Math 433, Fall 2000

Solutions to the first midterm

Problem 1. Prove that the binormal at a point M of a regular curve is the limit position of the common perpendicular to the tangents at M and a close point M' as M' approaches M.

Solution: First, discuss the direction. Let $M = \mathbf{r}(s_0)$ and $M' = \mathbf{r}(s) = \mathbf{r}(s + \Delta s)$. Then the common perpendicular to $\mathbf{t}_0 = \mathbf{t}(s_0)$ and $\mathbf{t} = \mathbf{t}(s)$ is $\mathbf{t}_0 \times \mathbf{t} = \mathbf{t}_0 \times (\mathbf{t}_0 + \mathbf{t}'(s_0)\Delta s + o(\Delta s)) = (\mathbf{t}_0 \times \mathbf{t}'(s_0))\Delta s + o(\Delta s)$. To avoid zero in the limit, divide by Δs; then the limit is $\mathbf{t}_0 \times \mathbf{t}'(s_0) = \varkappa \mathbf{b}$.

Now, consider the position of the line. We are interested in the value of u such that the equation $(\mathbf{r}_0 + u\mathbf{t}_0) + v(\mathbf{t}_0 \times \mathbf{t}) = \mathbf{r} + w\mathbf{t}$ has a solution in (v, w). (Why?) Take the cross-product by $\mathbf{t}$ and then, the dot product by $(\mathbf{t}_0 \times \mathbf{t})$. We obtain $u = (\mathbf{r} - \mathbf{r}_0, \mathbf{t}, \mathbf{t}_0 \times \mathbf{t})/(\mathbf{t}_0 \times \mathbf{t})^2$. (Note that the solution is known to exist, as close tangents are not parallel, and we only need to find it.) Now, as above, divide the numerator and the denominator by Δs and take limit as $\Delta s \to 0$; the limit value of u is $(0, \mathbf{t}, \varkappa\mathbf{b})/(\varkappa\mathbf{b})^2 = 0$.

Problem 2. Find the intrinsic equation of the plane curve which meets under a constant angle all the lines passing through a fixed point.

Solution: Let $\mathbf{r}(s)$ be the radius-vector of the curve (with the origin at the fixed point) and $a \neq 0$ the cosine of the fixed angle φ. (The case $\varphi = \pi/2$ obviously corresponds to a circle $\varkappa = \text{const.}$) Then one has $\mathbf{rt} = a|\mathbf{r}|$. This implies $|\mathbf{r}|' = \mathbf{rr}'/|\mathbf{r}| = a$, and the differentiation yields $\mathbf{t}^2 + \varkappa\,\mathbf{rn} = a^2$. Thus, one has

$$\mathbf{r} = (\mathbf{rt})\mathbf{t} + (\mathbf{rn})\mathbf{n} = a|\mathbf{r}|\mathbf{t} + \frac{a^2 - 1}{\varkappa}\,\mathbf{n} \quad \text{and, hence,} \quad |\mathbf{r}| = \pm\frac{\sqrt{1 - a^2}}{\varkappa}\,.$$

Since $|\mathbf{r}|' = a$, this gives us $(1/\varkappa)' = \pm a/\sqrt{1 - a^2}$ and, finally, $\varkappa = \pm\sqrt{1 - a^2}/a(s - C)$, where C is a constant. (*Remark*: of course, the constant is irrelevant here: it represents a shift of the initial point on the curve. Thus, one can as well write the solution as $\varkappa = \pm\tan\varphi/s$.)

Problem 3. Let $F(x, y)$ be a class C^2 function and a a regular value of F. Show that the curvature $\varkappa$ of the plane curve $F(x, y) = a$ at a point (x, y) is given by

$$\varkappa = -\frac{F_x^2 F_{yy} - 2F_x F_y F_{xy} + F_y^2 F_{xx}}{(F_x^2 + F_y^2)^{3/2}}\,.$$

Solution: Since (x, y) is a regular point of F, at this point either $F_y \neq 0$ or $F_x \neq 0$ and, hence, the curve can locally be represented as $y = y(x)$ or $x = x(y)$, respectively. In the former case (the latter case is absolutely similar) the implicit differentiation yields $F_x + F_y y' = 0$ and $F_{xx} + 2F_{xy}y' + F_{yy}{y'}^2 + F_y y'' = 0$. It remains to solve for y' and y'' and to use the formula $\varkappa = y''/(1 + {y'}^2)^{3/2}$.

Problem 4. The curvature $\varkappa(s)$ and the torsion $\tau(s)$ of a curve $\mathbf{r}(s)$ (where s is the arc length) are periodic with the same period T. Show that there is a rigid motion of the space $\mathbb{R}^3$ taking $\mathbf{r}(s)$ to $\mathbf{r}(s + T)$ (for all s).

Solution: Pick a point s_0. There is a rigid motion $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ taking $\mathbf{r}(s_0)$ to $\mathbf{r}(s_0 + T)$ and the natural frame at s_0 to that at $s_0 + T$. The original curve and its image under φ have the same intrinsic equation (as $\varkappa$ and τ are T-periodic), both pass through $\mathbf{r}(s_0 + T)$, and have the same natural frame at this point. Due to the uniqueness theorem, they coincide.