**Math 411, Spring 1997**

## Solutions to the Final Exam

**Problem 1.** Does there exist a continuous map $D^2 \to \partial D^2 = S^1$?

Solution: There sure does. Just map $D^2$ to a single point in $S^1$.

**Problem 2.** Find the fundamental group of a punctured real projective plane (i.e., real projective plane with a small open disk removed from it).

Solution: Punctured $\mathbb{R}p^2$ is homeomorphic to the Möbius band, which, in turn, is homotopy equivalent to $S^1$. Hence, $\pi_1 = \mathbb{Z}$.

**Problem 3.** (Oh, my God!) Give the **homotopy** classification of latin letters. (As usual, use Computer modern sans serif font):

$$\mathsf{A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$$

(Do not construct homotopy equivalences explicitly. Just explain why you think spaces are homotopy equivalent.)

Solution: Here are the classes:

| N | Class | Homotopy type | $\pi_1(X)$ |
|---|---|---|---|
| 1 | A D O P Q R | $S^1$ | $\mathbb{Z}$ |
| 2 | B | $S^1 \vee S^1$ | $\mathbb{Z} * \mathbb{Z}$ (free product) |
| 3 | C E F G H I J K L M N S T U V W X Y Z | pt | $\{1\}$ |

Classes are distinguished by their fundamental groups. Homotopy equivalences can be constructed as follows. Consider, for example, the letter A, which is $S^1$ with two segments attached. Each segment retracts to its end attached to the circle, and there is a homotopy between the retraction and identity map which is identity on that end. Hence, both the retraction and homotopy extend to the whole space and give a homotopy equivalence $\mathsf{A} \to S^1$. (The homotopy inverse map is the inclusion $S^1 \hookrightarrow \mathsf{A}$.)

**Problem 4.** Let $A \subset X$ be a subspace, $\mathrm{in}\colon A \to X$ the inclusion, and $a \in A$ a point.

(1) Does $\mathrm{in}_*\colon \pi_1(A, a) \to \pi_1(X, a)$ need to be a monomorphism?
(2) The same question assuming that $A$ is a retract of $X$.

Solution: (1) No. E.g., $A = S^1 \subset X = D^2$, with $\mathrm{in}_*\colon \mathbb{Z} \to 0$.

(2) Yes. Let $\rho\colon X \to A$ be a retraction. Then $\rho \circ \mathrm{in} = \mathrm{id}_A$ and, hence, $\rho_* \circ \mathrm{in}_* = (\rho \circ \mathrm{in})_* = \mathrm{id}$. Hence, $\mathrm{Ker}\,\mathrm{in}_* \subset \mathrm{Ker}(\rho_* \circ \mathrm{in}_*) = \{1\}$ and $\mathrm{in}_*$ is a monomorphism.