\to(0,1)$
6	$6,6,6,6,7,7$		576	512	2	39^2	$[4,0,88]$	$(1,0)$
	$1^\star,1^\star,2^\star,3,3,4,6,6$	48	$336+240$	256	4	21^2	$[4,0,20]$	$(1,0)\to(0,1)$
	$4,4,5,7$		568	384	6	49^2	$[4,0,108]$	$(1,0)$
							$[16,4,28]$	$(0,1)$
	$1^\star,4,4,7$	8	$16+544$	576	6	49^2	$[4,2,52]$	$(2,0)$
	$1^\star,1^\star,1^\star,4,4,7$	24	$96+464$	192	6	49^2	$[4,2,32]$	$(1,0)$
							$[8,2,16]$	$(0,1)$
	$2^\star,4,4,7$	32	$160+400$	576	6	49^2	$[4,2,28]$	$(1,0)$
7	$3,3,5,6,6,7$		552	128	2	21^2	$[4,0,92]$	$(1,0)$
							$[12,4,32]$	$(0,1)$
7	$3,4,5,5,6,6$		552	128	4	21^2	$[4,0,92]$	$(1,0)\to(0,1)$
							$[12,4,32]$	$(0,1)\to(0,2)$
7	$3,3,4,5,6$		552	128	4	21^2	$[4,0,92]$	$(1,0)\to(0,1)$
							$[12,4,32]$	$(0,1)\to(0,2)$
	$4,6,6,7$		544	512	8	39^2	$[16,8,28]$	$(0,1)\to(0,2)$
	$1^\star,4,4,6$	8	$16+528$	512	8	39^2	$[8,4,28]$	$(1,0)\to(2,0)$
	$1^\star,1^\star,4,4,6$	16	$48+496$	512	8	39^2	$[4,0,40]$	$(1,0)\to(0,1)$
	$1^\star,1^\star,2^\star,3,3,6,6,7$	48	$336+208$	64	2	21^2	$[4,2,24]$	$(1,0)$
							$[8,2,12]$	$(0,1)$

In the last two steps, each decorated pair (α_i, α_j) must be one of those in Table 5.

Then, for each decorated triple $(\alpha_1, \alpha_2, \alpha_3)$, we check if the corresponding lattice $\mathbf{S}_h[u_1, u_2, u_3]$ is hyperbolic, *cf.* §5, and, if it is, analyze it as explained in §3.3.

6.1. **Notation in Table 6.** The notation is similar to that of Table 5, see §5.1 and §4.4, but we omit the products $\circ$ in $\operatorname{discr}_2 \mathbf{S}_h$ and show the transcendental lattices $T(X)$, see (1.9), instead of $|\det T|$. Each lattice, together with its respective counts (r, c) (itemized according to the type of T), occupies a separate row.

Marked with a $*$ is the quartic with the maximal number 560 of real conics.

Remark 6.1. Recall that the numbers $\operatorname{rc}\mathcal{X}$ and $\operatorname{rc}\tilde{\mathcal{X}}_i$, see (1.2), may differ; they are listed in the last column as explained in (4.7), and in Table 6 they are further itemized according to the isomorphism type of the transcendental lattice.

It should be understood that the absolute numbers $\operatorname{rc}\mathcal{X}$ are the (first pair of) values (r, c) *common to all rows* related to Γ, whereas $\operatorname{rc}\tilde{\mathcal{X}}_i$ are the (second pair of) values $(\tilde{r}_i, \tilde{c}_i)$ listed separately in the row corresponding to the relative form (Γ, Ω_i). These latter pairs are to be summed up to obtain the total numbers $\operatorname{rc}\tilde{\mathcal{X}}(\Gamma)$. This convention is illustrated in the next example (see also Corollary 6.5 below).

Example 6.2. We illustrate Remark 6.1 by a few examples, referring to Table 6. Since the quartics are rigid, we speak about the (projective equivalence classes of) quartics rather than components. Denote by $\mathrm{K}(X)$ the set of subgraphs $\Omega \subset \operatorname{Fn} X$, so that the relative forms are the $\operatorname{Aut}(\operatorname{Fn} X)$-orbits on $\mathrm{K}(X)$.

In the first row, the two pairs are equal: $\operatorname{rc}\mathcal{X} = \operatorname{rc}\tilde{\mathcal{X}} = (1, 0)$; thus, there is a single quartic X with this Fano graph Γ, the group $\operatorname{Aut}_h X$ acts transitively on $\mathrm{K}(X)$, and X admits a real structure preserving a point $\Omega \in \mathrm{K}(X)$.

In the second row, $\operatorname{rc}\mathcal{X} = (1, 0)$, whereas $\operatorname{rc}\tilde{\mathcal{X}} = (0, 1)$. Hence, there is a single quartic X with transitive $\operatorname{Aut}_h X$-action on $\mathrm{K}(X)$, this quartic is real, but there is no real structure preserving a point $\Omega \in \mathrm{K}(X)$.

The two rows prefixed with [1] represent two real forms of the same graph:

$$\operatorname{rc}\mathcal{X} = (0, 2); \quad \operatorname{rc}\tilde{\mathcal{X}}_1 = (0, 2),\ \operatorname{rc}\tilde{\mathcal{X}}_2 = (0, 4) \implies \operatorname{rc}\tilde{\mathcal{X}} = (0, 6).$$

(From this point on, the verbal description is left to the reader.)

For the two rows prefixed with [2], $\operatorname{rc}\mathcal{X} = \operatorname{rc}\tilde{\mathcal{X}}_1 = \operatorname{rc}\tilde{\mathcal{X}}_2 = (2, 0)$, $\operatorname{rc}\tilde{\mathcal{X}} = (4, 0)$.

Finally, for the three rows prefixed with [7], we have $\operatorname{rc}\mathcal{X} = (1, 1)$: there is one real quartic, with $T \cong [4, 0, 92]$, and a pair of conjugate ones, with $T \cong [12, 4, 32]$. (Most likely, the quartics are Galois conjugate over an algebraic number field, but I would refrain from an affirmative statement.) The other counts

$$\operatorname{rc}\tilde{\mathcal{X}}_1 = (1, 1),\ \operatorname{rc}\tilde{\mathcal{X}}_2 = \operatorname{rc}\tilde{\mathcal{X}}_3 = (0, 3) \implies \operatorname{rc}\tilde{\mathcal{X}} = (1, 7)$$

can also be itemized further according to the two types of $T(X)$.

6.2. **Consequences of Table 6** (see also [8]). Unfortunately, there seems to be no simple way to interpret Table 6 in terms of Table 5. For example, some triples C_1, C_2, C_3 of clusters independent over $\mathbf{S}_h$ may project to dependent elements $\delta_2(C_i) \in \operatorname{discr}_2 \mathbf{S}_h$, generating a subgroup of length 2. For this reason, for a typical corank 3 extension $N \supset \mathbf{S}_h$, one cannot assert that any decorated (by the types of the clusters) length 2 subgroup in the image of δ_2 is among the groups listed in Table 5. Nor can one assert that $N = \mathbf{S}_h[u_1, u_2, u_3]$ for representatives $u_i \in C_i$ of *any* triple of independent clusters (*cf.* Corollary 5.3): some of these lattices do have proper geometric finite index extensions.

Still, one can state certain limited analogues of Corollaries 5.2 and 5.3.

Corollary 6.3. *Given a geometric corank* 3 *extension $N \supset \mathbf{S}_h$ generated over $\mathbb{Q}$ by lines and conics, the images $\delta_2(C)$ of all clusters $C \subset N$ generate a subgroup*

$$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \not\ni \hbar \bmod \mathbf{S}_h \quad \textit{or} \quad \mathbb{Z}/4 \times \mathbb{Z}/2 \times \mathbb{Z}/2$$

in $\operatorname{discr}_2 \mathbf{S}_h$. ◁

Corollary 6.4. *If a geometric corank 3 extension $N \supset \mathbf{S}_h$ is generated by lines and conics over $\mathbb{Q}$, then so it is over $\mathbb{Z}$.* $\triangleleft$

By definition, the fact that a connected component $\mathcal{R} \subset \mathcal{X}$ or $\tilde{\mathcal{X}}$ is real means that any $X \in \mathcal{R}$ is projectively equivalent to $\bar{X}$, *i.e.*, X has an anti-holomorphic automorphism, but the latter does not need to be involutive. (If $\operatorname{codim} \mathcal{X} \leqslant 2$, even less can be stated: X and $\bar{X}$ are related by an equiconical deformation; *cf.* [1, 16] for examples of real strata without real points in other similar problems.) Nevertheless, a case-by-case analysis shows that real components of codimension 3 do consist of real quartics X or, respectively, pairs (X, Ω).

Corollary 6.5. *Each real component $\mathcal{R} \subset \mathcal{X}$ (resp. $\mathcal{R} \subset \tilde{\mathcal{X}}$) of codimension 3 in $\mathcal{B}$ consists of real quartics X (resp. pairs (X, Ω)).* $\triangleleft$

7. Missing proofs

In this concluding section, we fill in a few missing details to complete the proofs of the principal results stated in §1.2.

7.1. **Proof of Theorem 1.3.** The only statement that needs proof is the fact that the unique Barth–Bauer quartic X with 800 conics (see the first row in Table 6) is given by Mukai's polynomial as in the theorem. (This was observed by X. Roulleau, private communication.) From the computation, we know that X admits a faithful symplectic action of Mukai's group M_{20}.

Let $\mathcal{M}_{20}$ be the 1-parameter family of (generically non-algebraic) $K3$-surfaces with a faithful symplectic action of $G := M_{20}$. All actions have isomorphic covariant lattices $(\operatorname{Fix} G_*)^\perp$ (see, *e.g.*, [17]; here, G_* is the induced action on H_2), and this common lattice S is found as the orthogonal complement $h^\perp \subset NS(X)$. We have

$$\operatorname{discr}_2 S \cong \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \oplus \left[\tfrac{1}{8}\right], \qquad \operatorname{discr}_5 S \cong \left[\tfrac{2}{5}\right],$$

so that $NS(X)$ is the index 2 extension of $S \oplus \mathbb{Z}h$ *via* $\tilde{\alpha}_1 + \hbar$ for a certain vector $\tilde{\alpha}_1 \in S^\vee$ such that $\tilde{\alpha}_1^2 = 1 \bmod 2\mathbb{Z}$.

Lemma 7.1. *The natural homomorphism $\mathrm{d}_S \colon O(S) \twoheadrightarrow \operatorname{Aut}(\operatorname{discr} S)$ is surjective.*

Proof. In the notation above, the group $\operatorname{Aut}(\operatorname{Fn} X)$ restricts to S and its image under d_S is an index 6 maximal subgroup of $\operatorname{Aut}(\operatorname{discr} S)$ which, expectedly, fixes the order 2 element

$$\alpha_1 := (\tilde{\alpha}_1 \bmod S) \in \operatorname{discr}_2 S.$$

Therefore, it suffices to show that the full group $O(S)$ acts transitively on the set of all six elements $\alpha_1, \ldots, \alpha_6 \in \operatorname{discr}_2 S$ of square $1 \bmod 2\mathbb{Z}$. To this end, pick $\tilde{\alpha}_i \in S^\vee$ such that $\alpha_i = \tilde{\alpha}_i \bmod S$ and consider the index 2 extension $N_i \supset S \oplus \mathbb{Z}h$, $h^2 = 4$, generated by $\tilde{\alpha}_i + \hbar$. A computation shows that each $N_i \ni h$ is geometric and its Fano graph $\operatorname{Fn}(N_i, h)$ consists of 800 conics and contains a copy of Ω. Thus, due to the uniqueness given by Table 6, there is a lattice isomorphism

$$N_i \xrightarrow{\cong} N_1 = NS(X)$$

preserving h, *i.e.*, an element of $O(S)$ taking α_i to α_1. $\square$

Given a primitive isometry $\iota\colon S\hookrightarrow \mathbf{L}$, the genus of $S^\perp$ (*cf.* Proposition 3.2) consists of a single isomorphism class:

$$S^\perp \cong \begin{bmatrix}4&0&2\\0&4&2\\2&2&12\end{bmatrix}.$$

Hence, by Proposition 3.2 and Lemma 7.1, the isometry is unique up to $O(S)$-equivalence (*cf.* Hashimoto [17]). The next corollary is proved in Bonnafé–Sarti [4] under somewhat stronger hypotheses, assuming a faithful action of the full group $\operatorname{Aut}_h X_4$ (which is an index 2 extension of M_{20}, see Table 6).

Corollary 7.2. *Up to projective transformation, there is a unique quartic surface $X_4\subset\mathbb{P}^3$ admitting a faithful symplectic action of the Mukai group M_{20}.*

Proof. It suffices to observe that $O^+(S^\perp)$ acts transitively on the set of the four square 4 vectors in $S^\perp$ and use an obvious analogue of Proposition 3.4. □

It follows that any quartic in $\mathbb{P}^3$ admitting a faithful symplectic action of M_{20} is projectively equivalent to the example found in Mukai [21]. This observation completes the proof of Theorem 1.3. □

For future references, we state an analogue of Corollary 7.2 for octics. For the proof, we merely observe that the group $O^+(S^\perp)$ also acts transitively on the set of the four square 8 vectors in $S^\perp$.

Corollary 7.3. *Up to projective transformation, there is a unique octic $K3$-surface $X_8\subset\mathbb{P}^5$ admitting a faithful symplectic action of the Mukai group M_{20}.* ◁

7.2. **Real conics** (*cf.* [11]). Given a real structure $\tau\colon X\to X$ on a $K3$-surface X, we denote by $\mathbf{L}_{\pm\tau}\subset\mathbf{L}=H_2(X;\mathbb{Z})$ the (±1)-eigenlattices of τ_*. We say that a real structure $\tau\colon X\to X$ is *uniform* if τ_* restricts to $-\operatorname{id}\colon NS(X)\to NS(X)$. Clearly, on a uniformly real $K3$-surface X, each (-2)-curve is real; the converse also holds provided that $NS(X)$ is rationally generated by (-2)-curves.

Let $\tau\colon X\to X$ be a real structure on a quartic $X\subset\mathbb{P}^3$. A conic $c\in NS(X)$ is real if and only if $\tau_*(c)=-c$; hence, τ_* restricts to $-\operatorname{id}$ on the primitive hull N of the sublattice generated by real conics. Clearly, N is geometric, as it is primitive in $NS(X)$, and by an obvious analogue of Proposition 2.9, there exists a quartic X', arbitrarily close to X, and a uniform real structure $\tau'\colon X'\to X'$ such that

- the conics on X' are in a bijection with the real conics on X,
- $NS(X')=N$ is rationally generated by conics.

Remark 7.4. Unlike [11], where only lines are considered, in general we cannot assert that $\operatorname{Fn}X'$ is a subgraph of $\operatorname{Fn}X$: some irreducible conics on X' may become reducible on X. Nor can we assert that X' is a Barth–Bauer quartic, even if X is, as we do not assume that X has a 16-tuple Ω of real Kummer conics.

Lemma 7.5 (*cf.* [11, Lemma 3.8])**.** *If $\tau\colon X\to X$ is a uniform real structure on a Barth–Bauer quartic* (*more generally, Kummer surface*) *X, then $\mathbf{L}_{+\tau}\cong\mathbf{U}(2)$ is an orthogonal direct summand in $T(X)$. Conversely, if $T(X)\cong\mathbf{U}(2)\oplus T'$, then X is equiconically deformation equivalent to a uniformly real quartic.*

Proof. We have $\mathbf{L}_{+\tau}\subset T(X)$ and, in view of (2.2), $\mathbf{L}_{+\tau}(\frac12)\subset T(X)(\frac12)$ are also even lattices. On the other hand, $\operatorname{discr}\mathbf{L}_{+\tau}$ is an elementary abelian 2-group (see [23]); hence, $\mathbf{L}_{+\tau}(\frac12)$ is unimodular and, as such, is an orthogonal direct summand in any overlattice. Besides, $\sigma_+(\mathbf{L}_{+\tau}) = 1$ (see [23]) and $\operatorname{rk}\mathbf{L}_{+\tau} \leqslant \operatorname{rk}\mathbf{T} = 6$; hence, using the classification of unimodular even lattices of small rank, $\mathbf{L}_{+\tau}(\frac{1}{2}) \cong \mathbf{U}$.

For the converse statement, letting $N := NS(X)$, we mimic the arguments of [13, 23]: the period ω_X in the period domain of marked N-polarized $K3$-surfaces is moved to $\omega_+ + i\omega_-$, where $\omega_+ \in \mathbf{U}(2) \otimes \mathbb{R}$ and $\omega_- \in T' \otimes \mathbb{R}$ are sufficiently generic vectors, $\omega_+^2 = \omega_-^2 > 0$, and $\mathbb{R}\omega_+ \oplus \mathbb{R}\omega_-$ is positively oriented. The new quartic has the same Néron–Severi lattice N and, by [23] and the global Torelli theorem, it has a real structure τ such that $\mathbf{L}_{+\tau}$ is the summand $\mathbf{U}(2)$ in the statement. □

Corollary 7.6. *If X is a Barth–Bauer quartic, the real conics on X (with respect to any real structure) span a sublattice of $NS(X)$ or rank at most* 19. ◁

7.3. **Proof of Theorem 1.6.** Due to Corollary 7.6, a rigid Barth–Bauer quartic cannot have all its conics real. However, in view of Remark 7.4, we cannot directly refer to Table 5, and we have to compute the set of real conics for each quartic X in Table 6 with $|\operatorname{Fn}_2 X| \geqslant 560$ and each real structure $\tau\colon X \to X$. The induced action $g_N := \tau_*$ on $N := NS(X)$ is found *via* (3.3), with an extra requirement that both $g_N \in O_h(N)$ and $g_T \in O(T) \smallsetminus O^+(T)$, *cf.* Proposition 3.4(3), must be involutive. The computation results in a single pair (X, τ), *viz.* the one marked with a $*$ in Table 6, and the corresponding graph of real conics is isomorphic to $\operatorname{Fn} Y$, $Y \in \mathcal{Y}$, where $\mathcal{Y}$ is the 1-parameter family marked with a $*$ in Table 5.

To complete the construction of the 1-parameter family as in the theorem, we observe that $T(Y) \cong \mathbf{U}(2) \oplus [76]$ for a generic $Y \in \mathcal{Y}$ and refer to Lemma 7.5.

Finally, we check that, for each conic $c \in \operatorname{Fn} Y$, there is another conic c' such that $c \cdot c' = 1$, implying that c has a real point. □

Remark 7.7. Using the fact that, for $N := NS(Y)$, the natural homomorphism $O_h(N) \twoheadrightarrow \operatorname{Aut}(\operatorname{discr} N)$ is surjective and analyzing the walls in the period domain, it is not difficult to show that the moduli space of real quartics as in Theorem 1.6 is connected. Technical details and analysis of the chirality are left to the reader.

7.4. **Proof of Proposition 1.7 and Corollary 1.8.** Proposition 1.7 is merely a combined statement of Proposition 4.8 and Corollaries 2.13, 5.3, and 6.4.

For Corollary 1.8, we observe that, due to Proposition 1.7, for any subgraph $\Omega \subset \Gamma$ as in the statement, the primitive hull $(\mathbb{Q}\Omega + \mathbb{Q}h) \cap \mathcal{F}(\Gamma)$ admits a primitive isometry to $\mathbf{L}$. Hence, it is isomorphic to $\mathbf{S}_h$ as in (2.6), which determines a unique 4-Kummer structure on Ω (*cf.* also Lemma 2.1). □

References

1. Ayşegül Akyol and Alex Degtyarev, *Geography of irreducible plane sextics*, Proc. Lond. Math. Soc. (3) **111** (2015), no. 6, 1307–1337. MR 3447795
2. W. Barth and Th. Bauer, *Smooth quartic surfaces with* 352 *conics*, Manuscripta Math. **85** (1994), no. 3-4, 409–417. MR 1305751
3. Th. Bauer, *Quartic surfaces with* 16 *skew conics*, J. Reine Angew. Math. **464** (1995), 207–217. MR 1340342
4. Cédric Bonnafé and Alessandra Sarti, *K3 surfaces with maximal finite automorphism groups containing M_{20}*, To appear, arXiv:1910.05955, 2020.
5. J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 290, Springer-Verlag, New York, 1988, With contributions by E. Bannai, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. MR 920369 (89a:11067)
6. Alex Degtyarev, *Lines on Smooth Polarized K3-Surfaces*, Discrete Comput. Geom. **62** (2019), no. 3, 601–648. MR 3996938

7. ______, *800 conics in a smooth quartic surface*, To appear, arXiv:2102.08163, 2021.
8. ______, *Conics in Kummer quartics: the tables*, http://www.fen.bilkent.edu.tr/~degt/papers/4Kummer_tab.pdf, 2021.
9. ______, *Conics in sextic surfaces in* $\mathbb{P}^4$, To appear, arXiv:2010.07412, 2021.
10. ______, *Tritangents to smooth sextic curves*, To appear, arXiv:1909.05657, 2021.
11. Alex Degtyarev, Ilia Itenberg, and Ali Sinan Sertöz, *Lines on quartic surfaces*, Math. Ann. **368** (2017), no. 1-2, 753–809. MR 3651588
12. Alex Degtyarev and Sławomir Rams, *Counting lines with Vinberg's algorithm*, To appear, arXiv:1803.03548, 2021.
13. I. V. Dolgachev, *Mirror symmetry for lattice polarized* $K3$ *surfaces*, vol. 81, 1996, Algebraic geometry, 4, pp. 2599–2630. MR 1420220
14. *GAP – Groups, Algorithms, and Programming, Version 4.10.1*, https://www.gap-system.org, Feb 2019.
15. Carl Friedrich Gauss, *Disquisitiones arithmeticae*, Springer-Verlag, New York, 1986, Translated and with a preface by Arthur A. Clarke, Revised by William C. Waterhouse, Cornelius Greither and A. W. Grootendorst and with a preface by Waterhouse. MR 837656 (87f:01105)
16. Çisem Güneş Aktaş, *Real representatives of equisingular strata of simple quartic surfaces*, Internat. J. Math. **30** (2019), no. 12, 1950063, 28. MR 4039101
17. Kenji Hashimoto, *Finite symplectic actions on the* $K3$ *lattice*, Nagoya Math. J. **206** (2012), 99–153. MR 2926486
18. Daniel Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, vol. 158, Cambridge University Press, Cambridge, 2016. MR 3586372
19. Vik. S. Kulikov, *Surjectivity of the period mapping for* $K3$ *surfaces*, Uspehi Mat. Nauk **32** (1977), no. 4(196), 257–258. MR 0480528 (58 #688)
20. Rick Miranda and David R. Morrison, *Embeddings of integral quadratic forms*, Electronic, http://www.math.ucsb.edu/~drm/manuscripts/eiqf.pdf, 2009.
21. Shigeru Mukai, *Finite groups of automorphisms of* $K3$ *surfaces and the Mathieu group*, Invent. Math. **94** (1988), no. 1, 183–221. MR 958597
22. V. V. Nikulin, *Kummer surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 2, 278–293, 471. MR 0429917
23. ______, *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177, 238, English translation: Math USSR-Izv. 14 (1979), no. 1, 103–167 (1980). MR 525944 (80j:10031)
24. I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič, *Torelli's theorem for algebraic surfaces of type* K3, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 530–572, English translation: Math. USSR-Izv. 5, 547–588. MR 0284440 (44 #1666)
25. Sławomir Rams and Matthias Schütt, *64 lines on smooth quartic surfaces*, Math. Ann. **362** (2015), no. 1-2, 679–698. MR 3343894
26. B. Saint-Donat, *Projective models of* K*-3 surfaces*, Amer. J. Math. **96** (1974), 602–639. MR 0364263 (51 #518)
27. Friedrich Schur, *Ueber eine besondre Classe von Flächen vierter Ordnung*, Math. Ann. **20** (1882), no. 2, 254–296. MR 1510168
28. B. Segre, *The maximum number of lines lying on a quartic surface*, Quart. J. Math., Oxford Ser. **14** (1943), 86–96. MR 0010431 (6,16g)
29. Gang Xiao, *Galois covers between* $K3$ *surfaces*, Ann. Inst. Fourier (Grenoble) **46** (1996), no. 1, 73–88. MR 1385511 (97b:14047)

Department of Mathematics, Bilkent University, 06800 Ankara, TURKEY
Email address: degt@fen.bilkent.edu.tr