Math 310 Topology, Spring 1998

Solutions to Midterm 2

Problem 1 (20 pts). Consider the circle S^1 and the intervals $(0,1)$, $(0,1]$, $[0,1)$, and $[0,1]$. Which of these spaces are homeomorphic? Why?

Solution: $[0,1)$ and $(0,1]$ are homeomorphic via $x \mapsto 1-x$. Other pairs are not homeomorphic. For a topological space X denote by $N(X)$ the set of its *nonseparating points*, i.e., points $x_0 \in X$ such that $X \smallsetminus \{x_0\}$ is connected. Clearly, $N(X)$ is a topological invariant. On the other hand, one has $N(S^1) = S^1$, $N(0,1) = \varnothing$, $N(0,1] = N[0,1) =$ point, and $N[0,1] =$ two points; they are all distinct.

Problem 2 $(15+15$ pts).

(1) Show that a connected normal space with more than one point is uncountable.
(2) Show that a connected regular space with more than one point is uncountable.

(*Hint for* (2): any countable space is Lindelöf.)

Solution: (1) Let X be normal. Pick two distinct points $x, y \in X$. There is a Urysohn function $f\colon X \to I$ with $f(x) = 0$ and $f(y) = 1$. Since X is connected, so is $f(X)$; connected subspaces of $\mathbb{R}$ are intervals; hence, $f(X) \supset I$ is uncountable and so is X.

(2) Let X be regular. If it is countable, it is Lindelöf and, hence, normal, which contradicts to (1). □

Problem 3 (20 pts). Show that if $A \subset \mathbb{R}^2$ is countable, then $\mathbb{R}^2 \smallsetminus A$ is path connected. Use this result to prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$.

Solution: Pick $x, y \in \mathbb{R} \smallsetminus A$. There are uncountably many lines through x; hence, there is a line $l \subset \mathbb{R} \smallsetminus A$ through x. There are uncountably many lines through y not parallel to l; hence, there is a line $m \subset \mathbb{R} \smallsetminus A$ through y not parallel to l. Then x and y belong to a path connected subspace $l \cup m \subset \mathbb{R} \smallsetminus A$.

If $f\colon \mathbb{R} \to \mathbb{R}^2$ is a homeomorphism, so is its restriction $\mathbb{R} \smallsetminus \{0\} \to \mathbb{R}^2 \smallsetminus \{f(0)\}$. However, the former space is disconnected and the latter, connected. □

Problem 4 (Weierstraß theorems) $(10+10$ pts). Let X be a topological space and $f\colon X \to \mathbb{R}$ a continuous function.

(1) (Intermediate value theorem.) Assume that X is connected, $a, b \in X$ are two points with $f(a) \leqslant f(b)$, and $C \in \mathbb{R}$ is a number such that $f(a) \leqslant C \leqslant f(b)$. Show that there is a point $c \in X$ such that $f(c) = C$.
(2) Show that if X is compact, then f is bounded and attains its minimal and maximal values.

Solution: (1) A continuous image of a connected space is connected. Hence, $f(X) \supset [a,b] \ni c$.

(2) A continuous image of a compact space is compact; hence, $f(X)$ is compact and, thus, bounded and closed. This means that $\sup f(X) < \infty$ and $f(X) \ni \sup f(X)$, and $\inf f(X) > -\infty$ and $f(X) \ni \inf f(X)$. □