PLANES IN CUBIC FOURFOLDS

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Abstract. We show that the maximal number of planes in a complex smooth cubic fourfold in $\mathbb{P}^5$ is 405, realized by the Fermat cubic only; the maximal number of real planes in a real smooth cubic fourfold is 357, realized by the so-called Clebsch–Segre cubic. Altogether, there are but three (up to projective equivalence) cubics with more than 350 planes.

1. Introduction

The study of linear spaces in projective hypersurfaces is a classical problem in algebraic geometry. The 27 lines on a smooth cubic surface in $\mathbb{P}^3$, going back to A. Cayley and G. Salmon in the 19th century [3], and at most 64 lines on quartic surfaces, going back to B. Segre [28], are two of the most famous examples. In the last decade, there has been a substantial progress in studying and counting lines and other low degree rational curves on polarized $K3$-surfaces, see [27, 10, 6, 9, 8, 7]. Moreover, thanks to the global Torelli theorem for $K3$-surfaces [26] and the surjectivity of the period map [17], it has also been possible to obtain a complete description of the surfaces with large configurations of lines or, sometimes, conics.

While varieties of lines play an important role in the geometry of hypersurfaces (especially cubic threefolds [4] and cubic fourfolds [14, 31]), much less is known about linear spaces of higher dimension.

In this paper, we study 2-planes in smooth cubic fourfolds $X \subset \mathbb{P}^5$. Planes in cubic fourfolds have already appeared in many contexts: they are a key ingredient in Voisin's proof of the global Torelli theorem [31], they define one of the so-called Hassett divisors in the moduli space of cubics [14], and they serve as important examples in connection with the rationality problem for cubic fourfolds.

As in the case of lines on $K3$-surfaces, the plane counting problem for smooth cubic fourfolds is not strictly enumerative in the sense that a generic cubic contains no planes at all. Therefore, one looks for estimates on the maximal number of planes in a smooth cubic, and, if possible, a description of the cubics realizing this maximum. In fact, the similarities between $K3$-surfaces and cubic fourfolds are much deeper, as cubic fourfolds have their own version of the global Torelli theorem [31] and surjectivity of the period map [18]. This can be used to recast the original geometric problem in purely lattice-theoretic terms and eventually obtain a complete characterisation of extremal cubics. Formally, our characterisation is in terms of the periods. However, thanks to the injectivity of the period map, we

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can identify the extremal cubics found by comparing their configurations of planes with those of known explicit examples.

Our main result is the following theorem.

Theorem 1.1 (see §9.1)**.** *Let $X \subset \mathbb{P}^5$ be a complex smooth cubic fourfold. Then, either X has at most 350 planes, or, up to projective equivalence, X is*

- *the Fermat cubic (with 405 planes), see §2.5 and (8.4), or*
- *the Clebsch–Segre cubic (with 357 planes), see §2.6 and (8.5), or*
- *the cubic (with 351 planes) described in (8.6).*

We also consider a similar problem for real planes in real cubics. Recall that a *real algebraic variety* is a complex algebraic variety X equipped with a *real structure*, *i.e.*, anti-holomorphic involution $c\colon X \to X$. A subvariety $P \subset X$ is said to be *real* if $c(P) = P$. One can show (*cf.* Theorem 2.1 and its proof) that any real structure on a cubic $X \subset \mathbb{P}^5$ in appropriate coordinates in $\mathbb{P}^5$ is induced from the coordinatewise complex conjugation. In these coordinates, both X and a real plane $P \subset X$ can be given by equations with real coefficients.

For the number of real planes, we have the following stronger bound.

Theorem 1.2 (see §9.2)**.** *The number of real planes in a real smooth cubic $Y \subset \mathbb{P}^5$ is at most 357, the equality holding if and only if Y is projectively equivalent over $\mathbb{R}$ to the Clebsch–Segre cubic (see §2.6).*

Note that the Clebsch–Segre cubic (2.10) can be regarded as a four-dimensional analogue of the Clebsch cubic surface, which also has all of its 27 lines real.

Remark 1.3. Theorem 1.1 provides a sharp upper bound on the total number of planes. Another interesting question is that on the possible values that can be taken by the plane count. It appears that, for smooth cubics, the list is much more sparse than those counting lines on polarized $K3$-surfaces. The values observed in our computation are

$$0\mathinner{..}225, 227, 229, 231, 233, 235, 237, 239, 241, 243, 245, 247, 249,$$
$$255, 257, 259, 261, 267, 273, 285, 297, 351, 357, 405,$$

but we do not assert that this list is complete.

As mentioned above, our approach is lattice-theoretic. More precisely, given an abstract graph, there is a way to decide whether it is realized by the configuration of planes in a smooth cubic fourfold. However, unlike the case of lines on a polarized $K3$-surface, we lack geometric intuition (*e.g.*, elliptic pencils) which would narrow the search down to a sufficiently small collection of graphs. For this reason, we take a detour and replant a (modified) abstract lattice of algebraic cycles to a Niemeier lattice (*i.e.*, one of the 24 positive definite even unimodular lattices of rank 24), the planes mapping to certain vectors of square 4. This approach has a number of benefits. First, instead of dealing with abstract graphs of *a priori* unbounded complexity, we merely consider subsets of several finite sets known in advance; in particular, this implies the fact (not immediately obvious) that the number of planes is uniformly bounded. Second, these finite sets have rich intrinsic structure that can be used in the construction of large realizable subsets. Finally, working with known sets, all symmetry groups can be expressed in terms of permutations, which makes the computation in `GAP` [13] very effective.

The idea of using Niemeier lattices is not new (*cf.* [16, 24, 25]). The novelty of our treatment is in the fact that the original lattice is odd and, therefore, it needs to be modified. Of course, one could have used embeddings to odd unimodular definite lattices of rang 24, but their number is overwhelming.

The paper is organized as follows. In §2, we fix the terminology and recall a few basic facts related to integral lattices and cubic fourfolds. Towards the end, in §2.5 and §2.6, we describe the geometric configurations of planes in the two extremal cubics, *viz.* Fermat and Clebsch–Segre. In §3, we replant the lattice of algebraic cycles of a cubic fourfold to a Niemeier lattice, thus reducing the original geometric problem to an arithmetic one; the algorithms used to solve the latter are outlined in §4. In §5–§8, the 24 Niemeier lattices are treated one by one; this is followed by the proofs of Theorems 1.1 and 1.2 in §9.

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2. Preliminaries

The principal goal of this section is to fix the terminology/notation and to cite, in a convenient form, a few fundamental results used in the sequel.

2.1. **Lattices** (see [23])**.** A *lattice* is a free abelian group L of finite rank equipped with a symmetric bilinear form $b\colon L \otimes L \to \mathbb{Z}$. Since the form b is assumed fixed (and omitted from the notation), we abbreviate $x \cdot y := b(x, y)$ and $x^2 := b(x, x)$. The *determinant* $\det L \in \mathbb{Z}$ is the determinant of the Gram matrix of b in any integral basis; L is called *nondegenerate* (*unimodular*) if $\det L \neq 0$ (respectively, $\det L = \pm 1$). The *inertia indices* $\sigma_\pm L$ are those of the quadratic space $L \otimes \mathbb{R} \to \mathbb{R}$, $x \otimes r \mapsto r^2x^2$.

A sublattice $S \subset L$ is called *primitive* if the quotient L/S is torsion free. A *d-polarized lattice* is a lattice equipped with a distinguished element h, $h^2 = d$. Morphisms in the category of (polarized) lattices are called *isometries*.

A lattice L is called *even* if $x^2 = 0 \bmod 2$ for all $x \in L$; otherwise, L is *odd*. A *characteristic vector* is an element $v \in L$ such that $x^2 = x \cdot v \bmod 2$ for all $x \in L$. If L is unimodular, a characteristic vector exists and is unique mod $2L$. If $v \in L$ is characteristic, the orthogonal complement $v^\perp \subset L$ is even.

For lattices of rank 1 and 2 we use the abbreviations

- $[a] := \mathbb{Z}x$, $x^2 = a$;
- $[a, b, c] := \mathbb{Z}x + \mathbb{Z}y$, $x^2 = a$, $x \cdot y = b$, $y^2 = c$.

The *hyperbolic plane* $\mathbf{U} := [0, 1, 0]$ is the unique unimodular even lattice of rank 2.

A nondegenerate lattice L admits a canonical inclusion

$$L \hookrightarrow L^\vee := \bigl\{x \in L \otimes \mathbb{Q} \bigm| x \cdot y \in \mathbb{Z} \text{ for all } y \in L\bigr\}$$

to the dual group $L^\vee$. The finite abelian group $\mathcal{L} := \operatorname{discr} L := L^\vee/L$ (q_L in [23]) is called the *discriminant group* of L. Clearly, $|\mathcal{L}| = (-1)^{\sigma_- L} \det L$. This group is equipped with the nondegenerate symmetric bilinear form

$$\mathcal{L} \otimes \mathcal{L} \to \mathbb{Q}/\mathbb{Z}, \quad (x \bmod L) \otimes (y \bmod L) \mapsto (x \cdot y) \bmod \mathbb{Z},$$

and, if L is even, its quadratic extension

$$\mathcal{L} \to \mathbb{Q}/2\mathbb{Z}, \quad x \bmod L \mapsto x^2 \bmod 2\mathbb{Z}.$$

We denote by $\mathcal{L}_p := \operatorname{discr}_p L := \mathcal{L} \otimes \mathbb{Z}_p$ the p-primary components of $\operatorname{discr} L$. The 2-primary component $\mathcal{L}_2$ is called *even* if $x^2 \in \mathbb{Z}$ for all order 2 elements $x \in \mathcal{L}_2$; otherwise, $\mathcal{L}_2$ is *odd*. The *determinant* $\det \mathcal{L}_p$ is the determinant of the "Gram matrix" of the quadratic form in any minimal set of generators. (This is equivalent to the alternative definition given in [23].) Unless $p = 2$ and $\mathcal{L}_2$ is odd (in which case the determinant is not defined or used), we have $\det \mathcal{L}_p = u_p/|\mathcal{L}_p|$, where u_p is a well-defined element of $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$.

The *length* $\ell(\mathcal{A})$ of a finite abelian group $\mathcal{A}$ is the minimal number of generators of $\mathcal{A}$. We abbreviate $\ell_p(\mathcal{A}) := \ell(\mathcal{A} \otimes \mathbb{Z}_p)$ for a prime p.

Given a lattice L and $q \in \mathbb{Q}$, we use the notation $L(q)$ for the same abelian group with the form $x \otimes y \mapsto q(x \cdot y)$, assuming that it is still a lattice. We abbreviate $-L := L(-1)$, and this notation applies to discriminant forms as well. The notation nL, $n \in \mathbb{N}$, is used for the orthogonal direct sum of n copies of L.

A cyclic group $\mathbb{Z}/b$ equipped with a quadratic form $1 \mapsto a \bmod 2\mathbb{Z}$ is denoted by

$$\left[\frac{a}{b}\right]; \quad \text{we assume that } a, b \in \mathbb{Z},\ \text{g.c.d.}(a, b) = 1,\ ab = 0 \bmod 2.$$

Another notation used in the description of the discriminants is $\mathcal{U} := \operatorname{discr} \mathbf{U}(2)$.

A *root* in an even lattice L is a vector of square ± 2. A *root system* is a positive definite lattice generated by roots. Any root system has a unique decomposition into orthogonal direct sum of irreducible components, which are of types $\mathbf{A}_n$, $n \geqslant 1$, $\mathbf{D}_n$, $n \geqslant 4$, $\mathbf{E}_6$, $\mathbf{E}_7$, or $\mathbf{E}_8$ (see, *e.g.*, [1]), according to their *Dynkin diagrams.*

A *Niemeier lattice* is a positive definite unimodular even lattice of rank 24. Up to isomorphism, there are 24 Niemeier lattices (see [22]): the *Leech lattice* Λ, which is root free, and 23 lattices *rationally* generated by roots. In the latter case, the isomorphism class of a lattice $N := N(D)$ is uniquely determined by that of its maximal root system D. For more details, see [5].

2.2. **Cubic fourfolds** (see [14, 18, 31])**.** Let X be a smooth cubic fourfold in $\mathbb{P}^5$. The middle Hodge numbers of X are $h^{1,3} = h^{3,1} = 1$ and $h^{2,2} = 21$. We will be concerned with the middle integral cohomology group $H^4(X) := H^4(X; \mathbb{Z})$; via the Poincaré duality isomorphism, this group is canonically identified with $H_4(X; \mathbb{Z})$. (Unless stated otherwise, all homology and cohomology groups are with coefficients in $\mathbb{Z}$.) With respect to the intersection form, $H^4(X)$ is the unique (up to isomorphism) odd unimodular lattice of signature $(21, 2)$. This lattice is canonically 3-polarized, and the distinguished class h_X, *viz.* the square of the hyperplane divisor of X, is characteristic. There is a lattice isomorphism

$$H^4(X) \simeq \mathbf{L} := 21[+1] \oplus 2[-1], \quad h_X \mapsto h := (1, \ldots, 1, 3, 3).$$

Alternatively,

$$\mathbf{L} \simeq 3[+1] \oplus 2\mathbf{U} \oplus 2\mathbf{E}_8, \quad h \mapsto (1, 1, 1) \in 3[+1].$$

In particular, the sublattice $\mathbf{L}^0 := h^\perp \subset \mathbf{L}$ of primitive classes decomposes as

$$\mathbf{L}^0 \simeq \mathbf{A}_2 \oplus 2\mathbf{U} \oplus 2\mathbf{E}_8.$$

The choice of a polarized lattice isomorphism $\phi\colon H^4(X) \to \mathbf{L}$ is called a *marking* of the cubic fourfold X, and we call (X, ϕ) a *marked cubic fourfold.*

By definition, the sublattice $M_X := H^{2,2}(X;\mathbb{C})\cap H^4(X)$ of integral Hodge classes is primitive in $H^4(X)$. The Hodge–Riemann relations imply that M_X is positive definite. By [32], the integral Hodge conjecture holds for X, so that M_X is generated (over $\mathbb{Z}$) by the classes of algebraic surfaces in X.

We denote by $T_X := M_X^\perp$ the *transcendental lattice* of X.

2.3. **The global Torelli theorem.** The *period* of a marked cubic fourfold (X,ϕ) is defined as the line $\omega_X = \phi(H^{3,1}(X;\mathbb{C})) \subset \mathbf{L}^0 \otimes \mathbb{C}$. Thus, denoting by $\mathcal{M}$ the moduli space of marked cubic fourfolds (X,ϕ), we may define the *period map*

$$\begin{aligned} \mathcal{P}\colon \quad \mathcal{M} \;&\to\; \mathcal{D} \;\subset \mathbb{P}(\mathbf{L}^0 \otimes \mathbb{C}), \\ (X,\phi) &\mapsto [\omega_X], \end{aligned}$$

where $\mathcal{D}$ is the *period domain*, *i.e.*, a distinguished connected component of

$$\bigl\{x \in \mathbb{P}(\mathbf{L}^0 \otimes \mathbb{C}) \bigm| x^2 = 0,\ x \cdot \bar{x} < 0\bigr\},$$

see [15]. The component is distinguished by the so-called *positive sign structure*, *i.e.*, a coherent choice of orientations of the maximal negative definite subspaces of $\mathbf{L} \otimes \mathbb{R}$. More generally, we can consider cubic fourfolds X polarized by a positive definite polarized sublattice $h \in K \subset \mathbf{L}$. This gives us a moduli space $\mathcal{M}_K$ of dimension $21 - \operatorname{rk} K$ and an associated period domain $\mathcal{D}_K$, which is a connected component of

$$\bigl\{x \in \mathbb{P}(K^\perp \otimes \mathbb{C}) \bigm| x^2 = 0,\ x \cdot \bar{x} < 0\bigr\}.$$

The following result is a version of the global Torelli theorem for cubic fourfolds, which is due to C. Voisin [31].

Theorem 2.1. *Let $X, Y \subset \mathbb{P}^5$ be two smooth cubic fourfolds. Then an isometry*

$$f^*\colon H^4(Y) \to H^4(X)$$

is induced by a holomorphic (*respectively, anti-holomorphic*) *projective isomorphism $f\colon X \to Y$ if and only if*

(1) *f^* is polarized,* i.e., $f^*(h_Y) = h_X$*,*
(2) *$f^*(\omega_Y) = \omega_X$ (respectively, $f^*(\omega_Y) = \overline{\omega}_X$).*

If an isomorphism f as above exists, it is unique.

Proof. The holomorphic statement is essentially found in [31] (see also [12]), and the anti-holomorphic counterpart is immediately obtained by composing with the complex conjugation. The uniqueness follows from the well-known fact that an automorphism $X \to X$ which acts as the identity on $H^4(X)$ must be the identity (which can be proved using the Lefschetz fixed-point theorem). □

Corollary 2.2 (*cf.* [10, Lemma 3.8] or [11])**.** *A smooth cubic $X \subset \mathbb{P}^5$ admits a real structure identical on M_X if and only if T_X contains a sublattice $-\mathbf{A}_1$ or $\mathbf{U}(2)$.* ◁

A major consequence of Theorem 2.1 is the fact that the period map $\mathcal{P}\colon \mathcal{M} \to \mathcal{D}$ is injective; its image was computed by R. Laza [18] and E. Looijenga [19].

Theorem 2.3 (Surjectivity of the period map, see [18, Theorem 1.1])**.** *A 3-polarized lattice $M \ni h$ admits an isometry φ onto $M_X \ni h_X$ for a smooth cubic $X \subset \mathbb{P}^5$ if and only if*

(1) *M is positive definite and h is a characteristic vector in M,*
(2) *M admits a primitive embedding into $\mathbf{L}$ such that $M^\perp$ is even,*

(3) *there is no element* $e \in M$ *such that* $e^2 = e \cdot h = 1$,
(4) *there is no element* $e \in M$ *such that* $e^2 = 2$ *and* $e \cdot h = 0$.

Under these assumptions, $\mathcal{M}_M \subset \mathcal{M}$ *has codimension* $\operatorname{rk}(M) - 1$. ◁

2.4. **Planes in cubic fourfolds.** Our main result is an upper bound on the number of planes in a smooth cubic fourfold. The following lemma appears in Starr's appendix to [2], where it is attributed to O. Debarre.

Lemma 2.4. *Any smooth cubic fourfold contains but finitely many planes.* ◁

Let X be a smooth cubic fourfold, $P \subset X$ a plane, and $p := [P] \in M_X$ its class. Clearly, $h_X \cdot p = 1$, and using the normal bundle sequence, we find that $p^2 = c_2(N_{P|X}) = 3$. Furthermore, given two distinct planes P_1, P_2 with classes p_1, p_2, one has the following trichotomy:

- $p_1 \cdot p_2 = 0$ if P_1 and P_2 are disjoint;
- $p_1 \cdot p_2 = 1$ if P_1 and P_2 intersect at a point;
- $p_1 \cdot p_2 = -1$ if P_1 and P_2 intersect in a line.

This has the following important consequence.

Lemma 2.5. *Each class* $p \in M_X$ *is represented by at most one plane.* ◁

A configuration of planes in a smooth cubic $X \subset \mathbb{P}^5$ is described by means of its *graph of planes* $\operatorname{Fn} X$: the vertices of $\operatorname{Fn} X$ are planes $P \subset X$, and two vertices P_1, P_2 are connected by a solid (respectively, dotted) edge whenever P_1 and P_2 intersect at a point (respectively, in a line). By $|\operatorname{Fn} X|$, we denote the number of vertices of this graph, *i.e.*, the number of planes in X.

The next proposition gives us a precise relationship between classes in M_X and planes in X.

Proposition 2.6. *Given a smooth cubic fourfold* $X \subset \mathbb{P}^5$, *the map* $P \mapsto [P]$ *establishes a bijection between* $\operatorname{Fn} X$ *and the set of classes* $p \in M_X$ *such that* $p^2 = 3$ *and* $h_X \cdot p = 1$.

Proof. The map $P \mapsto [P]$ is injective by Lemma 2.5, whereas the surjectivity is essentially stated in the first paragraph of [31, §3]. Here is a more direct argument.

Let $\pi \colon \mathcal{X} \to B$ be a local universal family of marked cubic fourfolds, with X as the fiber over $b_0 \in B$. The marking allows us to identify each lattice M_{X_b}, $b \in B$, with a sublattice of $\mathbf{L}$. Let $p \in M_X \subset \mathbf{L}$ be a class with $p^2 = 3$ and $h_X \cdot p = 1$, and let $B' \subset B$ denote the Hodge locus of p, parameterizing the fibers $\mathcal{X}_b$ for which the class p stays Hodge, *i.e.*, $p \in M_{\mathcal{X}_b}$. Note that

(2.7) the sublattice $K := \mathbb{Z}h + \mathbb{Z}p \simeq [3, 1, 3] \subset M_{\mathcal{X}_b}$ is necessarily primitive

(as its only proper finite index extension contains a vector as in Theorem 2.3(3)); hence, the closed codimension 1 subset $B' \subset B$ is irreducible (see [14, Theorem 3.2.3]). On the other hand, a simple dimension count shows that the cubic fourfolds containing a plane form an irreducible divisor in the parameter space of cubics.

Therefore, if $b \in B'$ is a very general point, then both

- the cubic $X' = \mathcal{X}_b$ contains a plane P' and
- the lattice $M_{X'}$ has rank 2, hence $M_{X'} = K$ by (2.7).

Since also

- $p \in M_{X'} = K$ is the only class such that $h_{X'} \cdot p = 1$ and $p^2 = 3$, we conclude that $[P'] = p$. Then, by specialization, in X the class p is also represented by an effective cycle of degree 1 in $\mathbb{P}^5$, hence a plane $P \subset X$. ☐

Corollary 2.8. *Given a real structure $c\colon X \to X$ on a smooth cubic fourfold X, a class $p \in M_X$ is represented by a real plane $P \subset X$ if and only if*

(1) $h_X \cdot p = 1$ *and* $p^2 = 3$,
(2) $c^*p = p$.

Proof. If (1) holds, then p is represented by a unique plane P by Proposition 2.6, and P satisfies $c(P) = P$ by (2). The converse is immediate. ☐

2.5. **The Fermat cubic.** Let $X \subset \mathbb{P}^5$ be the *Fermat cubic*, defined by the equation

$$x_0^3 + x_1^3 + \ldots + x_5^3 = 0. \tag{2.9}$$

One can easily see that X contains at least 405 planes. Indeed, consider one of the 5×3 splittings of the index set $\{0, \ldots, 5\}$ into three pairs, *e.g.*, $\{0, 1\}, \{2, 3\}, \{4, 5\}$, and pick three cubic roots ξ_1, ξ_2, ξ_3 of -1. Then, each of the planes

$$x_0 = \xi_1 x_1, \quad x_2 = \xi_2 x_3, \quad x_4 = \xi_3 x_5$$

clearly lies in X. The number of planes obtained in this way is $15 \times 3^3 = 405$.

A direct calculation (e.g., following the argument of Segre [29, pp. 122–123]) shows that X does not contain any other plane. Alternatively, this statement is an immediate corollary of Theorem 1.1; moreover, we assert that X is the only (up to projective equivalence) smooth cubic with 405 planes.

2.6. **The Clebsch–Segre cubic.** The *Clebsch–Segre cubic* is the cubic fourfold Y defined by the following equations in $\mathbb{P}^6$:

$$x_0 + x_1 + \ldots + x_6 = x_0^3 + x_1^3 + \ldots + x_6^3 = 0. \tag{2.10}$$

Note that the full symmetric group $\mathbb{S}_7$ acts on Y by permuting the coordinates.

According to K. Hulek and M. Schütt (private communication), Y contains at least 357 planes, constituting two $\mathbb{S}_7$-orbits. One orbit consists of the 105 Fermat-type planes

$$x_i + x_j = x_k + x_l = x_m + x_n = x_o = 0,$$

where i, j, k, l, m, n, o is a permutation of $0, 1, 2, 3, 4, 5, 6$. To describe the other orbit, we fix a sequence of three pairwise disjoint pairs in the index set, *e.g.*, $\{1, 2\}, \{3, 4\}, \{5, 6\}$. Consider a vector

$$(0, 1, -1, \varphi, -\varphi, 0, 0) \in \mathbb{C}^7,$$

where $\varphi^2 + \varphi - 1 = 0$, and denote by O its orbit under the dihedral group $\mathbb{D}_{10} \subset \mathbb{S}_7$ generated by the simultaneous transposition $1 \leftrightarrow 2$, $3 \leftrightarrow 4$ and the 5-cycle

$$0 \mapsto 1 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 0.$$

It is straightforward to check that the image $[O] \subset \mathbb{P}^6$ of O consists of five collinear points and the plane spanned by $[O]$ and $[0 : 0 : 0 : 0 : 0 : 1 : -1]$ lies in Y. This plane is stabilized by $\mathbb{D}_{10} \times \langle 5 \leftrightarrow 6 \rangle$; hence, its $\mathbb{S}_7$-orbit is of size 252, resulting in the total of 357 planes in Y.

By a direct computation, or as a consequence of Lemma 9.2, we conclude that the cubic Y given by (2.10) contains no other planes. Observe also that, immediately by the construction, all 357 planes in Y are real (as they are spanned by real points).

3. Reduction to the Niemeier lattices

The principal goal of this section is replanting a lattice M as in Theorem 2.3 to an appropriate Niemeier lattice. Then, in §3.3 and §3.4, we outline the strategy of our proof of Theorems 1.1 and 1.2.

3.1. **Replanting to a Niemeier lattice.** Consider a positive definite 3-polarized lattice $M \ni h$. Assume that h is characteristic in M and M contains at least one h-*plane*, *i.e.*, a vector l such that $l^2 = 3$ and $l \cdot h = 1$.

We denote by $S := S(M)$ the index 3 extension of the lattice $h^\perp \oplus \mathbb{Z}\hbar$, $\hbar^2 = 12$, by the vector $\frac{1}{3}(3l - h + \hbar)$; this extension does not depend on the choice of an h-plane l. A *plane* in S is a vector $l \in S$ such that $l^2 = l \cdot \hbar = 4$.

The following statement is immediate.

Proposition 3.1. *The lattice S constructed above has the following properties*:

(1) *S is even and positive definite*;
(2) $\hbar \in 4S^\vee$;
(3) *the orthogonal complements $h^\perp \subset M$ and $\hbar^\perp \subset S$ are identical*; *hence, in particular, there is a canonical bijection between their sets of roots.*

Furthermore, the map $x \mapsto \frac{1}{3}(3x - h + \hbar)$ establishes bijections between

(4) *h-planes in M and planes in S*;
(5) *vectors as in Theorem 2.3(3) and $e \in S$ such that $e^2 = 2$, $e \cdot \hbar = 4$.* ◁

Remark 3.2. In view of item (2) in Proposition 3.1, for any root $e \in S$, we have $e \cdot \hbar \in \{0, \pm 4\}$; hence, either $e \in \hbar^\perp$ or $\pm e$ is as in item (5). We are mainly interested in lattices S not containing either of these two classes of vectors; it follows that this condition is equivalent to the requirement that S should be root free.

Proposition 3.3. *If M admits a primitive embedding to $\mathbf{L}$ with even orthogonal complement, then S admits an embedding to a Niemeier lattice N such that the torsion of N/S is a 2-group.*

Proof. The proof relies upon Nikulin's theory of discriminant forms (see [23]). Let $\rho = \operatorname{rk} M$ and $T = M^\perp \subset \mathbf{L}$. Then, since $h^\perp \subset M$ is the orthogonal complement of $\mathbb{Z}h \oplus T$ in the unimodular lattice $\mathbf{L}$, we have

$$\operatorname{discr} h^\perp \simeq \langle \eta \rangle \oplus \operatorname{discr}(-T),$$

where $3\eta = 0$ and $\eta^2 = \frac{2}{3} \bmod 2\mathbb{Z}$; we use the assumption that M contains an h-plane, so that $h \notin 3M^\vee$ and, hence, the sublattice $\mathbb{Z}h \oplus T$ is primitive in $\mathbf{L}$. The passage from $h^\perp$ to S changes the discriminant to

$$\operatorname{discr} S = \left\langle \tfrac{1}{4}\hbar \right\rangle \oplus \operatorname{discr}(-T), \qquad \left(\tfrac{1}{4}\hbar\right)^2 = \tfrac{3}{4} \bmod 2\mathbb{Z},$$

which almost satisfies the hypotheses of [23, Theorem 1.12.2]. The only difficulty is in item (4) of *loc. cit.*: if $\ell(\operatorname{discr}_2 T) = \operatorname{rk} T$ and $\operatorname{discr}_2 T$ is even, then $\operatorname{discr}_2 S$ is also even and has a wrong determinant

$$\pm 3 |\operatorname{discr} S| \bmod (\mathbb{Q}_2^\times)^2$$

instead of $\pm|\operatorname{discr} S| \bmod (\mathbb{Q}_2^\times)^2$. However, since $\ell(\operatorname{discr}_2 S) = 24 - \rho \geqslant 3$, we may pass to an appropriate iterated index 2 extension and either make $\operatorname{discr} S$ odd or reduce its length. □

The construction above is invertible: starting from a pair $S \ni \hbar$, where S is a positive definite even lattice and $\hbar^2 = 12$, and assuming that $\hbar \in 4S^\vee$, one can construct a unique 3-polarized lattice $M \ni h$ such that $S = S(M)$. However, the converse of Proposition 3.3 does not hold, and we state it as an extra restriction.

Proposition 3.4. *Let S be a positive definite even lattice, and let $\hbar \in S \cap 4S^\vee$ be a vector of square 12. Then, the 3-polarized lattice $M \ni h$ obtained from $S \ni \hbar$ by the inverse construction admits a primitive embedding to $\mathbf{L}$ with even orthogonal complement if and only if:*

(1) $\operatorname{rk} S \leqslant 21$; *we denote* $\delta := 23 - \operatorname{rk} S \geqslant 2$, *and*

the discriminant $\mathcal{S} := \operatorname{discr} S$ has the following properties at each prime p:

(2) *if $p > 2$, then either $\ell(\mathcal{S}_p) < \delta$ or $\ell(\mathcal{S}_p) = \delta$ and $\det \mathcal{S}_p = |\mathcal{S}| \bmod (\mathbb{Q}_p^\times)^2$;*
(3) *if $p = 2$, then $\ell(\mathcal{S}_2) \leqslant \delta + 1$ and, in the case of equality $\ell(\mathcal{S}_2) = \delta + 1$, either $\mathcal{S}$ is odd or $\det \mathcal{S}_2 = \pm 3|\mathcal{S}| \bmod (\mathbb{Q}_2^\times)^2$.*

Proof. We use [23, Theorem 1.12.2]. Under the assumption that $\hbar \in 4S^\vee$, there is a splitting $\mathcal{S} = \left\langle \frac{1}{4}\hbar \right\rangle \oplus \mathcal{T}$, and we merely restate the restrictions on $\mathcal{T} \cong \operatorname{discr} M$ in terms of $\mathcal{S}$. □

3.2. **Admissible and geometric sets.** In view of Propositions 3.1 and 3.3, we can replace the lattice M_X of a smooth cubic $X \subset \mathbb{P}^5$ by the corresponding lattice $S(M_X)$ and construct the latter directly in an appropriate Niemeier lattice, as the span of its set of planes.

Thus, we fix a Niemeier lattice N and a vector $\hbar \in N$, $\hbar^2 = 12$, and consider the set of *planes*

$$\mathfrak{F}(\hbar) = \left\{ l \in N \;\middle|\; l^2 = 4,\ l \cdot \hbar = 4 \right\}.$$

For any subset $\mathfrak{L} \subset \mathfrak{F}(\hbar)$, we define its *span*

$$\operatorname{span}_2 \mathfrak{L} = (\mathbb{Z}_2 \mathfrak{L} + \mathbb{Z}_2 \hbar) \cap N \subset N$$

(where the intersection is in $N \otimes \mathbb{Z}_2$). The *rank* of $\mathfrak{L}$ is $\operatorname{rk} \mathfrak{L} := \operatorname{rk} \operatorname{span}_2 \mathfrak{L}$, and we say that $\mathfrak{L}$ is *generated* by a subset $\mathfrak{L}' \subset \mathfrak{L}$ if $\mathfrak{L} = \mathfrak{F}(\hbar) \cap \operatorname{span}_2 \mathfrak{L}'$.

By definition, the torsion of $N/\operatorname{span}_2 \mathfrak{L}$ is a 2-group and $\hbar \in 4(\operatorname{span}_2 \mathfrak{L})^\vee$. A finite index extension $S \supset \operatorname{span}_2 \mathfrak{L}$ in N is called *mild* if $\hbar \in 4S^\vee$ and S is root free, *cf.* Remark 3.2.

A subset $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ is called *complete* if $\mathfrak{L} = \mathfrak{F}(\hbar) \cap \operatorname{span}_2 \mathfrak{L}$; it is called *saturated* if the identity $\mathfrak{L} = \mathfrak{F}(\hbar) \cap S$ holds for any mild extension S of $\operatorname{span}_2 \mathfrak{L}$.

Definition 3.5. A subset $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ is called *admissible* if $\operatorname{span}_2 \mathfrak{L}$ is root free, *cf.* Remark 3.2. A complete admissible subset $\mathfrak{L}$ is *pseudo-geometric* if $S = \operatorname{span}_2 \mathfrak{L}$ satisfies conditions (1) and (2) in Proposition 3.4; it is called *geometric* if $\operatorname{span}_2 \mathfrak{L}$ admits a mild extension S satisfying all hypotheses of Proposition 3.4 and such that $\mathfrak{L} = \mathfrak{F}(\hbar) \cap S$.

Since the lattice N is positive definite, we have $-1 \leqslant l_1 \cdot l_2 \leqslant 3$ for any two distinct planes l_1 and l_2. If $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ is admissible, then

$$l_1 \cdot l_2 \in \{0, 1, 2\} \text{ for any distinct planes } l_1, l_2 \in \mathfrak{L}. \tag{3.6}$$

(Indeed, if $l_1 \cdot l_2 = 3$ or -1, then, respectively, $e := l_1 - l_2$ or $\hbar - l_1 - l_2$ is a root.) Thus, we can regard $\mathfrak{L}$ as a graph, connecting two vertices $l_1 \neq l_2$ by an edge of multiplicity $l_1 \cdot l_2 - 1$ (*cf.* the description of the graph of planes in §2.4).

Two 12-polarized Niemeier lattices $N_i \ni \hbar_i$, $i = 1, 2$, are called *equivalent* if the corresponding polarized lattices $\operatorname{span}_2 \mathfrak{F}(\hbar_i) \ni \hbar_i$ are isomorphic and both torsions $\operatorname{Tors}(N_i/\mathfrak{F}(\hbar_i))$ are 2-groups. Clearly, equivalent polarized lattices share the same collections of admissible and pseudo-geometric sets of planes. *A priori*, this is not true for geometric sets.

3.3. **Orbits, counts, and bounds.** Let $N \ni \hbar$ be a 12-polarized Niemeier lattice as in §3.2, and let $O(N) \supset R(N)$ be the full orthogonal group of N and its subgroup generated by reflections, respectively. We denote by $O_\hbar(N) \supset R_\hbar(N)$ the stabilizers of $\hbar$ in these two groups. The stabilizers act on $\mathfrak{F}(\hbar)$ and, hence, $\mathfrak{F}(\hbar)$ splits into $O_\hbar(N)$-*orbits* $\bar{\mathfrak{o}}_n$, each orbit splitting into $R_\hbar(N)$-orbits $\mathfrak{o} \subset \bar{\mathfrak{o}}_n$, called *combinatorial orbits*. The number of combinatorial orbits in an orbit $\bar{\mathfrak{o}}_n$ is denoted by $m(\bar{\mathfrak{o}}_n)$, and the set of all combinatorial orbits is denoted by $\mathfrak{O} := \mathfrak{O}(\hbar)$. This set inherits a natural action of the group

$$\operatorname{stab} \hbar := O_\hbar(N)/R_\hbar(N),$$

which preserves each orbit $\bar{\mathfrak{o}}_n$. (By an obvious abuse of notation, occasionally $\bar{\mathfrak{o}}_n$ is treated as a subset of $\mathfrak{O}$, whereas subsets of $\mathfrak{O}$ are treated as sets of planes.)

For a subset $\mathfrak{C} \subset \mathfrak{O}$, let

$$\mathfrak{B}(\mathfrak{C}) := \bigl\{\mathfrak{L} \cap \mathfrak{C} \bigm| \mathfrak{L} \subset \mathfrak{F}(\hbar) \text{ is pseudo-geometric}\bigr\}, \quad \mathfrak{b}(\mathfrak{C}) := \bigl\{|\mathfrak{L}| \bigm| \mathfrak{L} \in \mathfrak{B}(\mathfrak{C})\bigr\}.$$

Since the pseudo-geometric property is obviously inherited by complete subsets, in the computation we can confine ourselves to the pseudo-geometric sets $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ generated by $\mathfrak{L} \cap \mathfrak{C}$.

Following [9], define the *count* $c(\mathfrak{o})$ and *bound* $b(\mathfrak{o})$ of a single combinatorial orbit $\mathfrak{o} \in \mathfrak{O}$ *via*

$$c(\mathfrak{o}) := |\mathfrak{o}|, \qquad b(\mathfrak{o}) := \max \mathfrak{b}(\mathfrak{o}).$$

Usually, the bound $b(\mathfrak{o})$ is replaced with its rough estimate computed as explained in §4.1 below. Clearly, c and b are constant within each orbit $\bar{\mathfrak{o}}_n$. We extend these notions to subsets $\mathfrak{C} \subset \mathfrak{O}$ *by additivity*:

$$c(\mathfrak{C}) := \sum_{\mathfrak{o} \in \mathfrak{C}} c(\mathfrak{o}), \qquad b(\mathfrak{C}) := \sum_{\mathfrak{o} \in \mathfrak{C}} b(\mathfrak{o}).$$

Thus, we have a naïve *a priori* bound

$$|\mathfrak{L}| \leqslant b(\mathfrak{O}) = \sum m(\bar{\mathfrak{o}}_n) b(\mathfrak{o}), \quad \mathfrak{o} \subset \bar{\mathfrak{o}}_n. \tag{3.7}$$

Clearly, the true count $|\mathfrak{L} \cap \mathfrak{C}|$ is genuinely additive, whereas the true sharp bound $\max \mathfrak{b}(\mathfrak{C}) \leqslant b(\mathfrak{C})$ is only subadditive; thus, our proof of Theorem 1.1 will essentially consist in reducing (3.7) down to a certain preset goal γ. To this end, we will consider the set

$$\mathcal{B} = \mathcal{B}(\mathfrak{F}) := \bigl\{\mathfrak{L} \subset \mathfrak{F} \bigm| \mathfrak{L} \text{ is geometric}\bigr\}/O_\hbar(N)$$

and, for a collection of orbits $\mathfrak{C} = \bar{\mathfrak{o}}_1 \cup \ldots$ and integer $d \in \mathbb{N}$, let

$$\mathcal{B}_d(\mathfrak{C}) := \bigl\{[\mathfrak{L}] \in \mathcal{B} \bigm| \mathfrak{L} \text{ is generated by } \mathfrak{L} \cap \mathfrak{C} \text{ and } |\mathfrak{L} \cap \mathfrak{C}| \geqslant b(\mathfrak{C}) - d\bigr\}.$$

We will also consider the oversets $\tilde{\mathcal{B}}_d(\mathfrak{C}) \supset \mathcal{B}_d(\mathfrak{C})$ consisting of pseudo-geometric (rather than geometric) sets. The computation of these sets is discussed in §4.5.

3.4. **Idea of the proof.** Fix a 12-polarized Niemeier lattice $N \ni \hbar$ and a goal

$$|\mathfrak{L}| \geqslant \gamma \quad \text{(typically, } \gamma = 351) \tag{3.8}$$

We need to list all geometric sets $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ satisfying this inequality. Clearly, such sets may exist only if $b(\mathfrak{O}) \geqslant \gamma$, where $b(\mathfrak{O})$ is the naïve bound given by (3.7); otherwise, the pair $N \ni \hbar$ does not need to be considered.

In the few remaining cases, we use brute force and, for each combinatorial orbit $\mathfrak{o}$, compute the $R_\hbar(N)$-orbits on the set $\mathfrak{B}(\mathfrak{o})$. (Obviously, it suffices to consider one representative of each orbit $\bar{\mathfrak{o}}_n$; the rest is obtained by translation.) In particular, we obtain sharp bounds $b(\mathfrak{o})$ and sets of values

$$\mathfrak{b}(\mathfrak{o}) = \{b(\mathfrak{o}) > b'(\mathfrak{o}) > \ldots\}. \tag{3.9}$$

This may yield a better bound $b(\mathfrak{O})$ given by (3.7); this improved bound is used in the subsequent computation. If still $b(\mathfrak{O}) \geqslant \gamma$, we choose a collection of pairwise disjoint unions of orbits $\mathfrak{C}_1, \ldots, \mathfrak{C}_m \subset \mathfrak{O}$ and integers $d_1, \ldots, d_m \geqslant 0$ such that

$$d_1 + \ldots + d_m + m > b(\mathfrak{O}) - \gamma.$$

Then, clearly, any geometric set $\mathfrak{L}$ satisfying (3.8) is in the union

$$\mathcal{E} := \mathcal{B}_{d_1}(\mathfrak{C}_1) \cup \cdots \cup \mathcal{B}_{d_m}(\mathfrak{C}_m),$$

and the same assertion holds for pseudo-geometric sets, with $\mathcal{B}_d$ replaced with $\tilde{\mathcal{B}}_d$. We try to fix the choices so that the union $\mathcal{E}$ consists of relatively few sufficiently large sets; then, these exceptional sets are analyzed one by one using one of the arguments described below.

3.4.1. *Maximal sets* (see [9]). If a set $\mathfrak{L} \in \mathcal{E}$ is saturated and the rank $\operatorname{rk} \mathfrak{L} = 21$ is maximal, *cf.* Proposition 3.4(1), then $\mathfrak{L}$ has no proper geometric extension; hence, this set can be either disregarded, if $|\mathfrak{L}| < \gamma$, or listed as an exception in the respective statement.

3.4.2. *Extension by a maximal orbit* (see [9]). In many cases, a set $\mathfrak{L} \in \mathcal{B}_d(\mathfrak{C})$ has the property that

$$\sum \bigl(b(\mathfrak{o}) - b'(\mathfrak{o})\bigr) \geqslant b(\mathfrak{O}) - \gamma, \qquad \mathfrak{o} \in \mathfrak{O}_\delta := \bigl\{\mathfrak{o} \in \mathfrak{O} \bigm| |\mathfrak{L} \cap \mathfrak{o}| < b(\mathfrak{o})\bigr\},$$

see (3.9). This implies that any (pseudo-)geometric extension $\mathfrak{L}' \supset \mathfrak{L}$ satisfying (3.8) must have maximal intersection, $|\mathfrak{L}' \cap \mathfrak{o}| = b(\mathfrak{o})$, with at least one orbit $\mathfrak{o} \in \mathfrak{O}_\delta$. Trying these orbits one by one, we obtain larger sets, which are usually maximal, see §3.4.1. Note that here we always assume $\mathfrak{L} \cap \mathfrak{C}$ fixed, *i.e.*, we accept only those extensions $\mathfrak{L}' \supset \mathfrak{L}$ that have the property

$$\mathfrak{L}' \supset \mathfrak{L} \text{ is pseudo-geometric and } \mathfrak{L}' \cap \mathfrak{C} = \mathfrak{L} \cap \mathfrak{C}; \tag{3.10}$$

indeed, otherwise, we would have started with a larger set $\mathfrak{L}'' \in \mathcal{B}_d(\mathfrak{C})$, *viz.* the one generated by $\mathfrak{L}' \cap \mathfrak{C}$. Thus, for the computation, we merely extend the restricted pattern

$$\pi_\mathfrak{L} \colon \mathfrak{C} \to \mathbb{N}, \quad \mathfrak{o} \mapsto |\mathfrak{L} \cap \mathfrak{o}|$$

(see §4.5.1 below), by a single extra value $\mathfrak{o}' \mapsto b(\mathfrak{o}')$ for some orbit $\mathfrak{o}' \in \mathfrak{O}_\delta \smallsetminus \mathfrak{C}$ and perform one extra step of the algorithm. We make use of the symmetry of $\mathfrak{L}$, trying for $\mathfrak{o}'$ a single representative of each orbit of the action on $\mathfrak{O}_\delta \smallsetminus \mathfrak{C}$ of the $(\operatorname{stab} \hbar)$-stabilizer of $\pi_\mathfrak{L}$.

3.4.3. *Maximal orbit count.* For smaller sets $\mathfrak{L} \in \mathcal{B}_d(\mathfrak{C})$, we choose an appropriate *test set* $\mathfrak{T} \subset \mathfrak{O} \smallsetminus \mathfrak{C}$ (typically, also a union of orbits) and use the same techniques as in §3.4.2 to compute the set

$$\mathfrak{T}_\mu := \bigl\{\mathfrak{o} \in \mathfrak{T} \bigm| \mathfrak{L}' \cap \mathfrak{o} = b(\mathfrak{o}) \text{ for some extension } \mathfrak{L}' \supset \mathfrak{L} \text{ satisfying (3.10)}\bigr\}.$$

Then, if

$$\bigl(b(\mathfrak{C}) - |\mathfrak{L} \cap \mathfrak{C}|\bigr) + \sum \bigl(b(\mathfrak{o}) - b'(\mathfrak{o})\bigr) \geqslant b(\mathfrak{O}) - \gamma, \qquad \mathfrak{o} \in \mathfrak{T} \smallsetminus \mathfrak{T}_\mu,$$

see (3.9), we conclude that $\mathfrak{L}$ has no extensions satisfying (3.8) and (3.10).

4. Algorithms

In the first four sections, we describe a rough estimate on the bounds $b(\mathfrak{o})$ in the Niemeier lattices with many roots (and, hence, large combinatorial orbits). Then, in §4.5, we explain the algorithms used to compute the sets $\mathcal{B}_d(\mathfrak{C})$ in those few cases where the rough estimates do not suffice.

4.1. **Bounds *via* blocks** (see [9]). Let $N := N(D) = N(\bigoplus_k D_k)$ be a Niemeier lattice rationally generated by roots, where the *blocks* D_k, $k \in \Omega$, are the irreducible components of the *maximal* root system $D \subset N$ and Ω is the index set. Thus, we have $N \subset D^\vee = \bigoplus_k D_k^\vee$; the vectors in $\operatorname{discr} D = D^\vee/D = \bigoplus_k \operatorname{discr} D_k$ that are declared “integral” are described in [5]. (We also use the convention of [5] for the numbering of the discriminant classes of irreducible root systems.)

Let $|_k\colon N \to D_k^\vee$ be the orthogonal projection. For a vector $v \in N$, we often abbreviate $v_k := v|_k$, so that $v = \sum_k v_k$, $v_k \in D_k^\vee$. Define the *support*

$$\operatorname{supp} v := \bigl\{k \in \Omega \bigm| v_k \neq 0\bigr\}.$$

The group $R_\hbar(N)$ preserves each block D_k and, hence, we can also speak about the *support* $\operatorname{supp} \mathfrak{o} \subset \Omega$ of a combinatorial orbit $\mathfrak{o}$. (It is worth mentioning that, for each $k \in \Omega$, the squares $l^2, l_k^2 \in \mathbb{Q}$, products $l \cdot \hbar, l_k \cdot \hbar_k \in \mathbb{Q}$, and discriminant classes $l \bmod D \in \operatorname{discr} D$ and $l_k \bmod D_k \in \operatorname{discr} D_k$, $l \in \mathfrak{o}$, are also constant within each combinatorial orbit $\mathfrak{o}$.)

Fix a combinatorial orbit $\mathfrak{o}$ and define the *count* and *bound* of a block D_k *via*

$$c(D_k) := |\mathfrak{o}|_k|, \qquad b(D_k) := \max|\mathfrak{R}|,$$

where $\mathfrak{R} \subset \mathfrak{o}|_k$ is a subset satisfying the condition

$$\text{for } l', l'' \in \mathfrak{R}, \text{ one has } l'^2 - l' \cdot l'' = 0 \text{ (iff } l' = l''\text{), 2, 3, or 4.} \tag{4.1}$$

In other words, we bound the cardinality of subsets $\mathfrak{L} \subset \mathfrak{o}$ satisfying (3.6) and such that all planes $l \in \mathfrak{L}$ have the same fixed restriction to all other blocks $D_s \neq D_k$. Then, we have (*cf.* [9])

$$c(\mathfrak{o}) = \prod_k c(D_k), \qquad b(\mathfrak{o}) \leqslant c(\mathfrak{o}) \min_k \frac{b(D_k)}{c(D_k)}. \tag{4.2}$$

For smaller blocks ($\mathbf{A}_{\leqslant 7}$, $\mathbf{D}_{\leqslant 6}$, and most $\mathbf{E}$-type blocks) the individual counts $c(D_k)$ and bounds $b(D_k)$ are computed by brute force, and the resulting estimates (4.2) suite most our needs. For larger blocks, we use even more rough estimates, based on the standard representation of the $\mathbf{A}$- and $\mathbf{D}$-type root systems as sublattices of the odd unimodular lattice

$$\mathbf{H}_n := \bigoplus \mathbb{Z} e_i, \quad e_i^2 = 1, \quad i \in \mathcal{I} := \{1, \ldots, n\}.$$

(When working with this lattice, we let $\bar{1}_o := \sum_{i\in o} e_i$ for a subset $o \subset \mathcal{I}$.) Then, given a vector $\hbar_k = \sum_i \alpha_i e_i \in \mathbf{H}_n \otimes \mathbb{Q}$, we subdivide the block $D_k^\vee \subset \mathbf{H}_n \otimes \mathbb{Q}$ into "subblocks"

$$D_k(\alpha) := \bigl\{\textstyle\sum_i \beta_i e_i \in \mathbf{H}_n \otimes \mathbb{Q} \bigm| i \in \operatorname{supp}(\alpha)\bigr\}, \quad \operatorname{supp}(\alpha) := \bigl\{i \in \mathcal{I} \bigm| \alpha_i = \alpha\bigr\},$$

on which $\hbar_k$ is constant. We obtain combinatorial counts and bounds, in the sense of (4.1), for each subblock and use an obvious analogue of (4.2) to estimate $b(D_k)$. The technical details are outlined in the next few sections. We use, without further references, the following simple observation.

Lemma 4.3. *If $l \in \mathfrak{F}(\hbar)$ and $l \bmod D \neq 0 \in \operatorname{discr} D$, then each l_k, $k \in \Omega$, is a shortest vector in its discriminant class $l_k \bmod D_k \in \operatorname{discr} D_k$.*

Proof. Indeed, otherwise the *nontrivial* discriminant class $l \bmod D$ would contain a shorter vector v, necessarily of square $l^2 - 2 = 2$, contradicting to the assumption that all roots in N are in D. □

4.2. **Root systems $\mathbf{A}_n$.** A block D_k of type $\mathbf{A}_n$ is $\bar{1}_{\mathcal{I}}^\perp \subset \mathbf{H}_{n+1}$:

$$\mathbf{A}_n = \bigl\{\textstyle\sum_i \alpha_i e_i \in \mathbf{H}_{n+1} \bigm| \sum_i \alpha_i = 0\bigr\}.$$

One has $\operatorname{discr} \mathbf{A}_n = \mathbb{Z}/(n+1)$, with a generator of square $n/(n+1) \bmod 2\mathbb{Z}$, and the shortest representatives of the discriminant classes are vectors of the form

$$\bar{e}_o := \frac{1}{n+1}\bigl(|\bar{o}|\bar{1}_o - |o|\bar{1}_{\bar{o}}\bigr), \qquad \bar{e}_o^2 = \frac{|o||\bar{o}|}{n+1},$$

where $o \subset \mathcal{I}$ and $\bar{o}$ is the complement. We have $\bar{e}_{\bar{o}} = -\bar{e}_o$ and

$$\bar{e}_r \cdot \bar{e}_s = |r \cap s| - \frac{|r||s|}{n+1}.$$

In particular, if $|r| = |s|$, or, equivalently, e_r and e_s are in the same discriminant class, then

$$\bar{e}_r^2 - \bar{e}_r \cdot \bar{e}_s = \frac{1}{2}|r \mathbin{\triangle} s|, \tag{4.4}$$

where $\triangle$ is the symmetric difference. Hence, in the case when l_k is a shortest vector in its (nonzero) discriminant class, the bound $b(D_k(\alpha))$ can be estimated by the following lemma, applied to $S = \operatorname{supp}(\alpha)$.

Lemma 4.5. *Consider a finite set S, $|S| = n$, and let $\mathfrak{S}$ be a collection of subsets $s \subset S$ with the following properties:*

1. *all subsets $s \in \mathfrak{S}$ have the same fixed cardinality m;*
2. *if $r, s \in \mathfrak{S}$, then $|r \mathbin{\triangle} s| \in \{0, 4, 6, 8\}$.*

Then, for small (n, m), the maximum $\mathcal{A}_{m,n} := \max|\mathfrak{S}|$ is as follows:

$(n,m):$	$(n,1)$	$(n,2)$	$(6,3)$	$(7,3)$	$(8,3)$	$(9,3)$	$(8,4)$
$\mathcal{A}_{m,n}:$	1	$\lfloor n/2 \rfloor$	4	7	8	12	14

More generally,

$$\mathcal{A}_{3,n} \leqslant \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor; \qquad \mathcal{A}_{m,n} \leqslant \left\lfloor \frac{1}{m} \binom{n}{m-1} \right\rfloor \quad \text{for } m \geqslant 1.$$

Note that, if a collection $\mathfrak{S}$ is as in the lemma, then so is the collection $\{\bar{s} \mid s \in \mathfrak{S}\}$. Hence, $\mathcal{A}_{m,n} = \mathcal{A}_{n-m,n}$ and we can always assume that $2m \leqslant n$.

Proof of Lemma 4.5. The first two values are obvious; the others are obtained by listing all admissible collections.

The general estimate for $m=3$ follows from the observation that any two subsets in $\mathfrak{S}$ have at most one common point and, hence, each point of S is contained in at most $\lfloor\frac{1}{2}(n-1)\rfloor$ such subsets.

For the last bound, we merely observe that each $(m-1)$-element set $r\subset S$ is contained in at most one set $s\in\mathfrak{S}$. $\square$

There remains to consider a subblock $D_k(\alpha)$ of a block D_k containing vectors of the form $l_k=\bar{1}_r-\bar{1}_s$, where $r,s\subset\mathcal{I}$, $r\cap s=\varnothing$, and $|r|=|s|=1$ or 2. In the latter case, one must have $l_k\cdot\hbar_k=4$, and it follows that $|(r\cup s)\cap\operatorname{supp}(\alpha)|\leqslant 3$ for each $\alpha\in\mathbb{Q}$. Letting $r_\alpha:=r\cap\operatorname{supp}(\alpha)$, $s_\alpha:=s\cap\operatorname{supp}(\alpha)$, the bounds are as follows:

(1) if $|r_\alpha|+|s_\alpha|=1$, then, obviously, $b(D_k(\alpha))=1$;
(2) if $(|r_\alpha|,|s_\alpha|)=(2,0)$ or $(0,2)$, then distinct sets r_α (respectively, s_α) must be pairwise disjoint and, hence, $b(D_k(\alpha))=\lfloor\frac{1}{2}|\operatorname{supp}(\alpha)|\rfloor$;
(3) if $|r_\alpha|=|s_\alpha|=1$, then distinct sets r_α must also be pairwise disjoint and, hence, $b(D_k(\alpha))=|\operatorname{supp}(\alpha)|$;
(4) if $(|r_\alpha|,|s_\alpha|)=(2,1)$ or $(1,2)$, then $b(D_k(\alpha))=|\operatorname{supp}(\alpha)|=3$.

4.3. **Root systems $\mathbf{D}_n$.** A block D_k of type $\mathbf{D}_n$ can be defined as the maximal even sublattice in $\mathbf{H}_n$:

$$\mathbf{D}_n=\bigl\{\textstyle\sum_i\alpha_ie_i\in\mathbf{H}_n \bigm| \sum_i\alpha_i=0 \bmod 2\bigr\}.$$

If $n\geqslant 5$, the group $O(\mathbf{D}_n)$ is an index 2 extension of $R(\mathbf{D}_n)$: it is generated by the reflection against the hyperplane orthogonal to any of e_i. Hence, up to $O(\mathbf{D}_n)$, we can assume that, in the expression $\hbar_k=\sum_i\alpha_ie_i$, all coefficients $\alpha_i\geqslant 0$. We always make this assumption (and adjust the results afterwards) when describing the orbits and computing counts and bounds, as otherwise the description of combinatorial orbits is not quite combinatorial.

One has $\operatorname{discr}\mathbf{D}_n=\mathbb{Z}/2\oplus\mathbb{Z}/2$ (if n is even) or $\mathbb{Z}/4$ (if n is odd); the shortest vectors are

$$e_i,\ i\in\mathcal{I},\quad\text{and}\quad \bar{e}_o:=\frac{1}{2}(\bar{1}_o-\bar{1}_{\bar{o}}),\ o\subset\mathcal{I},\quad \bar{e}_o^2=\frac{n}{4}$$

(the class $\bar{e}_o \bmod \mathbf{D}_n$ depends on the parity of $|o|$) and we have a literal analogue of (4.4) for any pair $r,s\subset\mathcal{I}$. Thus, if $D_k\ni\bar{e}_o$, the bounds $b(D_k(\alpha))$ are estimated by Lemma 4.5 (if $\alpha\neq 0$) or Lemma 4.6 below (if $\alpha=0$), applied to $S=\operatorname{supp}(\alpha)$.

Lemma 4.6. *The maximal cardinality of a collection $\mathfrak{S}$ satisfying condition (2) of Lemma 4.5 is bounded* via

$$|\mathfrak{S}|\leqslant\max_{m\geqslant 0}\bigl(\mathcal{A}_{m,n}+\mathcal{A}_{m+2,n}+\mathcal{A}_{m+4,n}+\mathcal{A}_{m+6,n}+\mathcal{A}_{m+8,n}\bigr),$$

where $\mathcal{A}_{m,n}$ is as in Lemma 4.5 and we let $\mathcal{A}_{m,n}=0$ unless $0\leqslant m\leqslant n$.

Proof. It suffices to observe that all sets $s\in\mathfrak{S}$ have cardinality of the same parity and that $\bigl||s|-|r|\bigr|\leqslant 8$ for any pair $r,s\in\mathfrak{S}$. $\square$

The few remaining cases are listed below.

(1) If $D_k(\alpha)\ni\pm 2e_i$, $i\in\operatorname{supp}(\alpha)$, then $b(D_k(\alpha))=|\operatorname{supp}(\alpha)|$.

Assume that $l_k=\sum(\pm e_i)$, $i\in o\subset\mathcal{S}$, $|o|\leqslant 4$. If $\alpha=0$, then

(2) $|o\cap\operatorname{supp}(\alpha)|=0,\ 1,\ 2$ and $b(D_k(\alpha))\leqslant 1,\ 2,\ 4\lfloor\frac{1}{2}|\operatorname{supp}(\alpha)|\rfloor$, respectively,

similar to §4.2. (Here, the last number is a bound on the size of a union of (affine) Dynkin diagrams admitting an isometry to $\mathbf{D}_{|\operatorname{supp}(\alpha)|}$, *cf.* §4.4.1 below.) If $\alpha \neq 0$, the numbers of signs $\pm$ within $\operatorname{supp}(\alpha)$ are also fixed, and the options are as follows:

(3) $m := |o \cap \operatorname{supp}(\alpha)| \leqslant 3$ and all signs are the same: by an analogue of (4.4), a bound on $b(D_k(\alpha))$ is given by Lemma 4.5 applied to $S = \operatorname{supp}(\alpha)$;
(4) $|o \cap \operatorname{supp}(\alpha)| = 2$ and the signs differ: $b(D_k(\alpha)) = |\operatorname{supp}(\alpha)|$ as in §4.2(3);
(5) $|o \cap \operatorname{supp}(\alpha)| = 3$ and the signs differ: $b(D_k(\alpha)) = |\operatorname{supp}(\alpha)| = 3$.

4.4. **Other root lattices.** We use the description of §4.2 for the blocks $\mathbf{A}_n$, $n \geqslant 8$, and that of §4.3, for $\mathbf{D}_n$, $n \geqslant 7$. Below we outline a few more tricks that simplify the computations for other blocks, mainly those of type $\mathbf{E}_8$.

4.4.1. *Integral roots.* Assume that $\hbar_k = 0$ and l_k is an integral root, *i.e.*, $l_k^2 = 2$ and $l_k \in D_k$. By (4.1), the pairwise products of roots take values in $\{0, -1, -2\}$, *i.e.*, an admissible subset in D_k is a union Γ of (affine) Dynkin diagrams (including those of type $\tilde{\mathbf{A}}_1$) that is mapped isometrically to D_k. If $D_k \cong \mathbf{E}_8$, $\mathbf{E}_7$, or $\mathbf{D}_{\text{even}}$, *cf.* §4.3(2), then D_k is rationally generated by pairwise orthogonal roots and, hence, the maximal union Γ as above is $n\tilde{\mathbf{A}}_1$, $n = \operatorname{rk} D_k$. Thus, we have $b(D_k) = 2n$ in this case.

If $D_k \cong \mathbf{E}_8$ and $\hbar_k, l_k \in D_k$ are orthogonal integral roots, $\hbar_k^2 = l_k^2 = 2$, $l_k \cdot \hbar_k = 0$, we can apply the above argument to $\hbar_k^\perp \cong \mathbf{E}_7$ to obtain $b(D_k) = 14$. If $l_k \cdot \hbar_k = 1$, the bound is $b(D_k) \leqslant 4$, which is the maximal valency of a vertex in (any) affine Dynkin diagram. If $l_k \cdot \hbar_k = 2$, then $l_k = \hbar_k$ and $c(D_k) = b(D_k) = 1$.

4.4.2. *Planes contained in* $\mathbf{E}_8$. Assume that $D_k \cong \mathbf{E}_8$ and $l_k^2 = 4$, so that necessarily $l = l_k$ and $l_k \cdot \hbar_k = 4$ (and, hence, $\hbar_k^2 \geqslant 4$). Below we consider two cases where a long computation can be simplified.

If $\hbar_k^2 = 12$, then $\hbar = \hbar_k$ and all planes l_k are in the index 2 sublattice

$$\bigl\{x \in D_k \bigm| x \cdot \hbar_k = 0 \bmod 2\bigr\} \cong \mathbf{D}_8.$$

Up to automorphism of $\mathbf{D}_8$, we have $\hbar_k = 2(\bar{1}_o)$, $o = \{1, 2, 3\}$, and $l_k = \bar{1}_r \pm e_i \pm e_j$, where $r \subset o$, $|r| = 2$, and $i, j \in \mathcal{I} \smallsetminus o$. Arguing as in §4.3, we obtain $b(D_k) \leqslant 24$.

If $\hbar_k^2 = 8$ and $\hbar_k \in 2D_k$, then each plane has the form $l_k = \frac{1}{2}\hbar_k + r$, where r is a root in $\hbar_k^\perp \cong \mathbf{E}_7$. Hence, we have $b(D_k) = 14$ as in §4.4.1.

4.5. **The computation of $\mathcal{B}_d(\mathfrak{C})$.** Most sets $\mathcal{B}_d(\mathfrak{C})$ or $\tilde{\mathcal{B}}_d(\mathfrak{C})$ introduced in §3.3 are computed using the so-called *patterns*, as explained below in this section.

4.5.1. *Patterns* (see [9]). The *pattern* of a pseudo-geometric set $\mathfrak{L}$ is the function

$$\pi_{\mathfrak{L}} \colon \mathfrak{O} \to \mathbb{N}, \quad \mathfrak{o} \mapsto |\mathfrak{L} \cap \mathfrak{o}| \in \mathfrak{b}(\mathfrak{o}).$$

To compute a set $\mathcal{B}_d(\mathfrak{C})$, we start with listing, up to the action of the group $\operatorname{stab} \hbar$, the abstract restricted patterns $\pi \colon \mathfrak{C} \to \mathbb{N}$ satisfying the conditions

$$\pi(\mathfrak{o}) \in \mathfrak{b}(\mathfrak{o}), \qquad \sum \pi(\mathfrak{o}) \geqslant b(\mathfrak{C}) - d, \quad \mathfrak{o} \in \mathfrak{C}.$$

Then, using the precomputed collections $\mathfrak{B}(\mathfrak{o})$, we build a (pseudo-)geometric set $\mathfrak{L}$ orbit by orbit, as the completion of a union $\bigcup_{\mathfrak{o} \in \mathfrak{C}} \mathfrak{L}_{\mathfrak{o}}$, $\mathfrak{L}_{\mathfrak{o}} \in \mathfrak{B}(\mathfrak{o})$, $|\mathfrak{L}_{\mathfrak{o}} \cap \mathfrak{o}| = \pi(\mathfrak{o})$. In this computation, we add combinatorial orbits $\mathfrak{o} \in \mathfrak{C}$ one by one, use the group $R_\hbar(N)$ to reduce the overcounting, and, at each step, make sure that the partial set $\mathfrak{L}$ constructed is (pseudo-)geometric and that $\pi_{\mathfrak{L}}|_{\mathfrak{C}} = \pi$. Further details are found in [9]; we reuse the code developed there.

4.5.2. *Clusters* (see [9]). If the number $m(\bar{\mathfrak{o}}_n)$ of combinatorial orbits in an orbit $\bar{\mathfrak{o}}_n = \mathfrak{C}$ is large, listing all abstract patterns in §4.5.1 is difficult. In this case, we try to subdivide $\bar{\mathfrak{o}}_n$ into *clusters* $\mathfrak{c}_k$, not necessarily disjoint, consisting of whole combinatorial orbits and such that $\operatorname{stab}\hbar$ induces an at least 1-transitive action on the set of clusters. (The construction of clusters, usually "natural", is described in the respective proof sections case by case.) Then, we can compute abstract patterns and construct (pseudo-)geometric sets cluster by cluster, assuming the latter ordered by the decreasing of their "complexity" (see [9] for more details). Within each cluster, we still use patterns (compatible with the clusters already considered) and extend the (pseudo-)geometric set orbit by orbit, as in §4.5.1.

4.5.3. *Meta-patterns.* In some cases, the number u of clusters is still large (requiring too many steps in §4.5.2), whereas each cluster is relatively small, so that we can easily compute the set $\mathcal{B}_*(\mathfrak{c}_*)$ of $O(\hbar)$-orbits of geometric sets pseudo-generated by a single cluster. In these cases, we construct a geometric set

$$\mathfrak{L} = \mathfrak{L}_u \supset \mathfrak{L}_{u-1} \supset \ldots \supset \mathfrak{L}_0 = \varnothing$$

cluster by cluster, using the full symmetry group $O_\hbar(N)$ and a *meta-pattern*, *i.e.*, a set of values

$$\tilde{\pi} := \{\mathfrak{T}_1 \geqslant \ldots \geqslant \mathfrak{T}_u\}, \quad \mathfrak{T}_i \in \mathcal{B}_*(\mathfrak{c}_*),$$

ordered lexicographically by the decreasing of the pair $(|\mathfrak{t}_i|, |\mathfrak{T}_i|)$, $\mathfrak{t}_i \in \mathfrak{T}_i$, and such that $\sum |\mathfrak{t}_i| \geqslant b(\mathfrak{C}) - d$. At each step k, we pass from $\mathfrak{L}_{k-1}$ to the sets

$$\mathfrak{L}_k := \mathfrak{F}(\hbar) \cap \operatorname{span}_2(\mathfrak{L}_{k-1} \cup \mathfrak{t}_k),$$

taking for $\mathfrak{t}_k \in \mathfrak{T}_k$ a single representative of each orbit of the $O_\hbar(N)$-stabilizer of $\mathfrak{L}_{k-1}$; then, we select those sets $\mathfrak{L}_k$ that are pseudo-geometric and satisfy the condition $|\mathfrak{L}_k \cap \mathfrak{c}_i| = |\mathfrak{t}_i|$ for all $i = 1, \ldots, k$. This algorithm is similar to §4.5.1 (*cf.* [9]), except that clusters are used instead of combinatorial orbits and the full symmetry group $O_\hbar(N)$ is used instead of $R(N)$: at the beginning, preparing the meta-patterns, we do not associate the values $\mathfrak{T}_k$ with particular clusters $\mathfrak{c}_k$.

4.5.4. *Iterated maximal subsets.* Let $\mathfrak{C}$ be a complete *admissible* set. (Typically, we take for $\mathfrak{C}$ a union of orbits.) Clearly, any subset $\mathfrak{L} \subset \mathfrak{C}$ is also admissible. If $\mathfrak{L}$ is also complete, then either $\operatorname{span}_2 \mathfrak{L} \subset \operatorname{span}_2 \mathfrak{C}$ has positive corank or $\operatorname{span}_2 \mathfrak{C}/\operatorname{span}_2 \mathfrak{L}$ is a 2-group, nontrivial if $\mathfrak{L}$ is proper. It follows that any maximal (with respect to inclusion) proper complete subset $\mathfrak{L} \subset \mathfrak{C}$ is of the form

$$\mathfrak{L} = \mathfrak{C} \cap \operatorname{Ker} v, \qquad v \in \operatorname{Hom}(\operatorname{span}_2 \mathfrak{C}, \mathbb{F}_2) = \operatorname{span}_2 \mathfrak{C}/2\operatorname{span}_2 \mathfrak{C},$$

and these sets can easily be listed. (Usually, we also take into account the action on $\operatorname{span}_2 \mathfrak{C}/2\operatorname{span}_2 \mathfrak{C}$ of the $O_\hbar(N)$-stabilizer of $\mathfrak{C}$. Note that we do not assert that $\operatorname{span}_2 \mathfrak{C}/\operatorname{span}_2 \mathfrak{L} = \mathbb{Z}/2$.) Iterating this procedure (and eliminating repetitions at the intermediate steps), we can list all complete subsets $\mathfrak{L} \subset \mathfrak{C}$ and, in particular, compute the sets $\mathcal{B}_d(\mathfrak{C}) \subset \tilde{\mathcal{B}}_d(\mathfrak{C})$.

5. Lattices with few components

In this section we consider the Niemeier lattices with at most eight irreducible root components and prove the following theorem.

Theorem 5.1. *Let N be a Niemeier lattice other than $N(12\mathbf{A}_2)$, $N(24\mathbf{A}_1)$ or Λ. Then, for any $\hbar \in N$, $\hbar^2 = 12$, and any geometric set $\mathfrak{L} \subset \mathfrak{F}(\hbar)$, one has $|\mathfrak{L}| \leqslant 350$.*

TABLE 1. The lattice $N(6\mathbf{D}_4)$

1:	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$		735	**375**
1:	$[2]_\circ$	$[2]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	15	1	$\underline{1}$
2:	$[1]_3$	$[1]_3$	$[1]_3$	$[1]_3$	$[0]_0$	$[0]_0$	45	16	$\underline{8}$

Proof. For each 12-polarized Niemeier lattice $N \ni \hbar$, we list all $O_\hbar(N)$-orbits $\bar{\mathfrak{o}}_n$ and compute the number $m(\bar{\mathfrak{o}}_n)$ of combinatorial orbits $\mathfrak{o} \subset \bar{\mathfrak{o}}_n$, the count $c(\mathfrak{o})$, and the naïve bound on $|\mathfrak{L} \cap \mathfrak{o}|$ given by (4.2). Sometimes, this bound is improved by a brute force computation; the best bound obtained is denoted by $b(\mathfrak{o})$. The results are listed in several tables below, where the bounds $b(\mathfrak{o})$ confirmed by brute force are underlined.

Convention 5.2. For the components $\hbar_k \in D_k^\vee$ of $\hbar$ we use the notation $[\![\hbar_k^2]\!]_d$, where d is either the discriminant class of $\hbar_k^2$ (in the notation of [5]) or, if $\hbar_k \in D_k$, the symbol

$$0 \text{ (if } \hbar_k = 0), \quad \circ \text{ (if } \hbar_k^2 = 2), \quad \bullet \text{ (if } \hbar_k^2 = 4), \quad * \text{ (if } \hbar_k^2 = 6).$$

For the components l_k of a plane l, we use the notation $[l_k \cdot \hbar_k]_d$, where d has the same meaning as for $\hbar$. (By Lemma 4.3, l_k^2 is uniquely determined by the subscript.) *In the cases considered*, these data determine $(\hbar_k, l_k)$ up to $R(D_k)$. (The occasional superscripts are artifacts left over from the complete set of tables.)

Also shown in the tables is the naïve *a priori* estimate $b(\mathfrak{O})$ given by (3.7). For the vast majority of lattices $N \ni \hbar$ we have $b(\mathfrak{O}) \leqslant \gamma$, and these pairs are omitted. The few cases where $b(\mathfrak{O}) \geqslant \gamma$ are shown in bold, and we treat them separately. In the "trivial" cases equivalent to those already considered in another lattice N, we usually also omit the list of orbits $\bar{\mathfrak{o}}_n$.

5.1. **The lattice $N(6\mathbf{D}_4)$.** There are 36 pairs $N \ni \hbar$, and $b(\mathfrak{F}) \leqslant 375$ (see Table 1).

5.1.1. *Configuration* 1. We subdivide the orbit $\bar{\mathfrak{o}}_2$ into six clusters (not disjoint)

$$\mathfrak{c}_k := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_2 \mid k \notin \operatorname{supp} \mathfrak{o}\}, \quad k \in \Omega,$$

and use these clusters (see §4.5.2) to compute $\tilde{\mathcal{B}}_{24}(\bar{\mathfrak{o}}_2) = \varnothing$.

5.2. **The lattice $N(8\mathbf{A}_3)$.** There are 110 pairs $N \ni \hbar$, and $b(\mathfrak{F}) \leqslant 407$ (see Table 2).

5.2.1. *Configuration* 1. Denote $\mathcal{K} := \{k \in \Omega \mid \hbar_k^2 = 1\}$, and subdivide $\bar{\mathfrak{o}}_6$ into seven pairwise disjoint clusters

$$\mathfrak{c}_k := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_6 \mid l_k^2 = 1 \text{ for } l \in \mathfrak{o}\}, \quad k \in \mathcal{K}.$$

Using these clusters (see §4.5.2), we show that $\tilde{\mathcal{B}}_{56}(\bar{\mathfrak{o}}_6) = \varnothing$.

5.2.2. *Configuration* 2. Denote $\mathcal{K} := \{k \in \Omega \mid \hbar_k \text{ is a root}\}$, and subdivide $\bar{\mathfrak{o}}_6$ into four pairwise disjoint clusters

$$\mathfrak{c}_k := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_6 \mid k \notin \operatorname{supp} \mathfrak{o}\}, \quad k \in \mathcal{K}.$$

Using these clusters (see §4.5.2), we show that $\tilde{\mathcal{B}}_{24}(\bar{\mathfrak{o}}_6) = \varnothing$. ∎

TABLE 2. The lattice $N(8\mathbf{A}_3)$

1:	$[\![5]\!]_2^+$	$[\![1]\!]_2$	$[\![1]\!]_2$	$[\![1]\!]_2$	$[\![1]\!]_2$	$[\![1]\!]_2$	$[\![1]\!]_2$	$[\![1]\!]_2$		799	**407**
1:	$[4]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	1	1	$\underline{1}$
2:	$[3]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	7	4	$\underline{2}$
3:	$[2]_2$	$[1]_2$	$[1]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_0$	21	4	$\underline{2}$
4:	$[2]_2$	$[\frac{1}{2}]_3$	$[\frac{1}{2}]_3$	$[0]_0$	$[\frac{1}{2}]_3$	$[0]_0$	$[0]_0$	$[\frac{1}{2}]_1$	14	16	$\underline{8}$
5:	$[1]_2$	$[1]_2$	$[1]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_2$	$[0]_0$	7	1	$\underline{1}$
6:	$[\frac{3}{2}]_3$	$[1]_2$	$[\frac{1}{2}]_3$	$[\frac{1}{2}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{1}{2}]_3$	56	8	$\underline{4}$
7:	$[0]_0$	$[1]_2$	$[1]_2$	$[1]_2$	$[0]_0$	$[1]_2$	$[0]_0$	$[0]_0$	7	1	$\underline{1}$
2:	$[\![1]\!]_2$	$[\![1]\!]_2$	$[\![1]\!]_2$	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![1]\!]_2$	$[\![2]\!]_\circ$		743	**375**
1:	$[1]_2$	$[1]_2$	$[1]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_2$	$[0]_0$	1	1	$\underline{1}$
2:	$[1]_2$	$[1]_2$	$[0]_0$	$[1]_2$	$[1]_2$	$[0]_0$	$[0]_0$	$[0]_0$	12	4	$\underline{2}$
3:	$[1]_2$	$[\frac{1}{2}]_3$	$[\frac{1}{2}]_3$	$[0]_0$	$[1]_3$	$[0]_0$	$[0]_0$	$[1]_1$	48	4	$\underline{2}$
4:	$[1]_2$	$[0]_0$	$[0]_0$	$[1]_3$	$[1]_1$	$[1]_1$	$[0]_0$	$[0]_1$	32	2	$\underline{1}$
5:	$[0]_2$	$[0]_0$	$[0]_0$	$[1]_3$	$[1]_1$	$[1]_1$	$[0]_0$	$[1]_1$	8	4	$\underline{2}$
6:	$[\frac{1}{2}]_3$	$[\frac{1}{2}]_3$	$[0]_0$	$[1]_2$	$[0]_0$	$[1]_3$	$[0]_0$	$[1]_1$	48	8	$\underline{4}$
7:	$[0]_0$	$[0]_0$	$[0]_0$	$[2]_\circ$	$[2]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	6	1	$\underline{1}$
8:	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_2$	$[1]_2$	$[1]_2$	$[0]_0$	$[1]_2$	1	16	$\underline{8}$

6. THE LATTICE $N(12\mathbf{A}_2)$

The goal of this section is the following theorem.

Theorem 6.1. *For any $\hbar \in N(12\mathbf{A}_2)$, $\hbar^2 = 12$, and any geometric set $\mathfrak{L} \subset \mathfrak{F}(\hbar)$, one has $|\mathfrak{L}| \leqslant 350$.*

Proof. We proceed as in the previous section, considering $O(N)$-orbits of square 12 vectors $\hbar \in N(12\mathbf{A}_2)$ one by one. There are 29 orbits; $b(\mathfrak{F}) \leqslant 407$ (see Table 3).

6.1. **Configuration 1.** Let $\mathcal{K} := \{k \in \Omega \mid \hbar_k^2 = \frac{2}{3}\}$, and subdivide $\bar{\mathfrak{o}}_3$ into eleven clusters (not disjoint)

$$\mathfrak{c}_k := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_2 \mid l_k \cdot \hbar_k = \tfrac{2}{3}\}, \quad k \in \mathcal{K},$$

to show (see §4.5.2) that $\tilde{\mathcal{B}}_{56}(\bar{\mathfrak{o}}_3) = \varnothing$.

6.2. **Configuration 3.** Denote $\mathcal{K} := \{k \in \Omega \mid \hbar_k \text{ is a root}\}$, and subdivide $\bar{\mathfrak{o}}_2$ into three pairwise disjoint clusters

$$\mathfrak{c}_k := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_2 \mid k \notin \operatorname{supp} \mathfrak{o}\}, \quad k \in \mathcal{K}.$$

Using these clusters (see §4.5.2), we show that $\tilde{\mathcal{B}}_{20}(\bar{\mathfrak{o}}_2) = \varnothing$. More precisely, we find eight sets $\mathfrak{L} \subset \tilde{\mathcal{B}}_6(\mathfrak{c}_1)$, all of size $|\mathfrak{L}| = 72$. Seven sets are maximal, see §3.4.1. The eighth one is of rank 20, and adding a single maximal orbit (see §3.4.2) from another cluster produces a single maximal set of size 156.

Table 3. The lattice $N(12\mathbf{A}_2)$

1:	$[\![\frac{14}{3}]\!]^+_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$		803	**407**
1:	$[3]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	11	2	$\underline{1}$
2:	$[\frac{5}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[\frac{\bar{1}}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	55	2	$\underline{1}$
3:	$[\frac{5}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	55	8	$\underline{4}$
4:	$[\frac{4}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	55	4	$\underline{2}$
5:	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	11	1	$\underline{1}$
3:	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_1$	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![\frac{2}{3}]\!]_1$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_1$		747	**393**
1:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{1}{3}]_2$	$[0]_0$	$[0]_0$	$[1]_2$	$[0]_0$	$[0]_0$	$[0]_0$	54	2	$\underline{1}$
2:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[1]_2$	$[0]_0$	$[1]_1$	$[0]_0$	$[0]_0$	$[0]_0$	108	4	$\underline{2}$
3:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_1$	$[1]_1$	$[1]_2$	$[0]_0$	$[0]_0$	$[\frac{\bar{1}}{3}]_1$	36	2	$\underline{1}$
4:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_1$	$[1]_1$	$[0]_2$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_1$	36	1	$\underline{1}$
5:	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_2$	$[1]_2$	$[1]_1$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_2$	12	8	$\underline{4}$
6:	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[2]_\circ$	$[2]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	3	1	$\underline{1}$
4:	$[\![\frac{8}{3}]\!]^-_2$	$[\![\frac{8}{3}]\!]^-_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$		743	**379**
1:	$[2]_\circ$	$[2]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	1	4	$\underline{2}$
2:	$[\frac{4}{3}]_1$	$[\frac{4}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[\frac{\bar{1}}{3}]_2$	30	4	$\underline{2}$
3:	$[\frac{4}{3}]_1$	$[\frac{4}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{\bar{2}}{3}]_1$	$[0]_0$	$[\frac{2}{3}]_2$	10	1	$\underline{1}$
4:	$[\frac{4}{3}]_1$	$[\frac{4}{3}]_1$	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	5	16	$\underline{8}$
5:	$[\frac{4}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	30	8	$\underline{4}$
6:	$[\frac{4}{3}]_1$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	60	4	$\underline{2}$
7:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	5	4	$\underline{2}$
8:	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	12	2	$\underline{1}$
9:	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	5	1	$\underline{1}$
5:	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_1$	$[\![6]\!]^+_*$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![\frac{2}{3}]\!]_1$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_1$		855	**351**
1:	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[3]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	9	4	$\underline{2}$
2:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_1$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	9	4	$\underline{2}$
3:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[2]_2$	$[0]_0$	$[0]_1$	$[0]_0$	$[0]_0$	$[0]_0$	36	12	$\underline{4}$
4:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[2]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{\bar{2}}{3}]_2$	9	1	$\underline{1}$
5:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[2]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{\bar{1}}{3}]_2$	$[\frac{1}{3}]_2$	36	4	$\underline{2}$
6:	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[2]_2$	$[0]_2$	$[0]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	6	9	$\underline{3}$
7:	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	$[\frac{1}{3}]_2$	$[2]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	9	16	$\underline{8}$

A direct computation using patterns (see §4.5.1) shows that there are four sets $\mathfrak{L} \in \tilde{\mathcal{B}}_7(\bar{\mathfrak{o}}_5)$; they are all maximal (see §3.4.1) and of size $|\mathfrak{L}| = 44$.

Finally, let $\mathfrak{C} := \bar{\mathfrak{o}}_4 \cup \bar{\mathfrak{o}}_6$; this set is complete and admissible. Listing iterated maximal subsets (see §4.5.4), we find 19 sets $\mathfrak{L} \in \mathcal{B}_{13}(\mathfrak{C})$. All but one are ruled out by counting maximal orbits (see §3.4.3) with the test set $\mathfrak{T} = \bar{\mathfrak{o}}_2$. The exception is $\mathfrak{L} = \bar{\mathfrak{o}}_4$, and in this case we use patterns (see §4.5.1) to show that, for any extension $\mathfrak{L}' \supset \mathfrak{L}$ satisfying (3.10), we have

- $|\mathfrak{L}' \cap \bar{\mathfrak{o}}| \leqslant b(\bar{\mathfrak{o}}) - 4$ for $\bar{\mathfrak{o}} = \bar{\mathfrak{o}}_1$, $\bar{\mathfrak{o}}_3$ or any of the three clusters $\mathfrak{c}_k \subset \bar{\mathfrak{o}}_2$, and
- $|\mathfrak{L}' \cap \bar{\mathfrak{o}}_5| \leqslant b(\bar{\mathfrak{o}}_5) - 33$.

Remark 6.2. The sublattice $\operatorname{span}_2 \mathfrak{C} \subset N$ has no proper mild extensions, and its 2-discriminant violates the condition stated in Proposition 3.4(3). Therefore, when computing $\mathcal{B}_{13}(\mathfrak{C})$, we confine ourselves to subsets $\mathfrak{L} \subset \mathfrak{C}$ of corank at least one. The rest of the argument applies to pseudo-geometric sets as well.

6.3. **Configuration 4.** Denote $\mathcal{K} := \bigl\{k \in \Omega \bigm| \hbar_k^2 = \frac{8}{3}\bigr\}$, and subdivide $\bar{\mathfrak{o}}_5$ into two disjoint clusters

$$\mathfrak{c}_k := \bigl\{\mathfrak{o} \subset \bar{\mathfrak{o}}_5 \bigm| l_k \cdot \hbar_k = \tfrac{4}{3} \text{ for } l \in \mathfrak{o}\bigr\}, \quad k \in \mathcal{K},$$

to show (see §4.5.2) that $\mathcal{B}_{25}(\bar{\mathfrak{o}}_5) = \varnothing$. On the other hand, one has

$$\mathcal{B}_0(\bar{\mathfrak{o}}_1 \cup \bar{\mathfrak{o}}_3 \cup \bar{\mathfrak{o}}_4 \cup \bar{\mathfrak{o}}_7 \cup \bar{\mathfrak{o}}_8 \cup \bar{\mathfrak{o}}_9) = \mathcal{B}_0(\bar{\mathfrak{o}}_2) = \mathcal{B}_0(\bar{\mathfrak{o}}_6) = \varnothing$$

directly as in §4.5.1. (Here, we do use condition (3) in Proposition 3.4.)

6.4. **Configuration 5.** We have $\mathcal{B}_0(\mathfrak{O}) = \varnothing$ directly as in §4.5.1. □

7. The lattice $N(24\mathbf{A}_1)$

The goal of this section is the following theorem.

Theorem 7.1. *Let $\hbar \in N(24\mathbf{A}_1)$, $\hbar^2 = 12$, and let $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ be a geometric set. Then, unless*

- *$|\mathfrak{L}| = 357$ and $\mathfrak{L} = \mathfrak{S}^{\mathrm{i}}_{357}$, see (7.2),*

one has $|\mathfrak{L}| \leqslant 350$.

Proof. We proceed as in the previous sections. Each component $v_k \in D_k^\vee$, $k \in \Omega$, of a vector $v \in N$ is a multiple of the generator $r_k \in D_k$. To save space, we use the following notation for the coefficient α in $v_k = \alpha r_k$:

$$\cdot\ (\alpha = 0), \quad -\text{ or } =\ (\alpha = \pm\tfrac{1}{2}), \quad \circ\ (\alpha = \pm 1), \quad +\ (\alpha = \pm\tfrac{3}{2}), \quad \bullet\ (\alpha = \pm 2).$$

Here, $=$ is used only for l_k and only if $\hbar_k \cdot l_k < 0$; in all other cases, the signs of l_k and $\hbar_k$ agree, so that we have $\hbar_k \cdot l_k \geqslant 0$.

There are 13 orbits $\hbar \in N(24\mathbf{A}_1)$, and $b(\mathfrak{F}) \leqslant 759$ (see Table 4).

Fix a basis $\{r_k\}$, $k \in \Omega$, for $24\mathbf{A}_1$ consisting of roots. The kernel

$$N \bmod 24\mathbf{A}_2 \subset \operatorname{discr} 24\mathbf{A}_1 \cong (\mathbb{Z}/2)^{24}$$

of the extension is the Golay code $\mathcal{C}_{24}$ (see [5]). The map supp identifies codewords with subsets of Ω; then, $\mathcal{C}_{24}$ is invariant under complement and, in addition to $\varnothing$ and Ω, it consists of 759 octads, 759 complements thereof, and 2576 dodecads.

To simplify the notation, we identify the basis vectors r_k (assumed fixed) with their indices $k \in \Omega$. For a subset $\mathcal{S} \subset \Omega$, we let $\bar{1}_{\mathcal{S}} := \sum r$, $r \in \mathcal{S}$, and abbreviate $[\mathcal{S}] := \frac{1}{2}\bar{1}_{\mathcal{S}} \in N$ if $\mathcal{S} \in \mathcal{C}_{24}$ is a codeword.

TABLE 4. The lattice $N(24\mathbf{A}_1)$

1:	−−−−−−−−−−−−−−−−−−−−−−−−		759	**759**
1:	−−−−−··−··−·−···········	759	1	1
2:	○○○○○··○················		999	**507**
1:	○○······················	15	1	1
2:	−−−−−··=··−·−···········	6	4	2
3:	−−−−·−········−··−·····−	60	16	8
3:	++−−−··−··−·−···········		987	**507**
1:	○·○·····················	12	1	1
2:	−−−−−··−··=·=···········	15	1	1
3:	−−−−·−········−··−·····−	60	16	8
4:	+−−−−−−−−−−·−·−···−−−···		807	**471**
1:	○○······················	15	1	1
2:	−−−−−··−··−·=···········	105	1	1
3:	−−−−·−········−··−·····−	168	4	2
4:	·−−−·−−···−········−−···	15	1	1
5:	−−−−−○○−○·−·−·····○·····		875	**451**
1:	−−−−−··−··−·−···········	1	1	1
2:	−−−·−−···−········−−····	72	4	2
3:	−−···−−·−··−·····−·−····	64	8	4
4:	−·−−·−−−−·········=·····	8	1	1
5:	−·−=·−−=−·········−·····	12	1	1
6:	·····○○·················	6	1	1
7:	·····−−·−−····−···−−−···	3	16	8
6:	−−−−−−−−−−−○−○−···−−−···		751	**439**
1:	−−−−−··−··−·−···········	30	1	1
2:	−−−−··−··−·−··········−·	192	2	1
3:	−−−·······−−·−−····=····	96	1	1
4:	−−·−·······−−−··−−······	60	4	2
5:	···········○·○··········	1	1	1
7:	+−−−−○○−··−·−···········		799	**407**
8:	+−−−−−−−○−···−······−··−		803	**407**
9:	−−−−−−−−○−○○·−······−··−		747	**393**
10:	●○○·····················		715	**379**
1:	○··○····················	21	2	2
2:	−−−−−··−··−·−···········	21	32	16
3:	·○○·····················	1	1	1
11:	○○○○○○··················		735	**375**
12:	−−−−−○○−○○−·−···········		743	**375**

7.1. **Configuration 1.** We have $\operatorname{stab} \hbar = M_{24}$ (the Mathieu group, which is the symmetry group of the Golay code) and $\hbar = [\Omega]$. The set $\mathfrak{F}(\hbar) = \bar{\mathfrak{o}}_1$ is complete and admissible; listing iterated maximal subsets (see §4.5.4), after four steps of the algorithm we find out that there are no complete subsets $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ of rank $\operatorname{rk} \mathfrak{L} \leqslant 21$ and size $|\mathfrak{L}| \geqslant 385$ (*cf.* §8.1 below).

7.2. **Configuration 2.** We have $|\operatorname{stab} \hbar| = 11520$ and $\hbar = \bar{1}_{\mathcal{R}}$, where $|\mathcal{R}| = 6$ and $\mathcal{R}$ is a subset of an octad (*cf.* §7.11). We subdivide $\bar{\mathfrak{o}}_3$ into 15 pairwise disjoint clusters

$$\mathfrak{c}_o := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_3 \mid \operatorname{supp} \mathfrak{o} \supset o\}, \quad o \subset \mathcal{R}, \ |o| = 4,$$

and use these clusters (see §4.5.2) to show that $\mathcal{B}_{156}(\bar{\mathfrak{o}}_3) = \varnothing$. (The set $\mathfrak{b}(\mathfrak{o})$, $\mathfrak{o} \subset \bar{\mathfrak{o}}_3$, equals $\{0, \ldots, 4, 6, 8\}$, which simplifies the computation.)

7.3. **Configuration 3.** We have $|\operatorname{stab} \hbar| = 11520$ and $\hbar = [\mathcal{O}] + \bar{1}_{\mathcal{R}}$, where $\mathcal{O}$ is an octad and $\mathcal{R} \subset \mathcal{O}$ a 2-element set. Subdivide $\bar{\mathfrak{o}}_3$ into 15 pairwise disjoint clusters

$$\mathfrak{c}_o := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_3 \mid \operatorname{supp} \mathfrak{o} \supset o\}, \quad o \subset \mathcal{O} \smallsetminus \mathcal{R}, \ |o| = 2.$$

and use these clusters (see §4.5.2) to show that $\mathcal{B}_{156}(\bar{\mathfrak{o}}_3) = \varnothing$. (The set $\mathfrak{b}(\mathfrak{o})$, $\mathfrak{o} \subset \bar{\mathfrak{o}}_3$, equals $\{0, \ldots, 4, 6, 8\}$, which simplifies the computation. In spite of the apparent similarity, this polarized lattice is not equivalent to #2, see §7.2.)

7.4. **Configuration 4.** We have $|\operatorname{stab} \hbar| = 20160$ and $\hbar = [\mathcal{O}] + r$, where $\mathcal{O} \ni r$ is a codeword of length 16. Let $\mathcal{K} := \Omega \smallsetminus \mathcal{O}$.

We subdivide the orbit $\bar{\mathfrak{o}}_3$ into 28 pairwise disjoint clusters

$$\mathfrak{c}_k := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_3 \mid \operatorname{supp} \mathfrak{o} \supset k\}, \quad k \subset \mathcal{K}, \ |k| = 2,$$

and, using these clusters and meta-patterns (see §4.5.3), show that $\mathcal{B}_{50}(\bar{\mathfrak{o}}_3) = \varnothing$. (Note that $|\mathfrak{L} \cap \mathfrak{c}_k| \in \{0, \ldots, 8, 10, 12\}$ for each k and each geometric set $\mathfrak{L}$.)

Now, consider the complete admissible set $\mathfrak{C} := \bar{\mathfrak{o}}_1 \cup \bar{\mathfrak{o}}_2 \cup \bar{\mathfrak{o}}_4$. Listing iterated maximal subsets (see §4.5.4), we find 34 sets $\mathfrak{L} \in \mathcal{B}_{70}(\mathfrak{C})$, of which all but two are ruled out by counting maximal orbits (see §3.4.3), using $\mathfrak{T} = \bar{\mathfrak{o}}_3$ for the test set.

One of the two exceptions is a set $\mathfrak{L}$ of size 87, for which $|\mathfrak{T}_\mu| = 96$ (see §3.4.3 for the notation). Extending the pattern $\pi_{\mathfrak{L}}$ (see §3.4.2) *via* $\mathfrak{o} \mapsto b(\mathfrak{o}) = 2$ for each orbit $\mathfrak{o} \in \mathfrak{T}_\mu$ (and leaving the other values undefined) and applying the algorithm of §4.5.1, we show that $|\mathfrak{L}'| \leqslant 350$ for each extension $\mathfrak{L}' \supset \mathfrak{L}$ satisfying (3.10).

The other exception is $\mathfrak{L} = \bar{\mathfrak{o}}_2$, with $\mathfrak{T}_\mu = \mathfrak{T}$. Arguing as in §3.4.2 (extending $\pi_{\mathfrak{L}}$ by a single extra value $\mathfrak{o} \mapsto 1$ or 2 for some orbit $\mathfrak{o} \subset \bar{\mathfrak{o}}_3$ and analyzing the output), we observe that $|\mathfrak{L}' \cap \mathfrak{c}_k| \in \{0, 12\}$ for each cluster $\mathfrak{c}_k$ and each extension $\mathfrak{L}' \supset \mathfrak{L}$ satisfying (3.10). Using this observation and meta-patterns (see §4.5.3) with the set of values restricted accordingly, we show that $|\mathfrak{L}'| \leqslant 350$ with the exception of a single, up to $O_\hbar(N)$, saturated set $\mathfrak{S}^{\mathrm{i}}_{357}$ of rank 21 and size 357. This set is characterised by any of the eight patterns

$$\pi_{\mathfrak{c}}(\mathfrak{o}) = \begin{cases} 1, & \text{if } \mathfrak{o} \subset \bar{\mathfrak{o}}_2, \\ 2, & \text{if } \mathfrak{o} \subset \mathfrak{c}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathfrak{c} := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_3 \mid \operatorname{supp} \mathfrak{o} \not\supset k\}$ for some fixed $k \in \mathcal{K}$. Alternatively,

$$\mathfrak{S}^{\mathrm{i}}_{357} = \mathfrak{F}(\hbar) \cap \operatorname{span}([\mathcal{K}], \hbar - 4r, k)^\perp. \tag{7.2}$$

7.5. **Configuration 5.** We have $|\operatorname{stab}\hbar| = 2304$ and $\hbar = [\mathcal{O}] + \bar{1}_{\mathcal{R}}$, where $\mathcal{O}$ is an octad and $\mathcal{R} \subset \Omega \smallsetminus \mathcal{O}$ a 4-element set, so that there is an octad $o \supset \mathcal{R}$ such that $|o \cap \mathcal{O}| = 4$ (*cf.* §7.12). We subdivide $\bar{\mathfrak{o}}_3$ into four pairwise disjoint clusters

$$\mathfrak{c}_r := \bigl\{\mathfrak{o} \subset \bar{\mathfrak{o}}_3 \bigm| \operatorname{supp}\mathfrak{o} \not\ni r\bigr\}, \quad r \in \mathcal{R},$$

and use these clusters (see §4.5.2) to show that $\mathcal{B}_{60}(\bar{\mathfrak{o}}_3) = \varnothing$. Then, subdividing $\bar{\mathfrak{o}}_2$ into six pairwise disjoint clusters

$$\mathfrak{c}'_s := \bigl\{\mathfrak{o} \subset \bar{\mathfrak{o}}_2 \bigm| \operatorname{supp}\mathfrak{o} \supset s\bigr\}, \quad s \subset \mathcal{R},\ |s| = 2,$$

we show (see §4.5.2) that $\mathcal{B}_{39}(\bar{\mathfrak{o}}_2) = \varnothing$.

7.6. **Configuration 6.** We have $|\operatorname{stab}\hbar| = 11520$ and $\hbar = [\mathcal{O}] + \bar{1}_{\mathcal{R}}$, where $\mathcal{O}$ is a codeword of length 16 and $\mathcal{R} \subset \Omega \smallsetminus \mathcal{O}$ is a 2-element set. Let $\mathcal{K} := \Omega \smallsetminus (\mathcal{O} \cup \mathcal{R})$. We subdivide $\bar{\mathfrak{o}}_4$ into six clusters (not disjoint)

$$\mathfrak{c}_k := \bigl\{\mathfrak{o} \subset \bar{\mathfrak{o}}_4 \bigm| \operatorname{supp}\mathfrak{o} \ni k\bigr\}, \quad k \in \mathcal{K},$$

and use these clusters (see §4.5.2) to show that $\mathcal{B}_{35}(\bar{\mathfrak{o}}_4) = \varnothing$. Next, we subdivide $\bar{\mathfrak{o}}_2$ into twelve pairwise disjoint clusters

$$\mathfrak{c}'_s := \bigl\{\mathfrak{o} \subset \bar{\mathfrak{o}}_2 \bigm| \operatorname{supp}\mathfrak{o} \supset s\bigr\}, \quad s \in \mathcal{R} \times \mathcal{K},$$

and show (using §4.5.2) that $\mathcal{B}_{22}(\bar{\mathfrak{o}}_2) = \varnothing$. Finally, listing iterated maximal subsets (see §4.5.4) in the complete admissible set $\mathfrak{C} := \bar{\mathfrak{o}}_1 \cup \bar{\mathfrak{o}}_3 \cup \bar{\mathfrak{o}}_5$, we obtain $\mathcal{B}_{29}(\mathfrak{C}) = \varnothing$.

7.7. **Configuration 7.** We have $|\operatorname{stab}\hbar| = 336$ and $\hbar = [\mathcal{O}] + \bar{1}_{\mathcal{R}} + r$, where $\mathcal{O} \ni r$ is an octad and $\mathcal{R} \subset \Omega \smallsetminus \mathcal{O}$ a 2-element set. This configuration is equivalent to #1 in $N(8\mathbf{A}_3)$, see §5.2.1, where we treat pseudo-geometric sets as well.

7.8. **Configuration 8.** We have $|\operatorname{stab}\hbar| = 660$ and $\hbar = [\mathcal{O}] + r + s$, where $\mathcal{O} \ni r$ is a dodecad and $s \in \Omega \smallsetminus \mathcal{O}$. This configuration is equivalent to #1 in $N(12\mathbf{A}_2)$, see §6.1, where we treat pseudo-geometric sets as well.

7.9. **Configuration 9.** We have $|\operatorname{stab}\hbar| = 432$ and $\hbar = [\mathcal{O}] + \bar{1}_{\mathcal{R}}$, where $\mathcal{O}$ is a dodecad and $\mathcal{R} \subset \Omega \smallsetminus \mathcal{O}$ a 3-element set. This configuration is equivalent to #3 in $N(12\mathbf{A}_2)$, see §6.2, where we mainly treat pseudo-geometric sets. The only exception, explained in Remark 6.2, applies to the present case as well, as the sublattice $\operatorname{span}_2(\bar{\mathfrak{o}}_3 \cup \bar{\mathfrak{o}}_6) \subset N$ has no proper mild extensions.

7.10. **Configuration 10.** We have $|\operatorname{stab}\hbar| = 40320$ and $\hbar = \bar{1}_{\mathcal{R}} + 2r$, where $\mathcal{R}$ is a 2-element set and $r \notin \mathcal{R}$. Using patterns (see §4.5.1) we show that $\mathcal{B}_{28}(\bar{\mathfrak{o}}_2) = \varnothing$. (Note that $\mathfrak{b}(\mathfrak{o}) = \{0, \ldots, 8, 10, 16\}$ for $\mathfrak{o} \subset \bar{\mathfrak{o}}_2$.)

7.11. **Configuration 11.** We have $|\operatorname{stab}\hbar| = 2160$ and $\hbar = \bar{1}_{\mathcal{R}}$, where $|\mathcal{R}| = 6$ and $\mathcal{R}$ is *not* a subset of an octad (*cf.* §7.2). This configuration is equivalent to #1 in $N(6\mathbf{D}_4)$, see §5.1.1, where we treat pseudo-geometric sets as well.

7.12. **Configuration 12.** We have $|\operatorname{stab}\hbar| = 192$ and $\hbar = [\mathcal{O}] + \bar{1}_{\mathcal{R}}$, where $\mathcal{O}$ is an octad and $\mathcal{R} \subset \Omega \smallsetminus \mathcal{O}$ a 4-element set, so that there is *no* octad $o \supset \mathcal{R}$ such that $|o \cap \mathcal{O}| = 4$ (*cf.* §7.5). This configuration is equivalent to #2 in $N(8\mathbf{A}_3)$, see §5.2.2, where we treat pseudo-geometric sets as well. ☐

8. The Leech lattice

Let $N := \Lambda$ be the Leech lattice. We prove the following theorem.

Theorem 8.1. *Let $\hbar \in \Lambda$, $\hbar^2 = 12$, and let $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ be a geometric set $\mathfrak{L} \subset \mathfrak{F}(\hbar)$. Then, unless*

- $|\mathfrak{L}| = 405$ *and* $\mathfrak{L} = \mathfrak{M}^{\mathrm{i}}_{405}$, *see* (8.4), *or*
- $|\mathfrak{L}| = 357$ *and* $\mathfrak{L} = \mathfrak{S}^{\mathrm{ii}}_{357}$, *see* (8.5), *or*
- $|\mathfrak{L}| = 351$ *and* $\mathfrak{L} = \mathfrak{L}^{\mathrm{i}}_{351}$, *see* (8.6)

one has $|\mathfrak{L}| \leqslant 297$.

Proof. It is well known (see, *e.g.*, Theorem 28 in [5, Chapter 10]) that any nonzero class $[\hbar] \in \Lambda/2\Lambda$ is represented by a unique pair $\pm a \in \Lambda$, where $a^2 = 4, 6$ or $a^2 = 8$ and a is part of a fixed coordinate frame. Since, clearly, $a^2 = \hbar^2 \bmod 4$, and since Λ is positive definite and root free, for $\hbar^2 = 12$ we have either

(1) $\hbar = a + 2b$, where $a^2 = b^2 = 4$ and $a \cdot b = -2$ (*type* 6_{22} in *loc. cit.*), or
(2) $\hbar = a + 2b$, where $a^2 = 8$, $b^2 = 4$, and $a \cdot b = -3$ (*type* 6_{32} in *loc. cit.*)

Besides, a pair $a, b \in \Lambda$ as in item (1) or (2) is unique up to $O(\Lambda)$. Thus, there are two $O(\Lambda)$-orbits of square 12 vectors $\hbar \in \Lambda$ (see Theorem 29 in *loc. cit.*)

The two cases are considered below. We use the shortcut $F := \operatorname{span}_2 \mathfrak{F}(\hbar)$.

8.1. Configuration 1: $\hbar = a \bmod 2\Lambda$, $a^2 = 4$.

We have

$$|O_\hbar(F)| = 55180984320, \qquad |\mathfrak{F}| = 891, \qquad \operatorname{rk} F = 23, \qquad \operatorname{discr} F = \mathcal{U} \oplus \left\langle \tfrac{7}{4} \right\rangle,$$

and all index 8 extensions of $F \oplus \mathbb{Z}a$, $a^2 = 4$, are isomorphic and have vector $\hbar$ of the same type. Listing iterated maximal subsets (see §4.5.4) in the full set $\mathfrak{F}(\hbar)$, which is obviously complete and admissible, after four steps we obtain 17 complete sets $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ of rank $\operatorname{rk} \mathfrak{L} \leqslant 21$ and size $|\mathfrak{L}| \geqslant 285$. Using Nikulin's theory [23], one can easily obtain the following statement.

Lemma 8.2. *None of the* 17 *sets above has a proper root-free finite index extension* $S \supset \operatorname{span}_2 \mathfrak{L}$ (*even abstract, not necessarily lying in* Λ) *such that* $\hbar \in 4S^\vee$. ◁

Only eight of the 17 sets are geometric, namely, the sets $\mathfrak{M}^{\mathrm{i}}_{405}$, $\mathfrak{S}^{\mathrm{ii}}_{357}$, and $\mathfrak{L}^{\mathrm{i}}_{351}$ (the subscript indicating the number of planes) described below, two sets of size 297 (ranks 20 and 21), and three sets of size 285 (ranks 20, 21, and 21).

Remark 8.3. Technically, we work in F rather than in Λ itself, which gives us a slightly larger symmetry group: indeed, $O_\hbar(\Lambda)$ induces on F an index 6 subgroup of $O_\hbar(F) = \operatorname{Aut} \mathfrak{F}(\hbar)$ (computed by the `GRAPE` package [20, 21, 30] in `GAP` [13]). That is why we have to consider *abstract* finite index extensions.

The three large sets found can be described in terms of square 4 vectors in Λ, which are relatively easy to handle. Consider the lattice $U := \mathbb{Z}a + \mathbb{Z}b + \mathbb{Z}c$ with the Gram matrix

$$\begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix},$$

and let $V := U + \mathbb{Z}v$, $v^2 = 4$, be its extension such that $v \cdot a = 1$ and the other two products are as follows:

$$v \cdot b = 1, \qquad v \cdot c = -2 \quad \text{for } \mathfrak{M}^{\mathrm{i}}_{405}: \quad |\operatorname{Aut} \mathfrak{L}| = 349920, \quad T \cong -[6, 3, 6], \tag{8.4}$$

$$v\cdot b=1,\qquad v\cdot c=0\qquad \text{for } \mathfrak{S}^{\mathrm{ii}}_{357}:\quad |\operatorname{Aut}\mathfrak{L}|=10080,\qquad T\cong-[2,1,18], \tag{8.5}$$

$$v\cdot b=-2,\quad v\cdot c=1\qquad \text{for } \mathfrak{L}^{\mathrm{i}}_{351}:\quad |\operatorname{Aut}\mathfrak{L}|=31104,\qquad T\cong-[6,0,6]. \tag{8.6}$$

(For the reader's convenience, we also list the size of the group $\operatorname{Aut}\mathfrak{L}$ of the graph automorphisms of $\mathfrak{L}$, computed *via* `GRAPE`, and the *transcendental lattice* T, see §9 below.) Up to $O_\hbar(\Lambda)$, there is a unique isometry $V\hookrightarrow\Lambda$ such that $a+2b\mapsto\hbar$. Then, the set in question is

$$\mathfrak{F}(\hbar)\cap\bigl((\mathbb{Z}\hbar\oplus V^\perp)\otimes\mathbb{Q}\bigr).$$

8.2. **Configuration 2: $\hbar=a \bmod 2\Lambda$, $a^2=8$.** We have

$$O_\hbar(\Lambda)=O_\hbar(F)=M_{24},\qquad |\mathfrak{F}|=759,\qquad \operatorname{rk}F=24,\qquad \operatorname{discr}F=\langle\tfrac14\rangle\oplus\langle\tfrac74\rangle.$$

This lattice has two index 4 extensions: one of them is Λ, and the other, $N(24\mathbf{A}_1)$. Therefore, this configuration is equivalent to #1 in $N(24\mathbf{A}_1)$, see §7.1, where we treat pseudo-geometric sets as well. □

9. Proofs of the main results

In this section, we complete the proof of the principal results of the paper, *viz.* Theorems 1.1 and 1.2.

9.1. **Proof of Theorem 1.1.** Let $X\subset\mathbb{P}^5$ be a smooth cubic 4-fold such that $|\operatorname{Fn}X|>350$. Applying the replanting procedure (see §3.1) to the 3-polarized lattice $M_X\ni h_X$, we obtain a 12-polarized lattice $S\ni\hbar$ embedded to a Niemeier lattice N (see Proposition 3.3) and such that

- one has $\hbar\in 4S^\vee$,
- the torsion $\operatorname{Tors}(N/S)$ is a 2-group,
- S satisfies all conditions of Proposition 3.4.

By Propositions 2.6 and 3.1, there is a bijection between the set of planes in X and the set

$$\mathfrak{L}=\mathfrak{F}(\hbar)\cap S\subset\mathfrak{F}(\hbar)=\bigl\{l\in N \bigm| l^2=4,\ l\cdot\hbar=4\bigr\}$$

of $\hbar$-planes in S. Furthermore, $\mathfrak{L}$ is geometric, essentially, by the definition. Thus, Theorems 5.1, 6.1, 7.1, and 8.1 imply that we are in one of the following situations:

- $N\simeq N(24\mathbf{A}_1)$ and $\mathfrak{L}$ coincides with $\mathfrak{S}^{\mathrm{i}}_{357}$ up to $O_\hbar(N)$,
- $N\simeq\Lambda$ and $\mathfrak{L}$ coincides with $\mathfrak{M}^{\mathrm{i}}_{405}$ up to $O_\hbar(N)$,
- $N\simeq\Lambda$ and $\mathfrak{L}$ coincides with $\mathfrak{S}^{\mathrm{ii}}_{357}$ up to $O_\hbar(N)$,
- $N\simeq\Lambda$ and $\mathfrak{L}$ coincides with $\mathfrak{L}^{\mathrm{i}}_{351}$ up to $O_\hbar(N)$.

The set $\mathfrak{S}^{\mathrm{ii}}_{357}$ is graph isomorphic to $\mathfrak{S}^{\mathrm{i}}_{357}$; this is established by the `GRAPE` package [20, 21, 30] in `GAP` [13].

Proposition 9.1. *Let $\mathfrak{L}$ be one of the sets $\mathfrak{M}^{\mathrm{i}}_{405}$, $\mathfrak{S}^{\mathrm{ii}}_{357}$ or $\mathfrak{L}^{\mathrm{i}}_{351}$. Then, there exists a unique (up to $PGL(\mathbb{C},6)$) smooth cubic $X\subset\mathbb{P}^5$ such that $\operatorname{Fn}X$ is graph isomorphic to $\mathfrak{L}$.*

Proof. Let $\mathfrak{L}=\mathfrak{M}^{\mathrm{i}}_{405}$. The lattice $S=\operatorname{span}_2\mathfrak{L}\subset\Lambda$ is of rank 21 and has no proper mild extension, see Lemma 8.2. According to Proposition 3.4, the 3-polarized lattice $M\ni h$ obtained from $S\ni\hbar$ by the inverse replanting procedure (*cf.* §3.1) admits a primitive embedding to $\mathbf{L}$ with even orthogonal complement. Thus, the existence of a smooth cubic $X\subset\mathbb{P}^5$ such that $\operatorname{Fn}X$ is graph isomorphic to $\mathfrak{L}$ follows from the surjectivity of the period map (see Theorem 2.3).

According to the global Torelli theorem 2.1, the projective equivalence classes of smooth cubics $X \subset \mathbb{P}^5$ such that $\operatorname{Fn} X$ is graph isomorphic to $\mathfrak{L}$ (equivalently, the 3-polarized lattice $M_X \ni h_X$ is isomorphic to $M \ni h$, *cf.* Lemma 8.2) are in a natural bijection with the $O^+(\mathbf{L})$-orbits of primitive embeddings $M \hookrightarrow \mathbf{L}$ such that $M^\perp$ is even, where $O^+(\mathbf{L})$ is the group of auto-isometries of $\mathbf{L}$ preserving the positive sign structure.

The classification of embeddings can be obtained using Nikulin's theory [23]. For any embedding $M \hookrightarrow \mathbf{L}$ such that $M^\perp$ is even, the genus of the transcendental lattice $T = M^\perp$ is determined by the discriminant $\operatorname{discr} M \cong -\operatorname{discr} T$. In our case, this implies that $T \cong -[6, 3, 6]$. The isomorphism classes of embeddings under consideration are in a bijection with

$$O(M)\backslash \operatorname{Aut}(\operatorname{discr} M)/O^+(T).$$

Using `GRAPE`, one can check that the natural homomorphism

$$\operatorname{Aut} \mathfrak{L} = O_h(M) \to \operatorname{Aut}(\operatorname{discr} T)$$

is surjective, and to complete the proof of the uniqueness there remains to notice that T admits an orientation reversing autoisometry.

If $\mathfrak{L} = \mathfrak{L}^{\mathrm{i}}_{351}$, the proof is literally the same except that now $T \cong -[6, 0, 6]$.

Likewise, in the case $\mathfrak{L} = \mathfrak{S}^{\mathrm{ii}}_{357}$ the proof is literally the same except that now we have $T \cong -[2, 1, 18]$. In view of Lemma 8.2, we do not need to consider the graph isomorphic set $\mathfrak{S}^{\mathrm{i}}_{357} \subset N(24\mathbf{A}_1)$, as it results in the same lattice M. □

As an immediate consequence of the uniqueness given by Proposition 9.1, we conclude that $\mathfrak{M}^{\mathrm{i}}_{405}$ is graph isomorphic the configuration of planes in the Fermat cubic, see §2.5.

Lemma 9.2. *The configuration of planes in the Clebsch–Segre cubic Y, see §2.6, is graph isomorphic to $\mathfrak{S}^{\mathrm{i}}_{357} \simeq \mathfrak{S}^{\mathrm{ii}}_{357}$.*

Proof. The only other alternative would be $\operatorname{Fn} Y \simeq \mathfrak{M}^{\mathrm{i}}_{405}$. However, $\mathfrak{M}^{\mathrm{i}}_{405}$ does not admit a faithful action of $\mathbb{S}_7$ (as $7!$ does not divide $|\operatorname{Aut} \mathfrak{M}^{\mathrm{i}}_{405}| = 349920$). Alternatively, by Corollary 2.2 and (8.4), the Fermat cubic does not admit a real structure with respect to which the classes of the real planes span a lattice of rank 21 (*cf.* a more detailed argument in §9.2). □

Theorem 1.1 is an immediate consequence of Proposition 9.1 and Lemma 9.2. □

9.2. **Proof of Theorem 1.2.** Let $Z \subset \mathbb{P}^5$ be a smooth real cubic and $c\colon Z \to Z$ the real structure. By Corollary 2.8, the classes of the real planes in Z span over $\mathbb{Q}$ the sublattice $M^c_Z := M_Z \cap \operatorname{Ker}(1 - c^*)$. Perturbing, if necessary, the period of Z (see Theorem 2.3), we can assume that $M_Z = M^c_Z$ *i.e.*, *all* planes contained in Z are real. Then, if Z contains at least 357 such planes, Theorem 1.1 implies that Z is projectively equivalent to either the Fermat cubic of the Clebsch–Segre cubic. In the latter case, $T_Z \cong -[6, 3, 6]$ and the assumption that all planes in Z are real contradicts Corollary 2.2. □

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