HOMEWORKS

Homework 3 (due on 18/11)

Problem 1 (3-3: 7; surfaces of revolution of constant curvature). Let $x = \varphi(v)$, $z = \psi(v)$, $\varphi(v) \geq 0$, be a plane curve parametrized by the arc length v and let

$$\mathbf{r}(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, \psi(v))$$

be the surface obtained by revolving the curve about the z-axis. Show that

(1) φ satisfies $\varphi'' + K\varphi = 0$ and $\psi = \int \sqrt{1 - (\varphi')^2}\, dv$.

(2) Any surface of revolution of constant Gaussian curvature $K \equiv 1$ and orthogonal to the Oxy-plane is given by

$$\varphi(v) = C \cos v, \qquad \psi(v) = \int_0^v \sqrt{1 - C^2 \sin^2 v}\, dv,$$

where $C = \text{const}$ ($C = \varphi(0)$). Determine the domain of v and sketch the surfaces for $C = 1$, $C > 1$, and $C < 1$. Show that $C = 1$ gives a sphere.

(3) Any surface of revolution of constant Gaussian curvature $K \equiv -1$ is of one of the following three types (in appropriate coordinates):

1. $\varphi(v) = C \cosh v$, $\psi(v) = \int_0^v \sqrt{1 - C^2 \sinh^2 v}\, dv$,
2. $\varphi(v) = C \sinh v$, $\psi(v) = \int_0^v \sqrt{1 - C^2 \cosh^2 v}\, dv$,
3. $\varphi(v) = e^v$, $\psi(v) = \int_0^v \sqrt{1 - e^{2v}}\, dv$,

where $C = \text{const}$. Determine the domain of v and sketch the surfaces.

(4) Any surface of revolution with $K \equiv 0$ is a right circular cylinder, right circular cone, or plane. Write down the corresponding equations.

Homework 2 (due on 2/10)

Problem 1. Show that a simple closed plane curve is convex (in the sense that it lies to one side of each of its tangents) if and only if its interior is convex (in the sense that it contains the segment connecting any pair of its points).

Problem 2. Assume that the curvature κ of a simple closed plane curve C has isolated zeroes only. Then, show that C is convex (see Problem 1) if and only if it is *locally convex*, *i.e.*, it has no points of inflexion. (*Remark*: I do not know if it helps, but a point of inflexion is a point where κ changes sign. Why?)

Homework 1 (due on 23/09)

Problem 1 (1-3: 2). A disk of radius 1 rolls in the xy-plane without slipping along the x-axis. The curve described by a point in the circumference of the disk is called a *cycloid*.

(1) Find a parametric equation $\alpha(t)$ of the cycloid and determine its singular points (i.e., points where $\alpha' = 0$).

(2) Find the length of one arc of the cycloid (corresponding to one full turn of the disk).

Problem 2 (1-3: 9; a nonrectifiable curve). Use the geometric definition of arc length via broken lines given in class (see also Exercise 8).

Consider the plane curve $\alpha\colon [0,1] \to \mathbb{R}^2$, $\alpha(t) = (t, t\sin(\pi/t))$ if $t \neq 0$ and $\alpha(0) = (0,0)$. (This parameterization is continuous, but not differentiable!) Show that (with any reasonable definition) the length of the arc of the curve corresponding to the segment $1/(n+1) \leq t \leq 1/n$ is at least $2/(n+\frac{1}{2})$. Conclude that the length of the whole curve is infinite.

Problem 3 (the definition of curvature). Let $\alpha(t)$ be a vector valued function with $|\alpha| \equiv 1$. Pick a value t_0 and let $\varphi(t)$ be the angle between $\alpha(t)$ and $\alpha(t_0)$. (To remove the ambiguity and obtain a continuous function, we can assume that t is close to t_0 and request that $-\pi/2 \leq \varphi \leq \pi/2$.) Show that $|\alpha'(t_0)| = |\varphi'(t_0)|$.