**Math 433, Fall 2000**

## Solutions to Final Exam

**Problem 1.** Find the coefficients of the first and second fundamental forms, principal, Gaussian, and mean curvatures, the lines of curvature, and the asymptotic curves of *Enneper's surface*

$$\mathbf{x}(u,v) = \Big(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + u^2v,\ u^2 - v^2\Big).$$

Solution: This is just a direct calculation, so I skip it. Here is the answer:

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0; \qquad e = -g = 2, \quad f = 0; \qquad k_1 = -k_2 = \frac{2}{E}; \qquad H = 0, \quad K = -\frac{4}{E^2};$$

the lines of curvature are $u = \text{const}$ and $v = \textit{const}$; the asymptotic curves are $u \pm v = \text{const}$.

**Problem 2.** (1) A nonsingular linear map $A\colon V \to W$ of 2-dimensional inner product spaces is called a *similitude* if there is a constant $\lambda$ such that $(Ax, Ay) = \lambda(x, y)$ for each pair $x, y \in V$. (In other words, $A$ is a constant times an isometry.) Show that if $A$ is **not** a similitude, there is a **unique** (up to reordering and multiplication by $-1$) pair of orthonormal vectors $\mathbf{e}_1, \mathbf{e}_2 \in V$ such that $A\mathbf{e}_1$ and $A\mathbf{e}_2$ are orthogonal.

(2) Let $\varphi\colon S_1 \to S_2$ be a diffeomorphism of surfaces so that $d\varphi$ is never a similitude. Use part (1) to show that a neighborhood of each point of $S_1$ admits an orthogonal parametrization $\mathbf{r}_1$ such that $\varphi \circ \mathbf{r}_1$ is an orthogonal parametrization of $S_2$.

Solution: (1) is an easy exercise in linear algebra. Given an orthonormal basis $\mathbf{x}_1, \mathbf{x}_2$ of $V$, the vectors $\mathbf{e}_1, \mathbf{e}_2$ are found in the form $\mathbf{e}_1 = \mathbf{x}_1 \cos\alpha + \mathbf{x}_2 \sin\alpha$, $\mathbf{e}_2 = -\mathbf{x}_1 \sin\alpha + \mathbf{x}_2 \cos\alpha$ from the equation $(A\mathbf{e}_1, A\mathbf{e}_2) = 0$, which is homogeneous in $\cos\alpha, \sin\alpha$. In particular, it follows that $\alpha$ can be chosen a continuous function of the coefficients of the matrix of $A$.

(2) In view of therosteritem1 one can find two orthogonal differentiable unit vector fields $\mathbf{e}_1, \mathbf{e}_2$ on $S_1$ such that $d\varphi(\mathbf{e}_1)$ and $d\varphi(\mathbf{e}_2)$ are also orthogonal. The statement follows from the existence theorem applied to the line fields defined by $\mathbf{e}_1, \mathbf{e}_2$.

**Problem 3.** Let $\mathbf{x} = \mathbf{x}(u, v)$ be a regular parametrized surface. A *parallel surface* to $\mathbf{x}$ is a parametrized surface

$$\mathbf{y}(u,v) = \mathbf{x}(u,v) + aN(u,v),$$

where $a$ is a constant. Prove that:

(1) $\mathbf{y}_u \times \mathbf{y}_v = (1 - 2Ha + Ka^2)(\mathbf{x}_u \times \mathbf{x}_v)$, where $K$ and $H$ are the Gaussian and mean curvatures of $\mathbf{x}$ respectively;

(2) at regular points the Gaussian and mean curvatures of $\mathbf{y}$ are, respectively

$$\frac{K}{1 - 2Ha + Ka^2} \quad \text{and} \quad \frac{H - Ka}{1 - 2Ha + Ka^2};$$

(3) if $\mathbf{x}$ has constant mean curvature $H \equiv c = \text{const}$ and $a = 1/2c$, then the parallel surface has constant Gaussian curvature $K \equiv 4c^2$.

Solution: (1) Differentiate $\mathbf{y}$ to get $\mathbf{y}_u = \mathbf{x}_u + a(a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v)$ and $\mathbf{y}_v = \mathbf{x}_v + a(a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v)$, multiply, and use $a_{11}a_{22} - a_{12}a_{21} = K$ and $a_{11} + a_{22} = -2H$.

(2) Let $K'$ and $H'$ be the new curvatures, $F', \dots$, the new coefficients, etc. Denote, for shortness, $A = (1 - 2Ha + Ka^2)$. From (1) it follows that $N' = N$. Hence,

$$\begin{cases} N'_u = N_u = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v = a_{11}(\mathbf{y}_u - aN_u) + a_{21}(\mathbf{y}_v - aN_v), \\ N'_v = N_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v = a_{12}(\mathbf{y}_u - aN_u) + a_{22}(\mathbf{y}_v - aN_v). \end{cases}$$

Now just resolve this system in $N_u$, $N_v$ to get the new coefficients $a'_{ij}$ (The determinant of this system is nonzero at regular points. Guess why?)

$$\begin{cases} N'_u = \dfrac{a_{11} + Ka}{A}\mathbf{y}_u + \dfrac{a_{21}}{A}\mathbf{y}_v, \\ N'_v = \dfrac{a_{12}}{A}\mathbf{y}_u + \dfrac{a_{22} + Ka}{A}\mathbf{y}_v \end{cases}$$

and use $H' = -\dfrac{1}{2}(a'_{11} + a'_{22})$ and $K' = a'_{11}a'_{22} - a'_{21}a'_{12}$.

(3) follows immediately from (2).

**Problem 4.** Let $\alpha\colon I \to S \subset \mathbb{R}^3$ be a curve on a regular surface $S$. Consider the ruled surface generated by the family $\big[\alpha(t), N(\alpha(t))\big]$. Prove that $\alpha$ is a line of curvature in $S$ if and only if this ruled surface is developable.

SOLUTION: Let $N(t) = N(\alpha(t))$. Then $N' = dN(\alpha')$ by definition. If $\alpha$ is a line of curvature, then $\alpha' = \lambda N'$, which implies $(N, N', \alpha') = 0$, i.e., the surface is developable. If the surface is developable, then, by definition, $(N, N', \alpha') = 0$ and, since $\alpha' \perp N$ and $N' \perp N$, this implies $\alpha' \parallel N'$, i.e., $\alpha$ is a line of curvature.