Math 310 Topology, Spring 2000

Solutions to Midterm 1

Problem 1. Prove that $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to the 2-torus T^2. (Here $\mathbb{R}^2$ is regarded as a group and $\mathbb{Z}^2$, as a subgroup. Since $\mathbb{R}^2$ is also a topological space, the quotient group has topology as well.)

Solution: The square $[0,1]\times[0,1]\subset\mathbb{R}^2$ contains a representative of each coset. Hence, set theoretically the two quotients coincide. Of course, one should also check that so do the topologies; this is tedious, but straightforward.

Problem 2. Let Y be the half-plane $\{(x,y)\in\mathbb{R}^2 \mid x\geqslant 0\}$. Define $\tilde{X}=Y\times\mathbb{N}$ and $X=\tilde{X}/\sim$, where $\sim$ is the equivalence relation generated by $((0,y),n)\sim((0,y),m)$ for all $y\in\mathbb{R}$ and all pairs $m,n\in\mathbb{N}$. (I.e., we identify the border lines of the half-planes, sewing 'sheets' to a 'book'.) Let $\tilde{A}=\{((x,y),n)\in\tilde{X}\mid x>0,\ y<-1/n\}$ and A the image of $\tilde{A}$ in X. What are $\operatorname{Cl}A$, $\operatorname{Cl_s}A$, and $\operatorname{Cl_s}\operatorname{Cl_s}A$ (where $\operatorname{Cl_s}$ is the sequential closure)?

Solution: $\operatorname{Cl_s}A$ is the image of $\operatorname{Cl}_{\tilde{X}}\tilde{A}$, and $\operatorname{Cl}A=\operatorname{Cl_s}\operatorname{Cl_s}A$ is $\operatorname{Cl_s}A\cup(0,0)$. (In particular, this is an example of a sequential closure that is not sequentially closed!) Almost everything is more or less obvious here; let us prove that $(0,0)\notin\operatorname{Cl_s}A$. Indeed, among neighborhoods of $(0,0)$ are the images $U(\varepsilon;\varepsilon_1,\varepsilon_2,\dots)$ of the saturated open sets $(-\varepsilon,\varepsilon)\times[0,\varepsilon_1)\times\{1\}\cup(-\varepsilon,\varepsilon)\times[0,\varepsilon_2)\times\{2\}\cup\dots$, where $\varepsilon,\varepsilon_1,\dots$ are positive reals. Now let (x_n) be a sequence in A; we need to show that $(0,0)$ has a neighborhood not containing infinitely many members of the sequence. If (x_n) has infinitely many members in one sheet, say, $Y\times\{m\}$, then all these members are not in $U(1/m;1,1,\dots)$. If each sheet $Y\times\{m\}$ contains finitely many of the x_n's, then the intersection of the sequence with $(Y\times\{m\})$ is at a positive distance δ_m from the edge $\{0\}\times\mathbb{R}\times\{m\}$ and, hence, the neighborhood $V(1;\delta_1,\delta_2,\dots)$ is contains none of x_n.

Problem 3. Let $f,g\colon X\to Y$ be two continuous maps.

(1) True or false: the set $A=\{x\in X\mid f(x)=g(x)\}$ is closed?
(2) The same question under the assumption that Y is Hausdorff.

Solution: (2) True. Consider the map $h=f\times g\colon X\to Y\times Y$. Then, obviously, h is continuous and $A=h^{-1}(\Delta)$, where $\Delta=\{(y,y)\}\subset Y\times Y$ is the diagonal. The latter is closed.

(1) False. If Y is not Hausdorff, let $X=Y\times Y$ and take for f, g the two projections $X\to Y$. Then A is the diagonal.

Problem 4. Consider the following topologies on $\mathbb{R}$:

– the standard topology τ_1;
– the lower limit topology τ_2;
– the topology τ_3 defined by the base $\{(a,b)\mid a,b\in\mathbb{Q}\}$;
– the topology τ_4 defined by the base $\{[a,b)\mid a,b\in\mathbb{Q}\}$.

Compare these four topologies and find the closures and the interiors in respect to each of them of the sets $A=(0,\sqrt{2})$ and $B=(\sqrt{2},3)$.

Solution: Straightforward.