Math 310 Topology, Spring 1999

Solutions to Midterm 1

Problem 1. Enumerate all the nonhomeomorphic topologies on a three point set.

Solution: The answer is: discrete, anti-discrete, and

$$\begin{array}{llll} \varnothing, \{1\}; & \varnothing, \{1\}, \{1,2\}; & \varnothing, \{1\}, \{2\}, \{1,2\}; & \varnothing, \{1\}, \{2\}, \{1,2\}, \{1,3\}. \\ \varnothing, \{1,2\}; & \varnothing, \{1\}, \{2,3\}; & \varnothing, \{1\}, \{1,2\}, \{1,3\}; & \end{array}$$

Invariants are obvious.

Problem 2. True or false: $\operatorname{Cl} \bigcup A_\alpha = \bigcup \operatorname{Cl} A_\alpha$?

Solution: False. One has $\operatorname{Cl} \bigcup A_\alpha \supset \bigcup \operatorname{Cl} A_\alpha$ (the proof is the same as for two sets) but, in general, not vice versa. Example: the index set is $(0,1)$ and $A_\alpha = \{\alpha\} \subset \mathbb{R}^1$. Then $\operatorname{Cl} \bigcup A_\alpha = [0,1]$ and $\bigcup \operatorname{Cl} A_\alpha = (0,1)$.

Problem 3. Let X, Y be two ordered sets in the order topology. Show that a bijective order preserving function $f\colon X \to Y$ is a homeomorphism.

Solution: Since f is order preserving and bijective, the pull-back of an interval (y_1, y_2) is the interval (x_1, x_2), where $y_i = f(x_i)$, $i = 1, 2$. Thus, f is continuous. The same consideration applies to f^{-1}, which is also order preserving. Hence, f is a homeomorphism.

Problem 4. Let Y be an ordered set in the order topology, X an arbitrary topological space, and $f, g\colon X \to Y$ two continuous functions. Prove that:

(1) the set $\{x \in X \mid f(x) \leqslant g(x)\}$ is closed;
(2) the function $h\colon X \to Y$, $x \mapsto \min\{f(x), g(x)\}$, is continuous.

Solution: (1) Let A be the set in question, $B = X \smallsetminus A\{x \in X \mid f(x) > g(x)\}$, and $x \in B$. Let $a = g(x)$ and $b = f(x)$. Then $a < b$. Pick c_1, c_2, c_3 so that $c_1 < a < c_2 < b < c_3$. Then x is contained in B together with the open set $f^{-1}(c_2, c_3) \cap g^{-1}(c_1, c_2)$. (If c_1 does not exist, i.e., a is the minimal element of Y, replace $(c_1, \cdot$ with $[a, \cdot$. If c_3 does not exist, i.e., b is the maximal element of Y, replace $\cdot, c_3)$ with $\cdot, b]$. Finally, if c_2 does not exist, replace $\cdot, c_2)$ with $\cdot, b)$ and $(c_2, \cdot$ with $(a, \cdot$.)

Statement (2) follows from (1) and the pasting lemma:

$$h(x) = \begin{cases} f(x), & \text{if } f(x) \leqslant g(x), \\ g(s), & \text{if } g(x) \leqslant f(x). \end{cases}$$

Problem 5. Let X be a topological space and $A \subset X$ a subspace. True or false: $\operatorname{Cl}(X \smallsetminus A) = X \smallsetminus \operatorname{Int} A$?

Solution: True. One has $X \smallsetminus \operatorname{Int} A = X \smallsetminus \bigcup U = \bigcap (X \smallsetminus U)$, where U runs over all the open sets contained in A. Clearly, $X \smallsetminus U$ is closed iff U is open, and $X \smallsetminus U \supset X \smallsetminus A$ iff $U \subset A$; thus, $X \smallsetminus U$ runs over all the closed sets containing $X \smallsetminus A$ and $\bigcap (X \smallsetminus U) = \operatorname{Cl} A$.