Math 310 Topology, Spring 1999

Solutions to Midterm 2

Problem 1. Let X be Hausdorff and $A \subset X$ dense and locally compact. Prove that A is open. Deduce from this that any locally compact subspace of a Hausdorff space is relatively closed, i.e., open in its closure.

SOLUTION: Pick a point $a \in A$ and assume that each neighborhood U of a in X contains limit points of A which are not in A. Clearly, limit points of A which are in U are also limit points of $U \cap A$; hence, $U \cap A$ is not closed in X. On the other hand, a has a compact neighborhood $U \cap A$; since X is Hausdorff, such a neighborhood must be closed in X.

In general, if A is locally compact, it is dense and, hence, open in $\operatorname{Cl} A$.

Problem 2. Let X be a topological space and $A, B \subset X$ compact subspaces.

(1) Is $A \cup B$ compact?
(2) Is $A \cap B$ compact?
(3) The same question under the assumption that X is Hausdorff.

SOLUTION: (1) Yes. Given an open covering of $A \cup B$, it also covers A and B. Hence, there is a finite subcovering of A and a finite subcovering of B; their union is finite and covers $A \cup B$.

(2) No. Let Y be an infinite set and $X = Y \cup \{a, b\}$ (where a, b are two extra points), so that open are any subset of Y, $Y \cup \{a\}$, $Y \cup \{b\}$, and X itself. Let $A = Y \cup \{a\}$ and $B = Y \cup \{b\}$. Then A and B are compact (say, any open covering of A must contain A itself), while $A \cap P = Y$ is an infinite discrete space, hence, noncompact.

(3) Yes. Due to (1) only the intersection part needs proof. Since X is Hausdorff and B is compact, it is closed. Thus, $A \cap B$ is closed in A and, hence, compact.

Problem 3. Let X be a Hausdorff space, $\{K_\alpha\}_{\alpha \in \Lambda}$ a family of compact subspaces, and $U \subset X$ an open set containing $\bigcap_{\alpha \in \Lambda} K_\alpha$. Prove that U contains $\bigcap_{\alpha \in A} K_\alpha$ for some finite subset $A \subset \Lambda$.

SOLUTION: We can assume that X itself is compact (otherwise replace it with one of K_α); hence, so is $X \smallsetminus U$. Since X is Hausdorff, all K_α are closed and $\{X \smallsetminus K_\alpha\}$ is an open covering of $X \smallsetminus U$. Take a finite subcovering; the corresponding K_α's are what we need.

Problem 4. Prove that any collection of disjoint balls (of nonzero radii) in $\mathbb{R}^n$ is countable.

SOLUTION: Any ball contains a rational point; since the balls are disjoint, all the points are distinct and, thus, they enumerate the balls. The set of rational points is countable.

Problem 5. Let $X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots$ be an increasing sequence of subspaces so that each X_i is is closed in X_{i+1}. Let $X = \bigcup_{i=1}^{\infty}$ with the so called *weak* or *direct limit* topology: a set $F \subset X$ is closed if and only if $F \cap X_i$ is closed in X_i for each i.

(1) Show that each X_i is a closed subspace of X.
(2) Show that if each X_i is normal, so is X.

SOLUTION: (1) Each X_n is closed (this is obvious); why is it a subspace? If A is closed in X, then $A \cap X_n$ is closed in X_n by the definition of the topology in X. If A is closed in X_n, then $A \cap X_m$ is closed in X_m for $m \leqslant n$ (as $X_m \subset X_n$ is a subspace) and A is closed in X_m for $m \geqslant n$ as $X_n \subset X_m$ is a closed subspace. Thus, A is also closed in X.

(2) It suffices to prove that, given a closed set $A \subset X$ and a function $f\colon A \to I$, it admits a continuous extension g to X. Construct g by induction, using Tietze's extension theorem. Let $g_1\colon X_1 \to I$ be an extension of $f|_{A \cap X_1}$ to X_1. Assuming that $g_{n-1}\colon X_{n-1} \to I$ such that $g_{n-1} = f$ on $A \cap X_{n-1}$ is already constructed, denote by $f_n\colon X_{n-1} \cup (A \cap X_n) \to I$ the result of pasting g_{n-1} and $f|_{A \cap X_n}$ (it is continuous due to the pasting lemma) and let $g_n\colon X_n \to I$ be an extension of f_n to X_n. Finally, define $g\colon X \to I$ via $g(x) = g_n(x)$ if $x \in X_n$. This function is continuous **due to the fact that X has weak topology**: if $K \subset I$ is closed, then $g^{-1}(K) \cap X_n = g_n^{-1}(K)$ is closed in X_n for all n and, hence, $g^{-1}(K)$ is closed in X.