Math 310 Topology, Spring 2000 *You can use your* **textbooks** *only!*
Final Exam

Name: ______________________________
I.D.: ______________________________

Problem 1. A topological pair (X, A) is called a *Borsuk pair* (or a *cofibration*) if $X \times I$ retracts to $(X \times \{0\}) \cup (A \times I)$. Show that:

(1) (X, A) is a Borsuk pair if and only if it has the following *homotopy extension property*: given a topological space Y, a map $f\colon X \to Y$, and a homotopy $h\colon A \times I \to Y$ of the restriction $f|_A$, there is an extension of h to a homotopy of f.
(2) If (X, A) is a Borsuk pair, so is $(X \times I, A \times I)$.

Problem 2. Given a pair (X, A), a space Y, and a map $f\colon A \to Y$, define $X \cup_f Y$ to be the quotient space $X \sqcup Y/\{a \sim f(a) \text{ for } a \in A\}$.

(1) Let (X, A) be a Borsuk pair and $f\colon A \to Y$ and $g\colon A \to Y$ two homotopic maps. Show that $X \cup_f Y$ is homotopy equivalent to $X \cup_g Y$. (*Hint*: Consider the space $(X \times I) \cup_h Y$, where h is a homotopy between f and g.)
(2) Define X to be the unit circle S^1 with two 2-disks D^2 attached via the maps $f, g\colon \partial D^2 \to S^1$, $f\colon z \mapsto z^2$ and $g\colon z \mapsto z^3$. Show that X is homotopy equivalent to S^2. Is X homeomorphic to S^2?

Problem 3. Let X be a topological space and G a group (considered as a discrete topological space). A G-action on X is a continuous map $G \times X \to X$, $(g, x) \mapsto g(x)$, such that $1(x) = x$ and $g(h(x)) = (gh)(x)$.

(1) Show that the quotient projection $X \to \bar{X} = X/\{x \sim g(x),\ x \in X,\ g \in G\}$ is a covering if and only if the action is *properly discontinuous*, i.e., every point $x \in X$ has a neighborhood U such that all the images $g(U)$, $g \in G$ are pairwise disjoint.
(2) Let X be simply connected and the action properly discontinuous. What is $\pi_1(\bar{X})$?

Problem 4. Show that a Möbius band does not retract to its boundary.