Math 310 Topology, Spring 2000

Solutions to Midterm 2

Problem 1. A subspace $A \subset X$ is called a *retract* of X if there is a map $\rho\colon X \to A$ such that $\rho(a) = a$ for all $a \in A$. (Such a map is called a *retraction.*) Show that a retract of a Hausdorff space is closed. Show, further, that A is a retract of X if and only if the pair (X, A) has the following *extension property*: any map $f\colon A \to Y$ to any topological space Y admits an extension $X \to Y$.

Solution: (1) Let $x \notin A$ and $a = \rho(x) \in A$. Since X is Hausdorff, x and a have disjoint neighborhoods U and V, respectively. Then $\rho^{-1}(V \cap A) \cap U$ is a neighborhood of x disjoint from A. Hence, A is closed.

(2) Let A be a retract of X and $\rho\colon A \to X$ a retraction. Then for any map $f\colon A \to Y$ the composition $f \circ \rho\colon X \to Y$ is an extension of f. Conversely, if (X, A) has the extension property, take $Y = A$ and $f = \mathrm{id}_A$. Then any extension of f to X is a retraction.

Problem 2. A compact Hausdorff space X is called an *absolute retract* if whenever X is embedded into a normal space Y the image of X is a retract of Y. Show that a compact Hausdorff space is an absolute retract if and only if there is an embedding $X \hookrightarrow [0,1]^J$ to some cube $[0,1]^J$ whose image is a retract. (*Hint*: Assume the Tychonoff theorem and previous problem.)

Solution: A compact Hausdorff space is normal, hence, completely regular, hence, it admits an embedding to some $[0,1]^J$. Due to the Tychonoff theorem $[0,1]^J$ is compact. Since it is also Hausdorff, it is normal, and, by the definition of absolute retract, the image of X is a retract.

Let X be a retract of some $[0,1]^J$ and $\rho\colon [0,1]^J \to X$ a retraction. (I identify X with its image.) Let $i\colon X \to Y$ be an embedding of X to another normal space Y. Since X is compact, its image is closed and, due to the Tietze theorem, the map $i^{-1}\colon i(X) \to X \hookrightarrow [0,1]^J$ admits an extension $f\colon Y \to [0,1]^J$. Then the composition $i \circ \rho \circ f\colon Y \to i(X)$ is a retraction.

Problem 3. Classify up to homeomorphism the Latin letters (in computer modern sans serif font):

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

(*Hint*: In order to distinguish nonhomeomorphic spaces try to classify their points according to how their removal distorts the connectedness of the space and/or neighborhood of the point.)

Solution: The homeomorphism classes of the letters, along with the invariants which distinguish between them, are given in the following table: (See also comments below)

N	Class	Endpoints	$I_1(X)$	$I_2(X)$	$\#I_3(X)$	$\#I_4(X)$
1	A R	2	$\mathrm{pt} \cup \mathrm{pt} \cup I^\circ \cup I^\circ$	$\hat{I} \cup \hat{I}$	0	0
2	B	0	X itself	$\varnothing$	0	0
3	C G I J L M N S U V W Z	2	$\mathrm{pt} \cup \mathrm{pt}$	I°	0	0
4	D O	0	X itself	$\varnothing$	0	0
5	E F T Y	3	$\mathrm{pt} \cup \mathrm{pt} \cup \mathrm{pt}$	$I^\circ \cup I^\circ \cup I^\circ$	1	0
6	H K	4	$\mathrm{pt} \cup \mathrm{pt} \cup \mathrm{pt} \cup \mathrm{pt}$	$I^\circ \cup I^\circ \cup I^\circ \cup I^\circ \cup I^\circ$	2	0
7	P	1	$\mathrm{pt} \cup I^\circ$	$\hat{I}$	0	0
8	Q	2	$\mathrm{pt} \cup \mathrm{pt} \cup I^\circ$	$I^\circ \cup I^\circ$	1	0
9	X	4	$\mathrm{pt} \cup \mathrm{pt} \cup \mathrm{pt} \cup \mathrm{pt}$	$I^\circ \cup I^\circ \cup I^\circ \cup I^\circ$	0	1

Here pt, I°, and $\hat{I}$ stand, respectively, for a single point, open interval $(0,1)$, and semi-open interval $(0,1]$, and $\cup$ denotes disjoint union of topological spaces. In order to distinguish the classes, introduce the subsets $I_n(X)$ of the points $x \in X$ such that $X \smallsetminus \{x\}$ consists of n connected components. Also, let us call a point $x \in X$ an *endpoint* of X if $U \smallsetminus \{x\}$ is connected for any connected neighborhood U of x. Clearly, any homeomorphism $X \to Y$ must take $I_n(X)$ to $I_n(Y)$ and the endpoints of X to those of Y.

In view of the table the only problem left is to prove that classes 2 and 4 are different. For this purpose just note that any *pair* of distinct points divides letters D and O (which are obviously homeomorphic to S^1) into two intervals, while in B there is a pair of points which divides it into three intervals.

Remark. Those who just copied this solution from the 1997 midterm were supposed to also give a rigorous justification for their count of endpoints.

Problem 4. Show that a connected regular space consisting of more than one point is uncountable. (*Hint*: Show first that the claim holds for a normal space.)

SOLUTION: Any countable space is, obviously, Lindelöf, and a Lindelöf regular space is normal. Thus, it suffices to show that a connected normal space X is uncountable. Pick two distinct points $x, y \in X$. Since X is normal, points are closed and, due to the Urysohn lemma, there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$. The image $f(X)$ is connected (as so is X) and contains $\{0,1\}$; hence, it coincides with $[0,1]$ and is uncountable. On the other hand, the image of a countable set (under any map) is at most countable.